Solutions to exercises: week 4

Comp. 4.21 a) The limiting distribution over $S = \{1, 2, 3, 4, 5\}$ is $(p_1, p_2, p_3, p_4, p_5)$. A set of balance equations becomes

$$\begin{array}{lllllllllll}
8p_1 & = & & & 5p_2 & + & 5p_3, & & & & \\
15p_2 & = & 4p_1 & & & + & 5p_3 & + & 5p_4, & & \\
18p_3 & = & 4p_1 & + & 5p_2 & & & + & 5p_4 & + & 5p_5, \\
14p_4 & = & & & 5p_2 & + & 4p_3 & & & + & 5p_5, \\
10p_5 & = & & & & & 4p_3 & + & 4p_4, & & \\
1 & = & p_1 & + & p_2 & + & p_3 & + & p_4 & + & p_5.
\end{array}$$

b) The limiting distribution p is given by the solution to Equations (4.43) and (4.44). Using the R matrix from Computational Problem 4.1 and MATLAB or MAXIM, we get

$$p = [0.2528,\ 0.1981,\ 0.2064,\ 0.1858,\ 0.1569].$$

Comp. 4.24 a) The limiting distribution over $S = \{0, 1, 2, 3, \dots, 10\}$ is $(p_0, p_1, p_2, p_3, \dots, p_{10})$. A set of balance equations becomes

$$\begin{array}{lllr}
20p_i & = & 10p_{i+1}, & i = 0, 1, 2, \\
20p_i & = & 20p_{i+1}, & i = 3, 4, 5, 6 \\
20p_i & = & 30p_{i+1}, & i = 7, 8, 9, \\
1 & = & p_0 + p_1 + p_2 + p_3 + \cdots + p_{10}. &
\end{array}$$

b) Solving this set of equations with MATLAB or using MAXIM gives:

$$p = [0.0172\ 0.0343\ 0.0687\ 0.1373\ 0.1373$$
$$0.1373\ 0.1373\ 0.1373\ 0.0915\ 0.0610\ 0.0407].$$

Otherwise:
The CTMC in Conceptual Problem 4.2 is a birth and death process. We are given $\lambda = 20$ and $\mu = 10$. Using Equation (4.48) we get

$$\rho_0 = 1,\quad \rho_1 = 2,\quad \rho_2 = 4,\quad \rho_3 = 8,$$
$$\rho_4 = 8,\quad \rho_5 = 8,\quad \rho_6 = 8,\quad \rho_7 = 8,$$
$$\rho_8 = 16/3,\quad \rho_9 = 32/9,\quad \rho_{10} = 64/27.$$

The limiting distribution p is given by Equation (4.50). This yields:

$$p = [0.0172\ 0.0343\ 0.0687\ 0.1373\ 0.1373$$
$$0.1373\ 0.1373\ 0.1373\ 0.0915\ 0.0610\ 0.0407].$$

Comp. 4.25 From Concep. 4.8 is known: the state of the system $A(t)$ is a vector giving the state of the three machines at time t. Using w for "working", b for "blocked", and i for "idle", we get the following definition of $X(t)$,

$$X(t) = \begin{cases} 1 & \text{if } A(t) = \{w, w, w\} \\ 2 & \text{if } A(t) = \{w, b, w\} \\ 3 & \text{if } A(t) = \{w, i, w\} \\ 4 & \text{if } A(t) = \{w, w, i\} \\ 5 & \text{if } A(t) = \{w, i, i\} \\ 6 & \text{if } A(t) = \{b, w, w\} \\ 7 & \text{if } A(t) = \{b, b, w\} \\ 8 & \text{if } A(t) = \{b, w, i\}. \end{cases}$$

