# Solutions to exercises: week 7

**Comp. 6.41** Time unit: hour. $\lambda_1 = 24$, $\lambda_2 = 0$, $\mu_1 = 15$, $\mu_2 = 20, a_1 = a_2 = 24$.

a)
- Use the balance equations

$$\begin{aligned} 24p_1(0) &= 15p_1(1), \\ 24p_1(1) &= 30p_1(2), \\ 24p_1(n) &= 45p_1(n+1), \quad n = 2, 3, \ldots, \end{aligned}$$

  together with the normalizing equation $\sum_{n=0}^{\infty} p_1(n) = 1$.
- $p(n_1, n_2) = p_1(n_1) \cdot p_2(n_2)$.
- $L_1 = \sum_{i=0}^{\infty} ip_1(i) = 1.913$.
  $L_2 = \sum_{i=0}^{\infty} ip_2(i) = 1.294$.
  $W_1 = L_1/a_1 = 0.0797$ hour $= 4.782$ minutes.
  $W_2 = L_2/a_2 = 0.0539$ hour $= 3.235$ minutes.
- Expected number of occupied servers station 1: $a_1/\mu_1 = 1.6$.
  Expected number of occupied servers station 2: $a_2/\mu_2 = 1.2$.
  Mean service time per visit at stations 1: 4 minutes.
  Mean service time per visit at stations 2: 3 minutes.
- $L_1^q = L_1 - a_1/\mu_1 = 0.313$.
  $L_2^q = L_2 - a_2/\mu_2 = 0.094$.
  $W_1^q = W_1 - 4 = 0.782$ minutes.
  $W_2^q = W_2 - 3 = 0.235$ minutes.
- probability server in first station occupied: 8/15.
  probability server in second station occupied: 2/5.
- throughput first station: $a_1 = 24$ customers per hour.
  throughput second station: $a_2 = 24$ customers per hour.
- $L = L_1 + L_2 = 3.207$ customers.
  $W = W_1 + W_2 = 0.1336$ hour $= 8.017$ minutes.
- fraction in queue: $(0.782 + 0.235)/8.017 \approx 12.7\%$.
  fraction in service $(4 + 3)/8.017 \approx 87.3\%$.

b) For stability, we must have $s_1 \geq 2$ and $s_2 \geq 2$. Hence, we have the following three options for server allocations $(s_1, s_2)$: (2,4), (3,3), (4,2). Let $L_{(2,4)}, L_{(3,3)}$ and $L_{(4,2)}$ be the expected number of customers in the network for the three options. We get $L_{(2,4)} = 5.6603$, $L_{(3,3)} = 3.2070$, $L_{(4,2)} = 3.5355$. Hence the optimal server allocation is three servers at each station.

**Comp. 6.45** This is a Jackson network with

$$N = 3, \;\; \lambda_1 = \lambda = 20, \;\; \lambda_2 = \lambda_3 = 0,$$

$$\mu_1 = 12, \;\; \mu_2 = 20, \;\; \mu_3 = 15, \;\; s_1 = 4, \;\; s_2 = s_3 = 2,$$

$$P = \begin{bmatrix} 0 & .4 & .6 \\ 0.3 & 0 & 0 \\ 0.2 & 0 & 0 \end{bmatrix},$$

$$r_1 = 0, \;\; r_2 = 0.7, \;\; r_3 = 0.8.$$

The solution to the traffic equation is

$$a = [500/19 \;\; 200/19 \;\; 300/19] = [26.32 \;\; 10.53 \;\; 15.79].$$

a) We have

$$s_1\mu_1 = 48 > a_1, \;\; s_2\mu_2 = 40 > a_2, \;\; s_3\mu_3 = 30 > a_3.$$

Hence the network is stable.

b) Use the balance equations

$$\begin{aligned} \frac{200}{19} p_2(0) &= 20 p_2(1), \\ \frac{200}{19} p_2(n) &= 40 p_2(n+1), \quad n = 1, 2, 3, \ldots, \end{aligned}$$

together with the normalizing equation $\sum_{n=0}^{\infty} p_2(n) = 1$.

c) $p(3,1,1) = p_1(3) \cdot p_2(1) \cdot p_3(1) = 0.185 \cdot 0.307 \cdot 0.327 = 0.019.$

d) $L_1 = \sum_{i=0}^{\infty} i p_1(i) = 2.466.$
$L_2 = \sum_{i=0}^{\infty} i p_2(i) = 0.566.$
$L_3 = \sum_{i=0}^{\infty} i p_3(i) = 1.456.$
$W_1 = L_1/a_1 = 5.62$ minutes.
$W_2 = L_2/a_2 = 3.22$ minutes.
$W_3 = L_3/a_3 = 5.53$ minutes.

