Proposed content course on
"Local convergence, giants and small-world properties in preferential attachment models"

(based on Random Graphs and Complex Networks Volume 2)

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In this document, we propose an outline for a course on connectivity properties in preferential attachment models, based on Random Graphs and Complex Networks Volume 2. We focus on the local limit, its connectivity and small- and ultra-small-world properties in preferential attachment models. Given the fact that preferential attachment models are substantially more difficult to analyse, parts of this course may be technically challenging.

(1) Start with a recap of the configuration model, by discussing some of their properties in [II, Chapter 1]. Particularly [II, Section 1.3.5] is then convenient. Should you wish to brush up more details on preferential attachment models, you can discuss the basics in [I, Chapter 8].

(2) Discuss the theory of local convergence in [II, Chapter 2]. Cover the basic notation of rooted graphs and their metric properties in [II, Section 2.2], the notion of local convergence of deterministic graphs in [II, Section 2.3], and that of random graphs in [II, Section 2.4].

(3) Continue with consequences of local convergence in [II, Section 2.5]. Here, you can pick those consequences that you like best.

(3b) Time permitting you may discuss the relation between the size of the giant and the local limit in [II, Section 2.6]. This is useful as intuition, and the proof is relatively straightforward. Note, however, that many preferential attachment models are connected, so that the giant often contains almost all vertices. The proof for this is relatively straightforward.

(4) Discuss De Finetti's Theorem on infinite sequences of exchangeable random variables, Pólya urns, and their relations to scale-free trees, in [II, Section 5.2]. This section is highly useful to understand the main tool for dealing with preferential attachment models, namely, a Pólya-urn representation of preferential attachment models.

(5) Discuss local convergence of preferential attachment models in [II, Section 5.3].

(6) State and prove the first main ingredient in the local convergence proof of $\mathrm{PA}_n^{(m,\delta)}(d)$ in [II, Section 5.3.3], the Finite-graph Pólya version of preferential attachment models. See [II, Theorem 5.10]. There are two proofs, one based on Pólya urns with an urn for every vertex, and one where the graph probabilities are computed. The first is shorter and more intuitive, the second somewhat more involved, but less intuitive.

(7) Discuss the second main ingredient in the local convergence proof of $\mathrm{PA}_n^{(m,\delta)}(d)$ in [II, Section 5.4], which is the relation to the Pólya point tree. Again, here you can stick to an intuitive argument based on [II, Lemma 5.17 and Proposition 5.18], or go through the full (rather involved) analysis in the remainder of this section.

(8) Treat the connectivity of preferential attachment models in [II, Section 5.5], which is relatively straightforward.

(9) Close the course with the small-world properties of the preferential attachment model in [II, Chapter 8]. [II, Section 8.2] discussed the setting of distances in scale-free trees. [II, Section 8.3] gives an outline of the results, which are then proved in [II, Sections 8.4–8.7]. [II, Section 8.4] contains path-counting estimates that are interesting in their own right. These are used to prove logarithmic lower bounds on distances in [II, Section 8.5]. [II, Section 8.6] contains the doubly logarithmic distance lower bounds in preferential attachment models with infinite-variance degrees, and is technically quite demanding. [II, Section 8.7] contains the, arguably more interesting, doubly logarithmic upper bound in the infinite-variance degree case. [II, Section 8.8] contains weak bounds on diameters, which also implies logarithmic weak upper bounds on distances in the finite-variance degree case.

(10) Time permitting, you could discuss related distance results in [II, Section 8.9], preferential attachment models with global communities in [II, Section 9.3.4], and/or spatial preferential attachment models in [II, Section 9.5.4].