Solutions to exercises: week 6

Concep. 6.20 Consider the queue in front of server 1. The inter-arrival times are iid, each being a sum of two iid exp(λ) random variables. Hence it is a $G/M/1$ queue.

Concep. 6.21 The pdf of an U(a, b) random variable is

$$g(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

Hence

$$\tilde{G}(s) = \int_a^b e^{-sx} g(x) dx = \frac{e^{-as} - e^{-bs}}{s(b-a)}.$$

The key functional equation is $u = \tilde{G}(\mu(1-u))$, which reduces to

$$u = \frac{e^{-a\mu(1-u)} - e^{-b\mu(1-u)}}{\mu(1-u)(b-a)}.$$

Comp. 6.10 This is an $M/M/1/K$ queue with $\lambda = 1$ per hour, $\mu = 20/24 = 5/6$ per hour, $K = 10$. The machine is off whenever the warehouse is full. The long run fraction of the time the machine is off is given by $p_{10}(10) = 0.1926$.

Comp. 6.11 This is the same queue as in Computational Problem 6.10. The demands are lost when the warehouse is empty. The demands occur according to a Poisson process. Hence, according to PASTA, the long run probability that a demand sees the warehouse empty is given by $p_0(10)$. Hence the long run fraction of the demands lost are given by $p_0(10) = 0.0311$.

Comp. 6.12 This is the same queue as in Computational Problem 6.10. Let W be the expected time an item spends in the warehouse in steady state. Then the expected revenue from the sale is $100 - W$ dollars. Using the parameters given in Computational Problem 6.10, we get $W = 8.3114$ hours. Hence the expected sale price is \$91.6886. Now the rate at which items enter the warehouse is

$$\lambda \cdot (1 - p_K(K)) = 1 \cdot (1 - 0.1926) = 0.8074 \text{ per hour}.$$

Hence the revenue rate is $0.8074 \cdot 91.6886 = 74.0294$ dollars/hour.
Alternative solution: If there are i items on stock, the warehouse looses i dollar per hour. So the expected loss per hour is L dollar, where L is the expected number of items on stock. Hence the revenue rate is

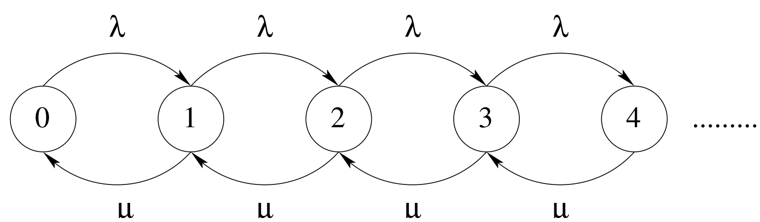
$$\lambda \cdot (1 - p_K(K)) \cdot 100 - L = 80.74 - 6.71 = 74.03 \text{ dollar per hour.}$$

Comp. 6.13 This is an $M/M/s/K$ queue with $\lambda = 60, \quad \mu = 10, \quad s = 6$. We need to determine K such that $p_K(K) \leq 0.05$. We have

$$p_{10}(10) = 0.1286, \quad \cdots, p_{22}(22) = 0.0506, \quad p_{23}(23) = 0.0481.$$

Hence the system needs a total of 23 lines. It currently has 10. Hence an additional 13 need to be installed. The expected queueing time increases from 0.0246*60=1.48 minutes to 0.1290*60=7.74 minutes.

Comp. 6.21 a) $M/M/1$ queue.



b)

c) The balance equations are

$$\begin{aligned} \lambda p_0 &= \mu p_1, \\ (\lambda + \mu) p_n &= \lambda p_{n-1} + \mu p_{n+1}, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

or, alternatively,

$$\lambda p_n = \mu p_{n+1}, \quad n = 0, 1, 2, \ldots.$$

Normalizing equation: $\sum_{n=0}^{\infty} p_n = 1$.

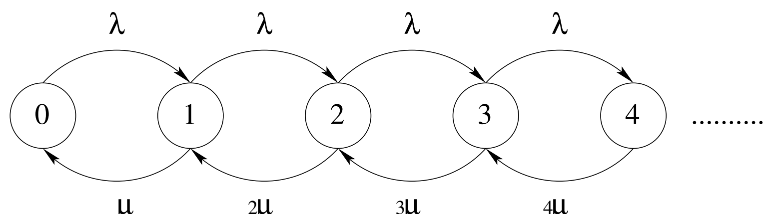
d) $\lambda < \mu$.

e) We have $\mu = 12$ items per hour. The fraction of demands lost is $1 - \rho$, where $\rho = \lambda/\mu$. Hence we must have $\rho \geq 0.9$, i.e., $\lambda \geq 0.9\mu = 10.8$ per hour. Thus, the mean production time is at most $60/10.8 = 5.556$ minutes. The mean number in the warehouse is then $\rho/(1-\rho) = 0.9/0.1 = 9$.

Comp. 6.27 This is an $M/M/20$ queue with $\lambda = 36, \quad \mu = 2$. The desired answer is $W_q = 0.1377$ hours $= 8.26$ minutes.