Then, performing the usual analysis of transition triggering events, we see that $\{X(t), t \geq 0\}$ is a CTMC with the following rate matrix:

$$R = \begin{bmatrix} 0 & \mu_2 & 0 & \mu_3 & 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & \mu_1 & 0 \\ \mu_1 & 0 & 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & \mu_3 \\ \mu_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From Comp. 4.16 is known: $\mu_1 = 1, \mu_2 = \frac{5}{6}, \mu_3 = 1$. The matrix R becomes:

$$R = \begin{bmatrix} 0 & \frac{5}{6} & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{6} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) The limiting distribution over $S = \{1,2,3,4,5,6,7,8\}$ is $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8)$. A set of balance equations becomes (see also Equations (4.43) and (4.44)), using the R matrix from Conceptual Problem 4.8 with the parameters from Computational Problem 4.16

$$\begin{array}{rcllllllll}
\frac{17}{6}p_1 & = & & & p_3 & & & & + \; p_7 & + \; \frac{5}{6}p_8 \\
2p_2 & = & \frac{5}{6}p_1 & & & & & & & \\
2p_3 & = & & p_2 & & + \; \frac{5}{6}p_4 & & & & \\
\frac{11}{6}p_4 & = & p_1 & & & & + \; p_5 & & & \\
p_5 & = & & & p_3 & & & & & \\
\frac{11}{6}p_6 & = & p_1 & & & & & & & \\
p_7 & = & & p_2 & & & & + \; \frac{5}{6}p_6 & & \\
\frac{5}{6}p_8 & = & & & & + \; p_4 & & + \; p_6 & & \\
1 & = & p_1 & + \; p_2 & + \; p_3 & + \; p_4 & + \; p_5 & + \; p_6 & + \; p_7 & + \; p_8
\end{array}$$

The limiting distribution p is given by the solution to this equations

$$p = [0.1540,\ 0.0642,\ 0.0868,\ 0.1314,\ 0.0868,\ 0.0840,\ 0.1342,\ 0.2585].$$

b) $p_1 = 0.1540$.

c) The last machine is working in states 1, 2, 3, 6, and 7. Hence the desired answer is $p_1 + p_2 + p_3 + p_6 + p_7 = 0.5232$.

d) Throughput Machine 1: $60(p_1 + p_2 + p_3 + p_4 + p_5) = 31.38$ parts per hour.
Throughput Machine 2: $50(p_1 + p_4 + p_6 + p_8) = 31.38$ parts per hour.
Throughput Machine 3: $60(p_1 + p_2 + p_3 + p_6 + p_7) = 31.38$ parts per hour.
Throughput system = 31.38 parts per hour.

Comp. 4.26 a) The phone-reservation system has s reservation agents and can put a maximum of H callers on hold. Each incoming call is directly handled by one agent if available, otherwise the caller is put on hold if possible. The completion rate of an individual call is μ. A caller who is served,is handled by one agent. Hence the transition rate μ_i from state i to state $i-1$ in this birth and death is equal to the product of the number of callers i and μ if $i \leq s$ and is equal to the product of s and μ if $i > s$, otherwise formulated: $\mu_i = \min(i, s)\mu$.

b) In Comp. 4.9 is given: $s = 8, H = 4, K = s + H = 12$. The arrival rate is 60 callers per hour, each call takes an average of 6 minutes to handle. The limiting distribution $p = (p_0, p_1, \ldots, p_{12})$ is given by the solution to Equations (4.43) and (4.44). These equations are, using the R matrix from Example 4.10, and the data from Computational Problem 4.9:

$$\begin{array}{rcll} 60p_0 & = & 10p_1 & \\ 70p_1 & = & 60p_0 & +20p_2 \\ 80p_2 & = & 60p_1 & +30p_3 \\ 90p_3 & = & 60p_2 & +40p_4 \\ 100p_4 & = & 60p_3 & +50p_5 \\ 110p_5 & = & 60p_4 & +60p_6 \\ 120p_6 & = & 60p_5 & +70p_7 \\ 130p_7 & = & 60p_6 & +80p_8 \\ 140p_8 & = & 60p_7 & +80p_9 \\ 140p_9 & = & 60p_8 & +80p_{10} \\ 140p_{10} & = & 60p_9 & +80p_{11} \\ 140p_{11} & = & 60p_{10} & +80p_{12} \end{array} \qquad p_0 + p_1 + \ldots + p_{12} = 1$$

The solution p is

$$p = [0.0023\ 0.0140\ 0.0421\ 0.0843\ 0.1264\ 0.1517$$
$$0.1517\ 0.1300\ 0.0975\ 0.0731\ 0.0548\ 0.0411\ 0.0309].$$

c) This probability is equal to the limiting probability $p_0 = 0.0023$.