e) Expected number of occupied servers station 1: $a_1/\mu_1 = 2.19.$
Expected number of occupied servers station 2: $a_2/\mu_2 = 0.53.$
Expected number of occupied servers station 3: $a_3/\mu_3 = 1.05.$
Mean service time per visit at stations 1: 5 minutes.
Mean service time per visit at stations 2: 3 minutes.
Mean service time per visit at stations 3: 4 minutes.

f) $L_1^q = L_1 - a_1/\mu_1 = 0.27$.
$L_2^q = L_2 - a_2/\mu_2 = 0.04$.
$L_3^q = L_3 - a_3/\mu_3 = 0.40$.
$W_1^q = W_1 - 5 = 0.62$ minutes.
$W_2^q = W_2 - 3 = 0.22$ minutes.
$W_3^q = W_3 - 4 = 1.53$ minutes.

g) 0.548, 0.263 and 0.527.

h) probability all servers in first station occupied: $\sum_{n=4}^{\infty} p_1(n) = 0.225$.

probability all servers in second station occupied: $\sum_{n=2}^{\infty} p_2(n) = 0.110$.

probability all servers in third station occupied: $\sum_{n=2}^{\infty} p_3(n) = 0.363$.

i) throughput first station: $a_1 = 26.32$ customers per hour.

throughput second station: $a_2 = 10.53$ customers per hour.

throughput third station: $a_3 = 15.79$ customers per hour.

j) $L = L_1 + L_2 + L_3 = 4.49$ customers.
$W = L/\lambda = 13.47$ minutes.

k) Customer visits on average $a_1/\lambda = 1.32$ times station 1, $a_2/\lambda = 0.53$ times station 2 and $a_3/\lambda = 0.79$ times station 3. Hence he spends on average 7.40 minutes in station 1, 1.69 minutes in station 2 and 4.36 minutes in station 3. The fraction of time he spends in station 3 is hence $4.36/13.47 \approx 32.4\%$.

l) Expected time in queue: $1.32 \cdot 0.62 + 0.53 \cdot 0.22 + 0.79 \cdot 1.53 = 2.14$ minutes.

Expected time in service $1.32 \cdot 5 + 0.53 \cdot 3 + 0.79 \cdot 4 = 11.33$ minutes.

Hence fraction in queue is 15.9%, fraction in service is 84.1%.

## Handout section 4

**Exercise 2** (a) Show that the given $v_1, v_2, v_3, v_4$ satisfy the traffic equations

$$v_1 = 0.8v_2 + 0.6v_3\,,$$
$$v_2 = 0.5v_1\,,$$
$$v_3 = 0.4v_1 + 0.2v_2 + v_4\,,$$
$$v_4 = 0.1v_1 + 0.4v_3\,.$$

(b) $p(k_1, k_2, k_3, k_4) = \dfrac{1}{46119}(1)^{k_1}(1)^{k_2}(4)^{k_3}(4)^{k_4} = \dfrac{1}{46119}(4)^{k_3}(4)^{k_4}$,

with $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0, k_4 \geq 0, k_1 + k_2 + k_3 + k_4 = 6$

(c) Stations 1 and 2 have the same *relative traffic intensities* $\dfrac{v_1}{\mu_1} = \dfrac{v_2}{\mu_2} = 1$. Also Stations 3 and 4 have the same relative traffic intensities.

(d)
- The absolute traffic intensities of the stations are $\rho_1 = \rho_2 = 0.2105$, $\rho_3 = \rho_4 = 0.8422$.
- The throughputs of the stations are 0.8420, 0.4210, 0.8422, 0.4211 repairs/hour.

(e) The mean number of cars in the stations 3 and 4 is 2.7367.

(f)
- The mean waiting times in the queue are 0.0627, 0.1254, 2.2495, 4.4989 hours.
- The mean times of a car in the stations are 0.3127, 0.6254, 3.2495, 6.4989 hours

(g) The throughput of the whole garage system is the sum of 0.8 times the throughput of station 2 and 0.6 times the throughput of station 3 and of course also equal to the throughput of station 1: 0.8420 repairs/hour. The mean time spent by a car in the system is therefore (Little) 7.1259 hours.

**Exercise 3** (a) With $v = (v_1, v_2, v_3, v_4)$, determine a solution of the traffic equations $v = vP$. Show that the relative traffic intensities $v_1/\mu_1, v_2/\mu_2, v_3/\mu_3, v_4/\mu_4$ are equal and show that there are 20 possible distributions of $K = 3$ customers over the four stations.

(b) $p_1(0) = 0.5, p_1(1) = 0.3, p_1(2) = 0.15, p_1(3) = 0.05$.

(c) $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0.5$.