1. Consider a Markov process with states 0, 1 and 2 and with the following transition rate matrix $Q$:

$$Q = \begin{pmatrix}
-\lambda & \lambda & 0 \\
\mu & -\lambda - \mu & \lambda \\
\mu & 0 & -\mu
\end{pmatrix}$$

where $\lambda > 0$ and $\mu > 0$.

a. Derive the parameters $v_i$ and $P_{ij}$ for this Markov process.

b. Determine the expected time to go from state 1 to state 0.
We have
\[ v_0 = \lambda, \quad v_1 = \lambda + \mu, \quad v_2 = \mu, \]
and
\[ P_{01} = P_{20} = 1, \quad P_{10} = 1 - P_{12} = \frac{\mu}{\lambda + \mu}. \]

Let \( T_i(i = 1, 2) \) denote the time to go from state \( i \) to 0. Then
\[ E[T_1] = \frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot E[T_2] = \frac{1}{\mu}. \]

6.1. Let us assume that the state is \((n, m)\). Male \( i \) mates at a rate \( \lambda \) with female \( j \), and therefore it mates at a rate \( \lambda m \). Since there are \( n \) males, mating occurs at a rate \( \lambda nm \). Therefore
\[ v_{n,m} = \lambda nm. \]
Since any mating is equally likely to result in a female as in a male, we have
\[ P_{(n,m):(n+1,m)} = P_{(n,m):(n+1,m)} = \frac{1}{2}. \]

6.3. This is not a birth and death process since we need more information than just the number working. We must also know which machine is working. We can analyze it by letting the states be
\[ b: \text{both machines are working} \]
\[ 1: 1 \text{ is working, 2 is down} \]
\[ 2: 2 \text{ is working, 1 is down} \]
\[ d1: \text{both are down, 1 is being repaired} \]
\[ d2: \text{both are down, 2 is being repaired.} \]

Then
\[ v_b = \mu_1 + \mu_2, \quad v_1 = \mu_1 + \mu, \quad v_2 = \mu_2 + \mu, \quad v_{d1} = v_{d2} = \mu, \]
and
\[ P_{b,1} = 1 - P_{b,2} = \frac{\mu_2}{\mu_1 + \mu_2}, \]
\[ P_{1,b} = 1 - P_{1,d2} = \frac{\mu}{\mu_1 + \mu}, \]
\[ P_{2,b} = 1 - P_{2,d1} = \frac{\mu}{\mu + \mu_2}, \]
\[ P_{d1,1} = P_{d2,2} = 1. \]
6.5.  

a. Yes.

b. It is a pure birth process.

c. If there are $i$ infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

\[ E[\text{time all infected}] = \frac{n(n-1)}{2\lambda} \sum_{i=1}^{n-1} \frac{1}{i(n-i)}. \]

6.6. Starting with $E[T_0] = \frac{1}{\lambda_0} = \frac{1}{\lambda}$, employ the identity

\[ E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}] \]

to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

a. $E[T_0] + \cdots + E[T_3]$.

1. Consider the following queueing model: customers arrive at a service station according to a Poisson process with rate $\lambda$. There are $c$ servers; the service times are exponential with rate $\mu$. If an arriving customer finds $c$ servers busy, then he leaves the system immediately.

   a. Model this system as a birth and death process.

   b. Suppose now that there are infinitely many servers ($c = \infty$). Again model this system as a birth and death process.

2. In Example 6.11 it is shown, using the backward equations, that

   $$P_{00}'(t) = \mu - (\mu + \lambda)P_{00}(t).$$

   a. Derive this result using the forward equations.

   b. Derive a differential equation for $P_{11}(t)$ in two ways: using the forward and backward equations.

   c. Suppose the machine is working at time 0. What is the probability that the machine is also working at time $t$?
a. The state space is \( \{0, 1, \ldots, c - 1, c\} \), and the birth rates are

\[
\lambda_n = \lambda, \quad n = 0, 1, \ldots, c - 1,
\]

and the death rates

\[
\mu_n = n\mu, \quad n = 1, 2, \ldots, c.
\]