d) i. All clerks are busy in states 8, 9, 10, 11, 12. Hence the desired answer is $p_8 + p_9 + p_{10} + p_{11} + p_{12} = .2975$.

ii. The system has to turn away calls when it is in state 12. Hence the desired answer is $p_{12} = 0.0309$.

iii. The expected number of busy clerks in steady state is given by:

$$\sum_{i=0}^{12} \min(i, 8)p_i = 5.8149.$$

d) The avarage number of occupied agents is equal to

$$0 * p_0 + 1 * p_1 + 2 * p_2 + \ldots + 7 * p_7 + 8 * (p_8 + p_9 + \ldots + p_{12}) = 5.8146$$

agents. Hence the fraction of time a reservation agent is busy is equal to $\frac{5.8146}{8} = 0.7268$.

f) The throughput is equal to $60 * (1 - p_{12}) = 58.15$ callers per hour. The throughput is also equal to the product of the average number of busy agents (5.8146) and the service rate of a agent (10 callers per hour), so is equal to 5.8146*10 = 58.15 callers.

Comp. 4.27 a) The capacity of the switch is 6 calls. Calls arrive at a rate of 4 calls per minute. Calls that arrive when the switch is full, are lost. A call has an average duration of 2 minutes. Calls are handled simultaneously. In the state $i (1 \leq i \leq 6)$ calls in the switch, the rate of a transition to state $i - 1$ is equal to $\mu_i = \frac{i}{2}$.

b) The limiting distribution $p = (p_0, p_1, \ldots, p_6)$ is given by the solution to Equations (4.43) and (4.44). Using the parameter values from Example 4.31, we get

$$R = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2.0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 3.0 & 0 \end{bmatrix}.$$

The equations become:

$$\begin{array}{rcllll} 4p_0 & = & & & 0.5p_1 & \\ 4.5p_1 & = & 4p_0 & +p_2 & & \\ 5p_2 & = & 4p_1 & +1.5p_3 & & \\ 5.5p_3 & = & 4p_2 & +2.0p_4 & & p_0 + p_1 + \ldots + p_6 = 1 \\ 6.0p_4 & = & 4p_3 & +2.5p_5 & & \\ 6.5p_5 & = & 4p_4 & +3.0p_6 & & \\ 3.0p_6 & = & 4p_5 & & & \end{array}$$

The solution is

$$p = [0.0011,\ 0.0086,\ 0.0343,\ 0.0913,\ 0.1827,\ 0.2923,\ 0.3898]$$

c) The probability that al lines are busy is equal to $p_6 = 0.3898$.

d) The desired answer is

$$\sum_{i=0}^{6} i p_i = 4.882.$$

e) The fraction of time a line is occupied is equal to $\frac{4.882}{6} = 0.8137$.

f) The throughput is equal to 2.44 calls per minute (= 6 lines * $\frac{1}{2}$ call per line per minute * 0.8137 fraction of time a line is occupied).

Comp. 4.37 Time unit: minute. $\lambda = 20, \mu_1 = 30, \mu_2 = 15, \mu_3 = 7.5$.

a) Balance equations

$$\begin{aligned}
20p_0 &= 30p_1 + 15p_2 + 7.5p_3, \\
50p_1 &= 20p_0 + 15p_4 + 7.5p_5, \\
35p_2 &= 30p_4 + 7.5p_6, \\
27.5p_3 &= 30p_5 + 15p_6, \\
65p_4 &= 20p_1 + 20p_2 + 7.5p_7, \\
57.5p_5 &= 20p_3 + 15p_7, \\
42.5p_6 &= 30p_7, \\
52.5p_7 &= 20p_4 + 20p_5 + 20p_6,
\end{aligned}$$

Normalizing equation

$$\sum_{i=0}^{7} p_i = 1.$$

b) The limiting distribution p is given by

$$p = [0.3777\ 0.1863\ 0.0916\ 0.0790\ 0.0939\ 0.0466\ 0.0517\ 0.0732].$$

Hence $p_0 = 0.3777$.

