Solutions to exercises: week 5

Concep. 5.1 Let the successive lifetimes of the batteries in the replacement problem of Example 5.2 be L_i. Define $N_0 = 0$ and define

$$N_{r+1} = \min\{i > N_r : L_i > 3\}, \quad r \geq 0.$$

Thus the N_rth replacement is the rth planned replacement. Hence the time between the $(r-1)$st and rth planned replacement is given by

$$T_r = \sum_{i=N_{r-1}+1}^{N_r-1} L_i + 3.$$

Since L_i are iid, and $L_i,\ i > N_r$ do not depend upon $L_i,\ i \leq N_r$, $\{T_r, r = 1, 2, 3..\}$ is a sequence of iid random variables. Hence $\{N(t), t \geq 0\}$ is a renewal process.

Concep. 5.7 Let T_n be the time between the arrival of the $(n-1)$st and nth task at the receiving station. $\{T_n, n \geq 1\}$ is given to be a sequence of iid random variables. Hence $\{Y(t), t \geq 0\}$ is a renewal process generated by $\{T_n, n \geq 0\}$. Let T'_n be the time between the arrival of the $(n-1)$st and nth batch (of size K) at the workshop. We have

$$T'_n = \sum_{i=(n-1)K+1}^{nK} T_i, \quad n \geq 1.$$

Hence$\{T'_n, n \geq 1\}$ is a sequence of iid random variables. Define the cost (C_n) incurred over the nth cycle as the number of items received in each cycle $(C_n = K)$. Then $\{(T'_n, C_n), n \geq 1\}$ is a sequence of iid bivariate random variables. Hence $\{Z(t), t \geq 0\}$ is a cumulative process. The long run rate at which jobs are received by the receiving station is given by Theorem 5.2 to be

$$\lim_{n\to\infty} \frac{Y(t)}{t} = \frac{1}{\tau},$$

and, the long run rate at which jobs are received by the workshop is given by Theorem 5.3,

$$\lim_{n\to\infty} \frac{Z(t)}{t} = \frac{\mathsf{E}(C_1)}{\mathsf{E}(T'_1)} = \frac{K}{K\tau} = \frac{1}{\tau}.$$

This is indeed to be expected since no jobs are lost or created from the receiving station to the workshop.

Comp. 5.1 The lifetime of a battery is given to be $Erl(k, \lambda)$ with $k = 3$, and $\lambda = 1$. Hence, the mean lifetime of a battery is $\tau = k/\lambda = 3$. From Theorem 5.2, the long run replacement rate is given by

$$\lim_{t\to\infty} = \frac{N(t)}{t} = \frac{1}{\tau} = \frac{1}{3}.$$

Comp. 5.2 Let $L \sim Erl(3, 1)$ be the lifetime of a battery. Then, the expected inter replacement time is given by

$$\begin{aligned}
\mathsf{E}(T) &= \mathsf{E}(\min(3, L)) \\
&= \int_0^3 x f_L(x)dx + \int_3^\infty 3 f_L(x)dx \\
&= \int_0^3 x e^{-x}\frac{x^2}{2}dx + 3\mathsf{P}(L > 3) \\
&= 3 - 13.5e^{-3} = 2.3279.
\end{aligned}$$

Thus, the long run replacement rate is given by

$$\lim_{t\to\infty} = \frac{N(t)}{t} = \frac{1}{\mathsf{E}(T)} = 0.4296.$$

Comp. 5.3 Let T_1 be the time of first planned replacement. Using the argument in Example 5.7, we get

$$\tau = \mathsf{E}(T_1) = \mathsf{E}(\min(L_1, 3)) + \tau\mathsf{P}(L_1 < 3) = 2.3279 + 0.5768\tau.$$

(Here we have used $\mathsf{E}(\min(L_1, 3)) = 2.3279$ from Computational Problem 5.2.) Solving for τ we get $\tau = 5.5008$. Hence, the long run planned replacement rate is given by

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mathsf{E}(T_1)} = 0.1818.$$

Alternative solution:

An arbitrary replacement is with probability $P(L_1 > 3) = 0.4232$ a planned replacement. Hence the long-run planned replacement rate is $(0.4296) \cdot (0.4232) = 0.1818$.

