Distribution of the ICI term in Phase Noise impaired OFDM systems

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Abstract

Orthogonality between the subcarriers of an orthogonal frequency division multiplexing (OFDM) system is affected by phase noise, which causes inter-carrier interference (ICI). The distribution of this interference term is studied in this paper. The distribution of the ICI for large number of carriers is derived and it is shown that the complex Gaussian approximation, generally applied in previous literature, is not valid and that the ICI term exhibits thicker tails. An analysis of the tail probabilities confirms these finding and shows that bit error probabilities are severely underestimated when the Gaussian approximation for the ICI term is used, leading to too optimistic design criteria. Results from a numerical study confirm the analytical findings and show the validity of the limit distribution, obtained under the assumption of a large number of subcarries, already for a modest number of subcarriers.

Index terms – Orthogonal frequency division multiplexing, phase noise, inter-carrier interference, limit distribution, stochastic integral, tail probabilities.

1 Introduction

The multicarrier technique orthogonal frequency division multiplexing (OFDM) [1] is chosen as basis for several wireless systems, because of its high spectral efficiency and ability to divide a dispersive multipath channel into parallel frequency flat subchannels. Furthermore, most systems apply a guard interval, a cyclic extension of the OFDM symbols, to protect the detected symbol against time delayed versions of the previous symbol, which could cause inter-symbol interference (ISI) [2]. OFDM is standardized for application in, amongst others, wireless local-area-networks (WLAN) [3], wireless metropolitan networks (WMAN), e.g. WiMAX [4], digital audio broadcasting (DAB) [5] and digital video broadcasting (DVB) [6].

The high spectral efficiency of OFDM is achieved by applying partly overlapping spectra for the different subcarriers. When the system is perfectly synchronized, these subcarriers are orthogonal. However, when due to imperfect local oscillators (LO) at either transmitter (TX) or receiver (RX) side of the system carrier frequency offset or phase noise (PN) occurs, the orthogonality between the subcarriers is lost and inter-carrier interference (ICI) occurs. These imperfections in the LOs will more and more appear to be a factor limiting performance of OFDM systems, when low-cost implementations or systems with high carrier frequencies are regarded, since it is in those cases harder to produce an oscillator with sufficient stability. Therefore, it is important to understand the influence of imperfect oscillators, i.e. phase noise, on the system performance.

The influence of PN on the performance of an OFDM system has been regarded in several publications, see e.g. [7–15]. They all show that the influence can be split into a multiplicative part, which is common to all subcarriers and therefore often referred to as common phase error (CPE), and an additive part, which is often referred to as inter-carrier interference (ICI). Although the CPE is identified as the main performance

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limiting factor for coherent detection based receivers, many adequate correction approaches for the CPE have been proposed previously, see, e.g. [8,16,17]. Some approaches for correction of the ICI have been proposed in [18,19], which, however, require high signal-to-noise (SNR) ratios to achieve reasonable performance and have a high complexity.

Thus, since any practical coherent OFDM receiver would have some kind of CPE correction, the main performance limiting factor in PN impaired systems is the ICI term. The properties of the ICI term are studied by several authors, see e.g. [7–11]. Many of these papers, implicitly or explicitly, assume the ICI term is distributed according to a complex Gaussian distribution, due to the central-limit-theorem. This is, however, as also noted by the authors of [14,15], an approximation which is only valid for some combinations of PN and subcarrier spacings. The authors of [14] show that the complex Gaussian approximation only holds for fast PN, i.e., PN which changes fast compared to the symbol time, and that error probabilities calculated under this Gaussian assumption are very inaccurate for other types of PN. The authors of [15] confirm these findings with numerical results from Monte-Carlo simulations, which show that good agreement with the Gaussian distribution is only achieved when the ratio of the 3 dB bandwidth of the power-spectral-density of the LO spectrum and the subcarrier spacing approaches 1. It is noted, however, that these values of PN correspond to such a severe system performance degradation, that for practical systems this ratio always will be chosen to be much smaller than 1.

In this contribution we extend previous work by analytically studying the distribution of the ICI term due to PN in OFDM systems. We will derive a limit distribution for the ICI term, which will be shown to exhibit thicker tails than the complex Gaussian distribution with the same variance. Subsequently, we will derive the tail probabilities of the ICI term for a system applying QPSK modulation. We show that previous approaches, based on the Gaussian assumptions, significantly underestimate these probabilities.