Comp. 6.28 a) This is an $M/M/\infty$ queue with $\lambda = 40$ and $\mu = 1/3$.

b)



c) The balance equations are

$$\begin{aligned}\lambda p_0 &= \mu p_1, \\ (\lambda + i\mu)p_i &= \lambda p_{i-1} + (i+1)\mu p_{i+1}, \quad i = 1, 2, 3, \ldots,\end{aligned}$$

or, alternatively,

$$\lambda p_i = (i+1)\mu p_{i+1}, \quad i = 0, 1, 2, \ldots.$$

Normalizing equation: $\sum_{i=0}^{\infty} p_i = 1$.

d) Use that

$$\begin{aligned}p_i &= \frac{\lambda}{i\mu} p_{i-1} \\ &= \frac{120}{i} p_{i-1} \\ &\vdots \\ &= (\frac{120^i}{i!}) p_0.\end{aligned}$$

From the normalizing equation we obtain $p_0 = e^{-120}$.

e)
- Mean number of cars in parking lot: $L = \lambda/\mu = 120$. (in steady state the number of cars in the lot is a Poisson random variable with mean $40/(1/3) = 120$).
  Mean waiting time of cars: $W = L/\lambda = 3$ minutes.
- Mean number of occupied parking places: 120.
  Mean parking time of cars: 3 minutes.

- Mean number of cars in the queue: 0.
  Mean queueing time of cars: 0 minutes.
- Fraction of time a certain parking place is occupied: 0 (due to the assumption that in the model there are infinitely many parking places)
- Probability that all places are occupied: 0 (due to the assumption that in the model there are infinitely many parking places)
- Throughput is 40 cars per hour.

Comp. 6.32 a) $M/G/1$ queue. More specifically, it is an $M/E_2/1$ because the service times of customers are Erlang distributed with 2 phases.

b)
- The service times are iid Erl(2,12). Hence $\tau = 2/12 = 1/6$ hour, $\sigma^2 = 2/(12)^2 = 1/72$ hour^2. Thus $s^2 = 1/72 + 1/36 = 3/72 = 1/24$ hour^2. We also have $\lambda = 3$. Hence $L = 0.875$.
  $W = L/\lambda = 0.875/3 = 0.2917$ hour $= 17.5$ minutes.
- Mean number of occupied servers: $\rho = 1/2$.
  Mean service time: $\tau = 1/6$ hour $= 10$ minutes.
- Mean number of customers in the queue: $L_q = L - \rho = 0.375$.
  Mean queueing time of customers: $W_q = W - \tau = 7.5$ minutes.
- Fraction of time server is occupied: $\rho = 1/2$.
- Throughput: 3 customers per hour.

Comp. 6.33 This is an $M/G/1$ queue. Let T_1 and T_2 be two iid Exp(6) random variables. Then the service time (in hours) of a single customer is $T = \max(T_1, T_2)$. The cdf of T is given by

$$F(x) = \mathsf{P}(T \le x) = \mathsf{P}(T_1 \le x, T_2 \le x) = (1 - e^{-6*x})^2.$$

Hence

$$\tau = \mathsf{E}(T) = \int_0^\infty xF'(x)dx = 0.25 \text{ hrs},$$

and

$$s^2 = \mathsf{E}(T^2) = \int_0^\infty x^2F'(x)dx = 7/72 \text{ hrs}^2.$$

Using $\lambda = 3$ we get $L = 2.5$. Thus the congestion is more in this setup. This is because one of the two servers is forced to be idle part of the time.

Comp. 6.35 Let T be a typical service time. We are given

$$\mathsf{P}(T=2)=0.5,\ \ \mathsf{P}(T=3)=0.2,\ \ \mathsf{P}(T=5)=0.3.$$

Hence $\tau = 2*0.5+3*0.2+5*0.3 = 3.1$ minutes and $s^2 = 4*0.5+9*0.2+25*0.3 = 11.3$ minutes^2. Using $\lambda = 18$ per hour, we get $L = 8.194$.

Comp. 6.36 a)

- Consider the queue in front of server 1. It is a $G/M/1$ queue with iid Erl$(2,10)$ interarrival times, and Exp(6) service times. The traffic intensity is $\rho = 5/6 < 1$. The functional equation becomes

$$u = \left(\frac{10}{10+6(1-u)}\right)^2,$$

with the required solution $\alpha = 0.7822$. Using Equation (6.42) we get $L = 3.826$ and $W = L/\lambda = 3.826/5 = 0.765$ hour $=$ 45.9 minutes.
- Mean number of occupied servers: $\rho = 5/6$.
  Mean service time: $1/\mu = 10$ minutes.
- Mean number of customers in queue: $L_q = L - \rho = 2.993$.
  Mean queueing time: $W_q = W - 1/\mu = 35.9$ minutes.
- Fraction of time server is occupied: $\rho = 5/6$.
- Throughput: 5 customers per hour.

b) Let $X(t)$ be the number of customers in the single line served by the two servers. Then $X(t)$ is an $M/M/2$ queue with $\lambda = 10,\ \ \mu = 6$. The expected number of customers in this system is 5.4545.

c) The expected number of customers in the first system is 7.6518. The expected number of customers in the second system is 5.4545. Hence pooling the two servers will reduce congestion.

Comp. 6.40 This is a $G/M/1$ queue with arrival rate $\lambda = 12$ per hour (constant inter arrival times of 5 minutes), and service rate $\mu = 15$ per hour. The solution to the functional equation is $\alpha = 0.6286$. Hence $L = 2.1540$, and the fraction of the time the server is busy is given by $\lambda/\mu = 12/15 = 0.8$. Hence the cost rate is $0.8*40+2.1540*2 = 36.3080$ dollars/hour. If each customer is charged c, the revenue rate is $12c$ per hour. Hence we must have $c \geq 36.3080/12 = 3.0257$. Thus each customer should be charged at least 3.03 dollars in order for the system to break even.