Hence the transition rate matrix \( Q \) is given by

\[
Q = \begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & \cdots & 0 \\
\mu & -(\lambda + \mu) & \lambda & 0 & 0 & \cdots & 0 \\
0 & 2\mu & -(\lambda + 2\mu) & \lambda & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & (c - 2)\mu & -(\lambda + (c - 2)\mu) & \lambda & 0 \\
0 & \cdots & 0 & (c - 1)\mu & -(\lambda + (c - 1)\mu) & \lambda & c\mu \\
0 & \cdots & 0 & 0 & 0 & -c\mu & -c\mu
\end{pmatrix}
\]

b. Now the state space does not stop at \( c \), i.e., it is \( \{0, 1, \ldots\} \) and the birth rates are

\[
\lambda_n = \lambda, \quad n = 0, 1, \ldots,
\]

and the death rates

\[
\mu_n = n\mu, \quad n = 1, 2, \ldots.
\]

3

a. The forward equations are

\[
P_{00}'(t) = \mu P_{01}(t) - \lambda P_{00}(t)
\]

\[
P_{01}'(t) = \lambda P_{00}(t) - \mu P_{01}(t).
\]

Using \( P_{01}(t) = 1 - P_{00}(t) \), we obtain the following differential equation,

\[
P_{00}'(t) = \mu - (\mu + \lambda)P_{00}(t).
\]

b. The backward equations are

\[
P_{11}'(t) = \mu P_{01}(t) - \mu P_{11}(t)
\]

\[
P_{01}'(t) = \lambda P_{11}(t) - \lambda P_{01}(t).
\]

Multiplying these equations by \( \lambda \) and \( \mu \), respectively, and then adding them, we obtain

\[
\lambda P_{11}'(t) + \mu P_{01}'(t) = 0.
\]
Hence, for some $c$,

$$\lambda P_{11}(t) + \mu P_{01}(t) = c.\)$$

Substituting $t = 0$ yields $c = \lambda$. Using the above equation, we get

$$P_{11}'(t) = \lambda - (\lambda + \mu)P_{11}(t).$$

The forward equations are

$$P_{11}'(t) = \lambda P_{10}(t) - \mu P_{11}(t)$$
$$P_{10}'(t) = \mu P_{11}(t) - \lambda P_{10}(t).$$

By substituting $P_{10}(t) = 1 - P_{11}(t)$, we obtain the same differential equation as before.

c. Solving the differential equation for $P_{00}(t)$ with initial condition $P_{00}(0) = 1$ yields

$$P_{00}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}, \quad t \geq 0.$$

6.8. The number of failed machines is a birth and death process with

$$\lambda_0 = 2\lambda, \quad \lambda_1 = \lambda, \quad \lambda_n = 0, n > 1, \quad \mu_1 = \mu_2 = \mu, \quad \mu_n = 0, n \neq 1, 2.$$ 

Now substitute into the backward equations.

6.10. Let

$$I_j(t) = \begin{cases} 0, & \text{if machine } j \text{ is working at time } t, \\ 1, & \text{otherwise.} \end{cases}$$

Also, let the state be $(I_1(t), I_2(t))$. This is clearly a continuous-time Markov chain with

$$v(0,0) = \lambda_1 + \lambda_2, \quad \lambda_{(0,0);(0,1)} = \lambda_2, \quad \lambda_{(0,0);(1,0)} = \lambda_1,$$
$$v(0,1) = \lambda_1 + \mu_2, \quad \lambda_{(0,1);(0,0)} = \mu_2, \quad \lambda_{(0,1);(1,1)} = \lambda_1,$$
$$v(1,0) = \mu_1 + \lambda_2, \quad \lambda_{(1,0);(0,0)} = \mu_1, \quad \lambda_{(1,0);(1,1)} = \lambda_2,$$
$$v(1,1) = \mu_1 + \mu_2, \quad \lambda_{(1,1);(0,1)} = \mu_1, \quad \lambda_{(1,1);(1,0)} = \mu_2.$$

By the independence assumption we have

$$P_{(i,j),(k,l)}(t) = P_{i,k}(t)Q_{j,l}(t),$$

where $P_{i,k}(t)$ is the probability that the first machine is in state $k$ at time $t$ given that it was at state $i$ at time 0; $Q_{j,l}(t)$ is defined similarly for the second machine. By example 4.11 we have

$$P_{0,0}(t) = \frac{\mu_1}{\lambda_1 + \mu_1} + \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t},$$
$$P_{1,0}(t) = \frac{\mu_1}{\lambda_1 + \mu_1} - \frac{\mu_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t}. $$
and by the same argument,

\[
P_{1,1}(t) = \frac{\lambda_1}{\lambda_1 + \mu_1} + \frac{\mu_1}{\lambda_1 + \mu_1} e^{-(\lambda_1 + \mu_1)t},
\]

\[
P_{0,1}(t) = \frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-(\lambda_1 + \mu_1)t}.
\]