The paper is organized as follows. Section 2 introduces the system model and shows the influence of PN. In Section 3 we examine the distribution and tails of the ICI term for large number of carriers. To confirm finding from the analytical study, numerical results are reported in Section 4. Finally, conclusions are drawn in Section 5.

2 System model

A transmitted OFDM signal is formed by applying the inverse discrete Fourier transformation (IDFT) to the complex data symbols and by adding a cyclic prefix to it by placing a copy of the last \( N_g \) samples in front of the symbol. This can be written as follows [1, 2]:

\[
u_{m,n} = \begin{cases} \sqrt{\frac{1}{N}} \sum_{k=0}^{N-1} s_{m,k} \exp \left( i \frac{2\pi(n-N_g)k}{N} \right) & N_g \leq n \leq N_{\text{tot}}-1 \\ u_{m,N+n} & 0 \leq n \leq N_g-1, \end{cases}
\]  

(1)

where we introduce

- \( m \): OFDM symbol index
- \( n \): sample index within the OFDM symbol, \{0,1,...,\( N_{\text{tot}} \}\)
- \( N \): number of subcarriers
- \( k \): subcarrier index, \{0,1,...,\( N \}\)
- \( u_{m,n} \): transmitted complex data signal during the \( m \)th symbol on the \( n \)th symbol
- \( N_g \): the guard interval length
- \( i \): the imaginary unit, \( i^2 = -1 \)
- \( N_{\text{tot}} \): OFDM symbol length \( N + N_g \)

Moreover, the transmitted complex data symbol during the \( m \)th symbol on the \( k \)th carrier is given by \( s_{m,k} \), which can be rewritten as \( s_{m,k} = R_{m,k} + iI_{m,k} \). We define \( R \) and \( I \) to be real zero-mean variables, which depend on the modulation scheme used for transmission. For instance, for BPSK modulation \( R \in \{-1,1\} \) with equal probability and \( I = 0 \). For QPSK modulation \( R \in \{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\} \) and \( I \in \{-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\} \), again with equal probability of occurrence.
The signal \( u \) is then up converted to radio frequency (RF) \( f_{RF} \) using the TX LO. Ideally this LO exhibits a spectrum which is given by a spectral line at \( f_{RF} \) and can be expressed by \( \exp(i2\pi f_{RF}t) \). However, any practical LO will exhibit a random phase process, i.e., phase noise, and it is thus modeled as \( \exp(i2\pi f_{RF}t + \theta_{TX}(t)) \), where \( \theta_{TX}(t) \) denotes the transmitter PN process, which is a function of time \( t \).

At the RX side, the signal is down converted to baseband, by multiplication with the receiver local oscillator process \( \exp(-i2\pi f_{RF}t + \theta_{RX}(t)) \), where \( \theta_{RX}(t) \) denotes the receiver PN process. Since we regard a sampled baseband model here, we further regard \( \theta_{TX,n} \) and \( \theta_{RX,n} \), which are the sampled versions of the TX and RX phase noise, respectively.

When we define \( n' = mN_{tot} + n \), the received baseband signal with additive white Gaussian noise (AWGN) is given by

\[
r_{m,n} = u_{m,n} \exp(i2\pi f_{RF}n'T + \theta_{TX,n'}) \exp(-i2\pi f_{RF}n'T + \theta_{RX,n'}) + w_{m,n},
\]

where the combined TX and RX phase noise term is given by \( \theta_{n} = \theta_{TX,n} - \theta_{RX,n} \) and \( w_{m,n} \) denotes the complex Gaussian, zero-mean additive receiver noise with a variance of \( \sigma^2_w \). Here we assumed there is no, or an ideal, wireless channel.