Comp. 5.9 Follow the computations in Example 5.14. Let L_1 be the lifetime of the first battery, T_1 be the time of first replacement, and C_1 be the cost of that replacement. From the solution to Computational Problem 5.2,

$$\mathsf{E}(T_1) = \mathsf{E}(\min(L_1, 3)) = 2.3279.$$

Furthermore,

$$\mathsf{E}(C_1) = 75 + 75\mathsf{P}(L_1 < 3) = 75 + 75 * .5768 = 118.26.$$

The long run cost rate under the "planned replacement" policy is given by

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathsf{E}(C_1)}{\mathsf{E}(T_1)} = \frac{118.26}{2.3279} = 50.80 \text{ dollars/year}.$$

Under the "replace upon failure" policy, we have

$$\mathsf{E}(T_1) = \mathsf{E}(L_1) = 3, \quad \mathsf{E}(C_1) = 75 + 75 = 150.$$

Hence the long run cost per year is given by

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{\mathsf{E}(C_1)}{\mathsf{E}(T_1)} = \frac{150}{3} = 50 \text{ dollars/year}.$$

Hence replacement upon failure is cheaper.

Concep. 6.8 Let $X(t)$ be the number of items in the warehouse at time t. The state space of $\{X(t), t \geq 0\}$ is $\{0, 1, 2, \cdots, K\}$. Assume that the machine always produces items, and if there is no space in the warehouse for a produced item, it is lost. Since the exponential distribution has memoryless property, this has the same effect as turning the machine off when the warehouse is full. Then the production process is $PP(\lambda)$ and acts as an arrival process to the warehouse. Assume that the items form a queue in the warehouse. Then the first item in the warehouse has to wait an Exp(μ) amount of time for the next demand before it is removed. Again, memoryless property of exponential distribution implies that lost demands do not have any effect. Hence the "service times" of the items are iid Exp(μ) random variables. Hence the number of items in the warehouse forms an $M/M/1/K$ queue.

Concep. 6.10 $\{X(t), t \geq 0\}$ is a stochastic process with state space $\{0, 1, 2, \cdots, K\}$. When $X(t) = i > 0$, a repair takes place with rate μ (in which case the state decreases by 1), and a failure occurs at rate $(K - i)\lambda$ (in which case the state increases by 1). If $X(t) = 0$, no repairs take place, while a failure occurs at rate $K\lambda$. Hence $\{X(t), t \geq 0\}$ is a birth and death process with birth rates

$$\lambda_i = (K - i)\lambda, \quad 0 \leq i \leq K,$$

and death rates

$$\mu_i = \mu, \quad 1 \leq i \leq K.$$

Concep. 6.11 $\{X(t), t \geq 0\}$ is a birth and death process with birth rates

$$\lambda_i = (K - i)\lambda, \quad 0 \leq i \leq K,$$

and death rates

$$\mu_i = \min(i, s)\mu, \quad 0 \leq i \leq K.$$

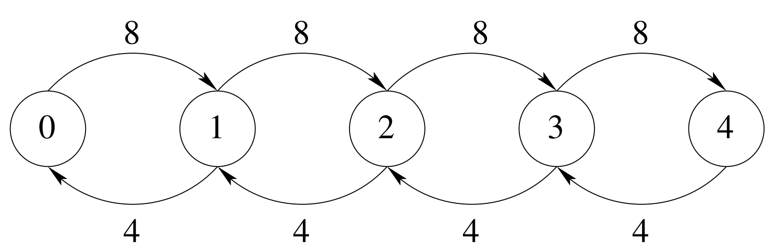
Comp. 6.1 $\lambda = 5, \quad \tau = 1.3$. Let s be the number of servers. From Theorem 6.4, the queue is stable if $s > \lambda\tau = 5 * 1.3 = 6.5$. Hence the minimum number of servers needed for stability is 7.

Comp. 6.2 From Example 6.5, the expected number of busy servers is given by $B = \min(\lambda\tau, s) = \min(6.5, s)$. Thus $B = s$ if $1 \leq s \leq 6$, and $B = 6.5$ if $s \geq 7$.