c) server 1: $p_1 + p_4 + p_5 + p_7 = 0.4000$.
server 2: $p_2 + p_4 + p_6 + p_7 = 0.3104$.
server 3: $p_3 + p_5 + p_6 + p_7 = 0.2505$.

d) $\lambda \cdot (1 - p_7) = 18.536$ customers per hour.
Alternative solution: $0.400 \cdot \mu_1 + 0.3104 \cdot \mu_2 + 0.2505 \cdot \mu_3 = 18.535$.

e) The cost rates are given by

$$c(0) = 0, \quad c(1) = 40 - c, \quad c(2) = 20 - c, \quad c(3) = 10 - c,$$
$$c(4) = 60 - 2c, \quad c(5) = 50 - 2c, \quad c(6) = 30 - 2c, \quad c(7) = 70 - 3c.$$

f) The long run cost rate of the system is given by

$$\sum_{i=0}^{7} c(i)p_i = 24.714 - 0.961c.$$

For long run profitability we must have $0.961c \geq 24.714,$. This implies $c > 25.717$.

Comp. 4.38 The cost rates in each state are given by

$$c(1) = 105, \quad c(2) = 70, \quad c(3) = 70,$$
$$c(4) = 70, \quad c(5) = 35, \quad c(6) = 70,$$
$$c(7) = 35, \quad c(8) = 35.$$

The limiting distribution p is given in Computational Problem 4.25

$$p = [0.1540,\ 0.0642,\ 0.0868,\ 0.1314,\ 0.0868,\ 0.0840,\ 0.1342,\ 0.2585].$$

The long run net revenue is given by

$$\sum_{i=1}^{8} c(i)p_i = 58.6079.$$

Comp. 4.44 Consider the system using server 1. We have $\lambda = 5$ per hour $\mu = 7.5$ per hour, and capacity = 10. Let $X(t)$ be the number of customers in the system at time t. Then $\{X(t), t \geq 0\}$ is a CTMC with state space $\{0, 1, \ldots, 10\}$ as described in Example 4.7. Each customer spends 8 minutes in service on the average, and pays \$10 for it. This means that the system earns money at rate of \$75 per hour that the server is busy. It has to pay the server at a rate of \$20 per hour whether it is busy or not. This revenue structure can be accounted for by the following revenue vector:

$$c(i) = \begin{cases} -20 & \text{if } i = 0, \\ 75 - 20 & \text{otherwise.} \end{cases}$$

Let p be the limiting distribution of $\{X(t), t \geq 0\}$. The long run revenue rate per hour is then given by

$$\sum_{i=0}^{10} c(i)p_i = 75(1 - p_0) - 20.$$

Using the results of Example 4.25, we get

$$p_0 = 0.3372.$$

Hence the long run revenue rate is

$$75(1 - 0.3372) - 20 = \ 29.71 \ \text{dollars/hour.}$$

Similar analysis with server 2 yields the long run revenue rate to be 33.44 dollars/hour. Hence it is more profitable to use the slower but cheaper server 2.

Comp. 4.45 a) Let m_i be the expected time to reach state 5 starting from state i. The equations of Theorem 4.11 are given by

$$\begin{aligned}
m_1 &= \frac{1}{8} + \frac{4}{8}m_2 + \frac{4}{8}m_3, \\
m_2 &= \frac{1}{15} + \frac{5}{15}m_1 + \frac{5}{15}m_3 + \frac{5}{15}m_4, \\
m_3 &= \frac{1}{18} + \frac{5}{18}m_1 + \frac{5}{18}m_2 + \frac{4}{18}m_4, \\
m_4 &= \frac{1}{14} + \frac{5}{14}m_2 + \frac{5}{14}m_3,
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
8m_1 &= 1 + 4m_2 + 4m_3, \\
15m_2 &= 1 + 5m_1 + 5m_3 + 5m_4, \\
18m_3 &= 1 + 5m_1 + 5m_2 + 4m_4, \\
14m_4 &= 1 + 5m_2 + 5m_3.
\end{aligned}$$

b) The solution is given by

$$m_1 = 0.7425, \quad m_2 = 0.6725, \quad m_3 = 0.5625, \quad m_4 = 0.5125.$$

The desired answer is $m_1 = 0.7425$.