Subsequently, the cyclic prefix is removed from \( r_{m,n} \) by disregarding the first \( N_g \) samples and the frequency domain complex data symbols are found by applying the discrete Fourier transform (DFT) to the remaining samples, and are given by

\[
x_{m,k} = \sqrt{\frac{1}{N}} \sum_{n=N_g}^{N_{tot}-1} r_{m,n} \exp\left(-\frac{2\pi(n-N_g)k}{N}\right). \tag{3}
\]

Using (1) and (2), equation (3) can be rewritten as

\[
x_{m,k} = g_{m,0}s_{m,k} + \sum_{l=0, l\neq k}^{N-1} g_{m,k-l}s_{m,l} + n_{m,k}, \tag{4}
\]

where the received signal is split into three parts: the first term represents the signal term multiplied with the common-phase-error (CPE), the second is the inter-carrier interference (ICI) term caused by PN and the last term, i.e., \( n_{m,k} \), models the AWGN in the frequency domain. The factor \( g_{m,k-l} \) models the influence of the PN and is given by

\[
g_{m,k-l} = \frac{1}{N} \sum_{n=0}^{N-1} \exp(i\theta_{mN_{tot}+N_g+n})e^{-\frac{2\pi ik(n-l)}{N}}. \tag{5}
\]

Here we define the increments of the phase noise process to be independent, which is a commonly used model for free-running oscillators [20, 21], so that

\[
\theta_{n+1} = \theta_n + \varepsilon_n = \sum_{j=1}^{n} \varepsilon_j, \tag{6}
\]

where \( \theta_0 = 0, \varepsilon_n \sim \mathcal{N}(0, \sigma^2_\varepsilon) \) and \( \sigma^2_\varepsilon = 4\pi \beta T \). Here \( \beta \) denotes the one-sided -3 dB bandwidth of the corresponding Lorentzian spectrum of the LO and \( T \) is the sample time. Since \( \beta \) is generally small compared the subcarrier spacing \( 1/(NT) \), it is useful to define \( \beta_n = \beta N T \), which is the normalized version of \( \beta \). Subsequently, we can rewrite \( \sigma^2_\varepsilon \) as

\[
\sigma^2_\varepsilon = \frac{4\pi \beta_n}{N} = \frac{\sigma^2}{N}, \tag{7}
\]

where \( \sigma^2 = 4\pi \beta_n \) is independent of \( N \).
Before detection is performed to $x_{m,k}$, generally correction of the CPE $g_{m,0}$ is applied. Approaches proposed in [8,10,16,17] could be applied to do this. Remaining nuisances in detection of the signal are the common AWGN and the ICI term. The influence AWGN on detection is well known, but the statistical properties of the ICI term remain an open issue in previous literature. Therefore, the remainder of this paper will focus on the properties of this term.

The ICI for the $k$th carrier in the $m$th symbol, here denoted as $\xi_{m,k}$, is given by

$$\xi_{m,k} = \sum_{l=0,l\neq k}^{N-1} g_{m,k-l} s_{m,l}$$

where $\chi_m = \exp\left(i\sum_{j=0}^{N_n-1} s_{m,l} s_{m,l}^{-1} \varepsilon_j\right)$, where the elements of $\varepsilon_j$ are i.i.d. according to $\mathcal{N}(0,\sigma^2)$ and where $\{\varepsilon_j\}_{j=0}^{N-1} = \{\varepsilon_{mN_{tot}+N_k+n}\}_{j=0}^{N-1}$ is independent of $\{\varepsilon_j\}_{j=0}^{N+N_k-1}$.

In the remainder of this paper we will, without loss of generality, regard the subcarrier $k = 0$ in symbol $m = 0$. When we then omit the subcarrier and symbol index for brevity, (8) reduces to

$$\xi_{0,0} = \xi = \frac{\chi_0}{N} \sum_{l=1}^{N-1} s_l \sum_{n=0}^{N-1} \exp\left(i\sum_{j=0}^{n} \varepsilon_j\right) \exp\left(\frac{2\pi nl}{N}\right).$$

The random variable $\chi_0 = \exp\left(i\sum_{j=0}^{N_k-1} \varepsilon_j\right)$ is independent of all other terms appearing in (9), and has norm 1 and, thus, gives rise to a rotation in the complex plane of the random variable

$$\frac{1}{N} \sum_{l=1}^{N-1} s_l \sum_{n=0}^{N-1} \exp\left(i\sum_{j=0}^{n} \varepsilon_j\right) \exp\left(\frac{2\pi nl}{N}\right).$$

For convenience, we will, therefore, in the sequel take $\chi_0 = 1$ and study (10), where we keep in mind that for the ICI, a random and independent rotation should be performed in the end.