Comp. 6.6 Time unit: hour. $\lambda = 8, \mu = 4$.

a) This is an $M/M/1/K$ queue with $K = 4$.

b)



c) Balance equations

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
12p_1(4) &= 8p_0(4) + 4p_2(4), \\
12p_2(4) &= 8p_1(4) + 4p_3(4), \\
12p_3(4) &= 8p_2(4) + 4p_4(4), \\
4p_4(4) &= 8p_3(4),
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
8p_1(4) &= 4p_2(4), \\
8p_2(4) &= 4p_3(4), \\
8p_3(4) &= 4p_4(4).
\end{aligned}$$

Normalizing equation: $p_0(4) + p_1(4) + p_2(4) + p_3(4) + p_4(4) = 1$.

d) The limiting distribution is given by

$$p(4) = \left[\frac{1}{31}, \frac{2}{31}, \frac{4}{31}, \frac{8}{31}, \frac{16}{31} \right].$$

e) $p_4(4) = \frac{16}{31}$

f) $1 - p_0(4) = \frac{30}{31}$

g) The arrival rate of entering customers is $\lambda(1 - p_4(4)) = \frac{120}{31}$ customers per hour. Furthermore, $L = \sum_{i=0}^{4} i p_i(4) = \frac{98}{31}$. Hence

$$W = \frac{L}{\lambda(1 - p_4(4))} = 0.8167 \text{ hours} = 49 \text{ minutes}$$

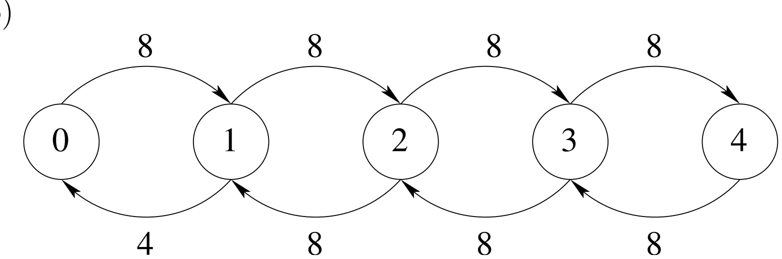
h) $W^q = W - \frac{1}{4} = 0.5667$ hours $= 34$ minutes.

Comp. 6.7 The fraction of the customers lost is given by $p_4(4) = 0.5161$. Hence the fraction of the customers that enter is given by $1 - p_4(4) = 0.4839$. Hence the rate at which customers enter is $0.4839 * 8 = 3.8712$ per hour. Each entering customer pays 12 dollars. Hence the long run revenue rate is

$$3.8712 * 12 = 46.4544 \quad \text{dollars/hour}.$$

Comp. 6.8 a) This is an $M/M/s/K$ queue with $s = 2$ and $K = 4$.

b)



c) Balance equations

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
12p_1(4) &= 8p_0(4) + 8p_2(4), \\
16p_2(4) &= 8p_1(4) + 8p_3(4), \\
16p_3(4) &= 8p_2(4) + 8p_4(4), \\
8p_4(4) &= 8p_3(4),
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
8p_0(4) &= 4p_1(4), \\
8p_1(4) &= 8p_2(4), \\
8p_2(4) &= 8p_3(4), \\
8p_3(4) &= 8p_4(4).
\end{aligned}$$

Normalizing equation: $p_0(4) + p_1(4) + p_2(4) + p_3(4) + p_4(4) = 1$.

d) The limiting distribution is given by

$$p(4) = \left[\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}\right].$$

e) $p_4(4) = \frac{2}{9}$

f) $0 \cdot p_0(4) + \frac{1}{2} \cdot p_1(4) + 1 \cdot (p_2(4) + p_3(4) + p_4(4)) = \frac{7}{9}$

g) $\lambda(1 - p_4(4)) = \frac{56}{9}$ (alternatively: $\frac{7}{9} \cdot 2 \cdot 4 = \frac{56}{9}$)

h) The arrival rate of entering customers is $\lambda(1 - p_4(4)) = \frac{56}{9}$ customers per hour. Furthermore, $L = \sum_{i=0}^{4} i p_i(4) = \frac{20}{9}$. Hence

$$W = \frac{L}{\lambda(1 - p_4(4))} = 0.3571 \ \text{hours} = 21.43 \ \text{minutes}$$

and

$$W^q = W - \frac{1}{4} = 0.1071 \text{ hours} = 6.43 \text{ minutes}$$

i) The new rate of revenue is given by

$$12 * \lambda * (1 - p_4(4)) = 74.6667 \text{ dollars/hour.}$$