Critical behavior in inhomogeneous random graphs

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Critical behavior Erdős-Rényi random graph

Erdős-Rényi random graph is random subgraph of complete graph on $n$ vertices where each of $\binom{n}{2}$ edges is occupied with probab. $p$.

Phase transition: (Erdős and Rényi (60))
For $p = \left(1 + \varepsilon\right)/n$, largest component is
(a) $\Theta_p(\log n)$ for $\varepsilon < 0$;
(b) $\Theta_p(n)$ for $\varepsilon > 0$;

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = \left(1/n\right)(1 + \lambda n^{-1/3})$, largest component is $\Theta_p(n^{2/3})$.

Extension: Aldous (97): Weak convergence of ordered clusters.

Key question:
How much remains valid when we let go of homogeneity vertices?
Rank-1 inhomogeneous random graphs

Vertex set \([n] := \{1, 2, \ldots, n\}\).

Attach edge with probability \(p_{ij}\) between vertices \(i\) and \(j\), where

\[ p_{ij} = \frac{w_i w_j}{l_n}, \]

and

\[ l_n = \sum_{i=1}^{n} w_i, \]

and different edges are independent.

Note that \(w_i\) is expected degree vertex \(i\).

When \(w_i = \lambda\), we retrieve Erdős-Rényi random graph with \(p = \lambda/n\).

Assume throughout talk \(w_i^2/l_n \leq 1\) for all \(i \in [n]\).
Choice of weights

Take \( \mathbf{w} = (w_1, \ldots, w_n) \) as

\[
    w_i = [1 - F]^{-1}(i/n),
\]

where \( F(x) \) is distribution function.

Interpretation: number of vertices \( i \) with \( w_i \leq x \) is close to \( F(x) \).

Simple example:

\[
    F(x) = \begin{cases} 
        0 & \text{for } x < a, \\
        1 - (a/x)^{\tau-1} & \text{for } x \geq a, 
    \end{cases}
\]

in which case

\[
    [1 - F]^{-1}(u) = a(1/u)^{-1/(\tau-1)}, \quad \text{so that} \quad w_j = a(n/j)^{1/(\tau-1)}.\]
Critical value

Bollobás-Janson-Riordan (2007): Let $W \sim F$, then

- largest component $\sim \rho n$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$;
- largest component $o(n)$ for $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] \leq 1$.

Thus, critical value IRG is

$$\nu = 1.$$ 

In simple example $F(x) = 1 - (a/x)^{\tau-1}$ for $x \geq a$

$$\mathbb{E}[W] = \frac{a(\tau - 1)}{\tau - 2}, \quad \mathbb{E}[W^2] = \frac{a^2(\tau - 1)}{\tau - 3},$$

so that critical case arises when $a = (\tau - 3)/(\tau - 2)$:

$$\nu = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} = \frac{a(\tau - 2)}{\tau - 3} = 1.$$
Power-law degrees

Let
\[ 1 - F(x) \sim cx^{-(\tau-1)} \quad \text{for } x \text{ sufficiently large.} \]

Further, let \(|C_{\text{max}}|\) denote largest connected component.

**Theorem 1. (vdH 09)** Assume that \(\nu = 1\).

(a) Let \(\tau > 4\). Then, there exists \(b > 0\) such that for all \(\omega \geq 1\)
\[ P\left( \frac{1}{\omega} n^{2/3} \leq |C_{\text{max}}| \leq \omega n^{2/3} \right) \geq 1 - \frac{b}{\omega} \quad \text{as } n \to \infty. \]

(b) Let \(\tau \in (3, 4)\). Then, there exists \(b > 0\) such that for all \(\omega \geq 1\)
\[ P\left( \frac{1}{\omega} n^{(\tau-2)/(\tau-1)} \leq |C_{\text{max}}| \leq \omega n^{(\tau-2)/(\tau-1)} \right) \geq 1 - \frac{b}{\omega} \quad \text{as } n \to \infty. \]
Scaling limit for $\tau > 4$

Let $\mu = \mathbb{E}[W]$, $\sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Consider

$$B^\lambda_s = \sigma B_s + s\lambda - s^2\sigma^2/(2\mu),$$

where $B$ is standard Brownian motion. Let

$$R^\lambda_s = B^\lambda_s - \min_{0 \leq u \leq s} B^\lambda_s.$$

Aldous (1997): Excursions of $R^\lambda$ can be ranked in increasing order as $\gamma_1(\lambda) > \gamma_2(\lambda) > \ldots$.