### 3 Properties of the ICI term for large $N$

In this section, we will study the ICI term

$$\xi = \frac{1}{N} \sum_{l=1}^{N-1} s_l \sum_{n=0}^{N-1} \exp\left(i\sum_{j=0}^{n} \varepsilon_j\right) \exp\left(\frac{2\pi ln}{N}\right).$$

In Section 3.1, we will study the convergence of $\xi$ when $N \to \infty$. In Section 3.2, we will study the distribution of the limit of $\xi$ when $\sigma$ is small, that is, when the interference is quite small, and in Section 3.3, we prove some facts about the tail of the distribution when $\sigma$ is small. The consequences for the probability of bit error are illustrated in Section 3.4.
3.1 Convergence of the ICI term for large $N$

Before stating the main convergence result, we need some notation and a key observation. We let $\{B_t\}_{t \geq 0}$ be a standard Brownian motion. Then, we have that

\[ \{\varepsilon_j^i\}_{j=0}^{N-1} \cong \left\{ \sigma \left( \frac{B_{i+1} - B_i}{N} \right) \right\}_{j=0}^{N-1}, \]

where $X \cong Y$ when the random variables $X$ and $Y$ have the same distribution. Therefore, we have that

\[ \xi \cong 1 \sum_{l=1}^{N-1} s_l \sum_{n=0}^{N-1} e^{\frac{i2\pi ln}{N}} \exp \left( i \sum_{j=0}^{n} \sigma \left( \frac{B_{i+1} - B_i}{N} \right) \right). \]

(13)

In the sequel, we will use $\varepsilon_j^i = \sigma \left( \frac{B_{i+1} - B_i}{N} \right)$, and will identify the limit of the right-hand side of (13), which, for convenience, we again write as $\xi$.

We also define

\[ \zeta = \sum_{l \in \mathbb{Z} : l \neq 0} \frac{\sigma s_l}{2\pi l} \int_0^1 e^{i\sigma B_l \left[ e^{2\pi lt} - 1 \right]} dB_t, \]

which will turn out to be the limit of $\xi$ when $N \to \infty$.

Some words of caution are necessary here. The integral on the right-hand side of (14) is a so-called stochastic integral, which can be defined properly. In fact, the construction of such integral uses limits of the form in (43), and these ideas can be used to make (43) rigorous, as we will perform in more detail in Appendix A. Before continuing with the analysis, we list some properties of stochastic integrals that we will rely on. Firstly, we will use the rules that, for functions $f, g: [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that

\[ \mathbb{E} \left[ \int_0^1 |f(t, B_t)|^2 dt, \int_0^1 |g(t, B_t)|^2 dt < \infty \right], \]

we have that

\[ \mathbb{E} \left[ \int_0^1 f(t, B_t) dB_t \right] = 0, \]

(15)

and

\[ \mathbb{E} \left[ \int_0^1 f(t, B_t) dB_t \int_0^1 g(t, B_t) dB_t \right] = \int_0^1 \mathbb{E} [f(t, B_t) g(t, B_t)] dt. \]

(16)

Secondly, we will use for any function $f: [0, 1] \to \mathbb{R}$, we have that

\[ \int_0^1 f(t) dB_t \]

has a normal distribution with mean zero and variance $\int_0^1 f(t)^2 dt$. References for these statements can be found in [22, Chapter 13] or [23].

We will prove the following main result:

**Theorem 3.1.** When $N \to \infty$, for any $\sigma > 0$ fixed, $\xi$ in the right-hand side of (13) converges in probability to $\zeta$, defined in (14).

The convergence in Theorem 3.1 can be clearly seen in simulations, for $N$ is 64, 128, 256 and 512. See Figure 1 below.

**Proof.** We recall that $X_N$ converges in probability to $X$, and write $X_N \overset{p}{\to} X$, when, for every $\varepsilon > 0$,

\[ \lim_{N \to \infty} \mathbb{P}(|X_N - X| > \varepsilon) = 0. \]

(18)
We will make frequent use of the fact that if \( X_N \) and \( Y_N \) converge in probability to \( X \) and \( Y \), then also \( X_N + Y_N \) converges to \( X + Y \) in probability.