Of course, similar expressions for the second machine are obtained by replacing \((\lambda_1, \mu_1)\) by \((\lambda_2, \mu_2)\). We then get \(P_{(i,j),(k,l)}(t)\) by formula (??). For instance,

\[
P_{(0,0),(0,0)}(t) = P_{0,0}(t)Q_{0,0}(t) = \frac{\mu_1 + \lambda_1 e^{-(\lambda_1 + \mu_1)t}}{\lambda_1 + \mu_1} \cdot \frac{\mu_2 + \lambda_2 e^{-(\lambda_2 + \mu_2)t}}{\lambda_2 + \mu_2}.
\]
6.12.

a. If the state is the number of individuals at time $t$, we get a birth and death process with

\[
\lambda_n = n\lambda + \theta, \quad n < N, \\
\lambda_n = n\lambda, \quad n \geq N, \\
\mu_n = n\mu.
\]

b. Let $P_i$ be the long-run probability that the system is in state $i$. Since this is also the proportion of time the system is in state $i$, we are looking for

\[
\sum_{i=3}^{\infty} P_i.
\]

We have

\[
P_k \mu_k = P_{k-1} \lambda_{k-1}, \quad k = 1, 2, \ldots.
\]

This yields

\[
P_1 = \frac{\theta}{\mu} P_0 = \frac{1}{2} P_0, \\
P_2 = \frac{\theta + \lambda}{2\mu} P_1 = \frac{1}{2} P_1 = \frac{1}{4} P_0, \\
P_3 = \frac{\theta + 2\lambda}{3\mu} P_2 = \frac{1}{2} P_2 = \frac{1}{8} P_0,
\]

and for $k \geq 3$,

\[
P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1} = \frac{k-1}{k} \frac{1}{2} P_{k-1} = \cdots = \frac{3}{k} \left(\frac{1}{2}\right)^{k-3} P_3.
\]

Hence

\[
\sum_{k=3}^{\infty} P_k = P_3 \sum_{k=3}^{\infty} \frac{3}{k} \left(\frac{1}{2}\right)^{k-3} = 24 P_3 \sum_{k=3}^{\infty} \frac{1}{k} \left(\frac{1}{2}\right)^{k}.
\]

Since

\[
\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\log(1-x),
\]

we get

\[
\sum_{k=3}^{\infty} P_k = P_3(24 \log 2 - 15).
\]

Because all probabilities add up to 1, we have

\[
P_0 \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}(24 \log 2 - 15) \right) = 1.
\]
So

\[ P_0^{-1} = 3 \log 2 - \frac{1}{8}, \]

and thus finally,

\[ \sum_{k=3}^{\infty} P_k = \frac{24 \log 2 - 15}{24 \log 2 - 1} \approx 0.105. \]

6.13. With the number of customers in the shop as the state, we get a birth and death process with

\[ \lambda_0 = \lambda_1 = 3, \quad \mu_1 = \mu_2 = 4. \]

Therefore

\[ P_1 = \frac{3}{4} P_0, \quad P_2 = \frac{3}{4} P_1 = \left(\frac{3}{4}\right)^2 P_0. \]

And since \( P_0 + P_1 + P_2 = 1 \), we get

\[ P_0 = \frac{16}{37}. \]

a. The average number of customers in the shop is

\[ P_1 + 2P_2 = \frac{30}{37}. \]

b. The proportion of the customers that enter the shop is

\[ \frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = \frac{28}{37}. \]

c. Now \( \mu = 8 \), so

\[ P_0 = \frac{64}{97}. \]

So the proportion of the customers that enter the shop is

\[ 1 - P_2 = \frac{88}{97}. \]

The rate of added customers is therefore

\[ \lambda \left(\frac{88}{97}\right) - \lambda \left(\frac{28}{37}\right) \approx 0.45. \]

The business he does would improve by 0.45 customers per hour.