Let $|\mathcal{C}_1(\lambda)| \geq |\mathcal{C}_2(\lambda)| \geq |\mathcal{C}_3(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-1/3})w_i$ arranged in increasing order.

Theorem 2. (BvdHvL 09a) Assume that $\nu = 1$, and $\mathbb{E}[W^3] < \infty$. Then

$$\left(n^{-2/3}|\mathcal{C}_i(\lambda)|\right)_{i \geq 1} \xrightarrow{d} (\gamma_i(\lambda))_{i \geq 1}.$$
Scaling limit for $\tau \in (3, 4)$

Let $|C_1(\lambda)| \geq |C_2(\lambda)| \geq |C_3(\lambda)| \ldots$ denote sizes of components with weights $\tilde{w}_i = (1 + \lambda n^{-(\tau-3)/(\tau-1)}) w_i$ arranged in increasing order.

Theorem 3. (BvdHvL 09b) Assume that $\nu = 1$, and $\tau \in (3, 4)$. Then,

$$\left( n^{-(\tau-2)/(\tau-1)} |C_{(i)}(\lambda)| \right)_{i \geq 1} \overset{d}{\longrightarrow} \left( H_i(\lambda) \right)_{i \geq 1}. $$

Moreover, for every $i, j$ fixed

$$\mathbb{P}(i \leftrightarrow j) \rightarrow q_{ij}(\lambda) \in (0, 1).$$

Limits $H_i(\lambda)$ correspond to hitting times of 0 of a certain fascinating ‘thinned’ Lévy process.
Cluster exploration for $\tau > 4$

Take $\lambda = 0$.

- For all ordered pairs of vertices $(i, j)$, let $U(i, j)$ be i.i.d. $U(0, 1)$.
- Choose vertex $v(1)$ with probability proportional to $w$, so that

$$\mathbb{P}(v(1) = i) = \frac{w_i}{l_n}.$$  

Children of $v(1)$ are those vertices $j$ for which

$$U(v(1), j) \leq \frac{w_{v(1)}w_j}{l_n}.$$  

Label children of $v(1)$ as $v(2), v(3), \ldots v(c(1) + 1)$ in increasing order of their $U(v(1), \cdot)$ values.

- Move to $v(2)$, explore all of its children, and label them as before.

Once we finish exploring one component, move onto next component by choosing starting vertex in size-biased manner amongst remaining vertices.
Size-biased reordering

Size-biased order \( v^*(1), v^*(2), \ldots, v^*(n) \) is random reordering of vertex set \([n]\) where

- \( v^*(1) = i \) with prob. \( w_i/l_n \);
- given \( v^*(1), \ldots, v^*(i-1), v^*(i) = j \in [n] \setminus \{v^*(1)\} \) with prob. proportional to \( w_j \).

Key ingredient proof:

\[
(v(i))_{i \in [n]} \quad \text{is size-biased reordering.}
\]

Number of new neighbors \( c(i) \) of \( v(i) \) is close to

\[
c(i) = \text{Poi} \left( w_{v(i)} \sum_{j \in [n] \setminus \{v(1), \ldots, v(i)\}} w_j/l_n \right).
\]
Connected components

Recall number of new neighbors of $v(i)$ is close to

$$c(i) = \text{Poi}\left( w_{v(i)} \sum_{j \in [n] \setminus \{v(1), ..., v(i)\}} w_j / l_n \right).$$

Denote cluster exploration process $Z_n$ by $Z_n(0) = 0$ and

$$Z_n(i) = Z_n(i - 1) + c(i) - 1.$$

Denote first hitting time of $-j$ by

$$\eta(j) = \min\{ i : Z_n(i) = -j \}.$$

Then, all connected component sizes are given by successive excursions from past minima

$$C^*(j) = \eta(j) - \eta(j - 1).$$
Scaling limit of cluster exploration

Process \( t \mapsto n^{-1/3} Z_n(s n^{2/3}) \) is close to Brownian motion with changing drift given by

\[
\mathbb{E}[n^{-1/3} Z_n(s n^{2/3})] \sim s \lambda - s^2 \sigma^2 / (2\mu),
\]

while

\[
n^{-1/3} Z_n(s n^{2/3}) - (s \lambda - s^2 \sigma^2 / (2\mu)) \xrightarrow{d} B_s.
\]

Suggests that rescaled cluster sizes converge to successive excursions from past minima of process

\[
B_s^\lambda = B_s + s \lambda - s^2 \sigma^2 / (2\mu).
\]

Weak convergence of exploration process follows from

functional martingale central limit theorem.
References