We will start by rewriting the sum

\[
\sum_{n=0}^{N-1} e^{i\sum_{j=0}^{n} \varepsilon_j} e^{i\frac{2\pi in}{N}}. \tag{19}
\]

For this, we will use partial summation, which states that for any two sequences of numbers \( \{a_n\}_{n=1}^m \) and \( \{b_n\}_{n=1}^m \), we have

\[
\sum_{n=0}^{m-1} a_n (b_{n+1} - b_n) = a_m b_m - a_0 b_0 - \sum_{n=0}^{m-1} (a_{n+1} - a_n) b_{n+1}. \tag{20}
\]

We apply this to

\[
a_n = e^{i\sum_{j=0}^{n} \varepsilon_j}, \quad b_{n+1} - b_n = e^{i\frac{2\pi in}{N}},
\]

so that

\[
b_n = \sum_{j=0}^{n-1} e^{i\frac{2\pi i j}{N}}. \tag{22}
\]

For this choice, we can compute that \( b_0 = b_N = 0 \), and

\[
b_n = \frac{e^{i\frac{2\pi in}{N}} - 1}{e^{i\frac{2\pi in}{N}} - 1}. \tag{23}
\]

Therefore, we arrive at

\[
\sum_{n=0}^{N-1} e^{i\sum_{j=0}^{n} \varepsilon_j} e^{i\frac{2\pi in}{N}} = \frac{1}{e^{i\frac{2\pi i}{N}} - 1} \sum_{n=0}^{N-1} e^{i\varepsilon_{n+1}} \sum_{j=0}^{n} e^{i\varepsilon_j} \left[ e^{i\frac{2\pi in}{N}} - 1 \right]. \tag{24}
\]

Now, \( \varepsilon_{n+1} \) is small, since it has variance \( \sigma_\varepsilon^2 = \frac{\sigma^2}{N} \). Therefore, we can expand \( e^{i\varepsilon_{n+1}} - 1 \approx i\varepsilon_{n+1} \) to arrive at

\[
\sum_{n=0}^{N-1} e^{i\sum_{j=0}^{n} \varepsilon_j} e^{i\frac{2\pi in}{N}} \approx \xi' = \frac{i}{e^{i\frac{2\pi i}{N}} - 1} \sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i\sum_{j=0}^{n} \varepsilon_j} \left[ e^{i\frac{2\pi in}{N}} - 1 \right]. \tag{25}
\]

Indeed, in Appendix A, we will prove that, \( \eta_N \), which is defined to be the difference between \( \xi \) and \( \xi' \), converges to zero in probability, i.e.,

\[
\eta_N = \xi - \xi' \xrightarrow{p} 0, \tag{26}
\]

in probability. Therefore, to prove the claim, it now suffices to prove that \( \xi' \xrightarrow{p} \zeta \).

When we use the periodicity of \( e^{i\frac{2\pi in}{N}} \) and \( e^{i\frac{2\pi j}{N}} \), we can more conveniently rewrite this as

\[
\xi' \approx \sum_{0<|l|\leq N/2} \frac{is_l}{N[e^{i\frac{2\pi l}{N}} - 1]} \sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i\sum_{j=0}^{n} \varepsilon_j} \left[ e^{i\frac{2\pi in}{N}} - 1 \right], \tag{27}
\]

where, for \( l < 0 \), we define \( s_l = s_{N-l} \). Clearly, \( \{s_l\}_{0<|l|\leq N/2} \) now is an i.i.d. sequence of random variables with the same distribution as \( s_1 \). Here we note that we have, for \( N \) even, counted a term too many. However, for this term, we have that \( l = \frac{N}{2} \), so that the difference is equal to, with \( l = \frac{N}{2} \),

\[
\frac{is_l}{N[e^{i\frac{2\pi l}{N}} - 1]} \sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i\sum_{j=0}^{n} \varepsilon_j} \left[ e^{i\frac{2\pi in}{N}} - 1 \right] = \frac{is_l}{2N} \sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i\sum_{j=0}^{n} \varepsilon_j} [(-1)^n - 1] \xrightarrow{p} 0, \tag{28}
\]