6.15. With the number of customers in the system as the state, we get a birth and death process with

\[ \lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_i = 0, i \geq 4, \quad \mu_1 = 2, \mu_2 = \mu_3 = 4. \]
Therefore the balance equations reduce to

\[ P_1 = \frac{3}{2} P_0, \quad P_2 = \frac{3}{4} P_1 = \frac{9}{8} P_0, \quad P_3 = \frac{3}{4} P_2 = \frac{27}{32} P_0. \]

And therefore,

\[ P_0 = \left(1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right)^{-1} = \frac{32}{143}. \]

a. The fraction of potential custoemrs that enter the system is

\[ \frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = \frac{116}{143}. \]

b. With a server working twice as fast we would get

\[ P_1 = \frac{3}{4} P_0, \quad P_2 = \frac{3}{4} P_1 = \left(\frac{3}{4}\right)^2 P_0, \quad P_3 = \left(\frac{3}{4}\right)^3 P_0, \]

and

\[ P_0 = \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3\right)^{-1} = \frac{64}{175}. \]

So that now

\[ 1 - P_3 = \frac{148}{175}. \]

**6.17.** Say the state is 0 if the machine is up, say it is in state i when it is down due to a type i failure, i = 1, 2. The balance equations for the limiting probabilities are as follows.

\[
\begin{align*}
\lambda P_0 &= \mu_1 P_1 + \mu_2 P_2, \\
\mu_1 P_1 &= \lambda p P_0, \\
\mu_2 P_2 &= \lambda (1-p) P_0, \\
P_0 + P_1 + P_2 &= 1.
\end{align*}
\]

These equations are easily solved to give the results

\[ P_0 = (1 + \lambda p / \mu_1 + \lambda (1-p) / \mu_2)^{-1}, \quad P_1 = \lambda p P_0 / \mu_1, \quad P_2 = \lambda (1-p) P_0 / \mu_2. \]
6.18. There are \( k + 1 \) states: state 0 means the machine is working, state \( i \) means that it is in repair phase \( i \), \( i = 1, \ldots, k \). The balance equations for the limiting probabilities are

\[
\begin{align*}
\lambda P_0 &= \mu_k P_k, \\
\mu_1 P_1 &= \lambda P_0, \\
\mu_i P_i &= \mu_{i-1} P_{i-1}, \quad i = 2, \ldots, k,
\end{align*}
\]

and the normalization equation is

\[ P_0 + \cdots + P_k = 1. \]

To solve, note that

\[ \mu_i P_i = \mu_{i-1} P_{i-1} = \mu_{i-2} P_{i-2} = \cdots = \lambda P_0. \]

Hence,

\[ P_i = (\lambda / \mu_i) P_0, \]

and, upon summing,

\[ 1 = P_0 \left( 1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i} \right). \]

Therefore,

\[ P_0 = \left( 1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i} \right)^{-1}, \quad P_i = \left( \frac{\lambda}{\mu_i} \right) P_0, \quad i = 1, \ldots, k. \]

a. \( P_i \).

b. \( P_0 \).

6.22. The number in the system is a birth and death process with parameters

\[ \lambda_n = \lambda / (n + 1), \quad n \geq 0; \quad \mu_n = \mu, \quad n \geq 1. \]

From the equation above (6.20),

\[ 1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda / \mu)^n / n! = e^{\lambda / \mu} \]

and from (6.20),

\[ P_n = P_0 (\lambda / \mu)^n / n! = e^{-\lambda / \mu} (\lambda / \mu)^n / n!, \quad n \geq 0. \]

6.23. Let the state denote the number of machines that are down. This yields a birth and death process with

\[ \lambda_0 = \frac{3}{10}, \quad \lambda_1 = \frac{2}{10}, \quad \lambda_2 = \frac{1}{10}, \quad \lambda_i = 0, \quad i \geq 3, \]
and

\[ \mu_1 = \frac{1}{8}, \quad \mu_2 = \frac{2}{8}, \quad \mu_3 = \frac{2}{8}. \]

The balance equations reduce to

\[ P_1 = \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0, \]
\[ P_2 = \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0, \]
\[ P_3 = \frac{1/10}{2/8} P_2 = \frac{4}{10} P_2 = \frac{192}{250} P_0. \]

Hence, using \( P_0 + P_1 + P_2 + P_3 = 1 \), yields

\[ P_0 = \left( 1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250} \right)^{-1} = \frac{250}{1522}. \]

a. Average number not in use is

\[ P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}. \]

b. Proportion of time both repairmen are busy is

\[ P_2 + P_3 = \frac{336}{761}. \]