\(6\)
so that this change is negligible. From now on, we will for convenience define
\[ \xi' = \sum_{0 < |l| \leq N/2} \frac{i s_l}{N[e^{i \frac{2\pi l}{N}} - 1]} \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i \sum_{j=0}^{n} \varepsilon'_j} [e^{i \frac{2\pi n}{N}} - 1]. \] (29)

When \( N \to \infty \), we see that for every \( l \) fixed,
\[ N[e^{i \frac{2\pi l}{N}} - 1] \to 2\pi i l, \] (30)
so that it is natural to assume that
\[ \xi' \approx \sum_{l \in \mathbb{Z}, l \neq 0} \frac{s_l}{2\pi l} \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i \sum_{j=0}^{n} \varepsilon'_j} [e^{i \frac{2\pi n}{N}} - 1]. \] (31)

To make this precise, we define
\[ S(t) = \sum_{l \in \mathbb{Z}, l \neq 0} \frac{s_l}{2\pi l} [e^{i \frac{2\pi l}{N}} - 1]. \] (32)
This random sum is a well-defined random variable, and, in particular, has finite second moment. We will show in Appendix A that
\[ \zeta = \int_0^1 e^{i \sigma B_t} S(t) dB_t, \] (33)
and we will also prove that \( \xi' \overset{P}{\to} \zeta \), by using (33). First, with
\[ S_N(t) = \sum_{0 < |l| \leq N/2} \frac{i s_l}{N[e^{i \frac{2\pi l}{N}} - 1]} [e^{i \frac{2\pi n}{N}} - 1], \] (34)
so that
\[ \xi' = \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i \sum_{j=0}^{n} \varepsilon'_j} S_N \left( \frac{n}{N} \right). \] (35)

We will prove that \( \xi \overset{P}{\to} \zeta \) in two steps, namely, with \( \delta_N \) and \( \gamma_N \) given by
\[ \delta_N = \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i \sum_{j=0}^{n} \varepsilon'_j} \left[ S_N \left( \frac{n}{N} \right) - S \left( \frac{n}{N} \right) \right], \] (36)
and
\[ \gamma_N = \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i \sum_{j=0}^{n} \varepsilon'_j} S \left( \frac{n}{N} \right) - \zeta, \] (37)
we have that
\[ \xi = \zeta + \eta_N + \delta_N + \gamma_N. \] (38)

Therefore, it suffices to prove (26) and (33) and
\[ \delta_N \overset{P}{\to} 0, \] (39)
as well as
\[ \gamma_N \overset{P}{\to} 0. \] (40)
Together, these claims prove Theorem 3.1.
We now comment on these steps. We first note that
\[
\sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i \sum_{j=n}^{n+1} \varepsilon_j} S \left( \frac{n}{N} \right)
\]
is a Riemann-sum approximation to the integral
\[
\zeta = \sigma \int_0^1 e^{i \sigma B_t} S(t) dB_t,
\]
so that, in probability,
\[
\sum_{n=0}^{N-1} \varepsilon_{n+1} e^{i \sum_{j=n}^{n+1} \varepsilon_j} S \left( \frac{n}{N} \right) \rightarrow \sigma \int_0^1 e^{i \sigma B_t} S(t) dB_t.
\]
This would prove (40). Equations (26), (33), (39) and (40) will be proved in Appendix A. The above completes the proof of Theorem 3.1 subject to (26), (33), (39) and (40).

In the next section, we will investigate the distribution of the limit
\[
\zeta = \sum_{l \in \mathbb{Z}; l \neq 0} \frac{\sigma s_l}{2 \pi l} \int_0^1 e^{i 2 \pi l t} - 1 dB_t.
\]

3.2 Approximation of the ICI for small $\sigma$

When $\sigma$ is quite small, it seems reasonable to assume that we can replace $e^{i \sigma B_t}$ in (44) by 1. If we do so, then we end up with
\[
\zeta \approx \sum_{l \in \mathbb{Z}; l \neq 0} \frac{\sigma s_l}{2 \pi l} \int_0^1 [e^{i 2 \pi l t} - 1] dB_t = \sum_{l \in \mathbb{Z}; l \neq 0} \frac{\sigma s_l}{2 \pi l} \left[ - B_1 + \int_0^1 e^{i 2 \pi l t} dB_t \right].
\]
We note that $Z = -B_1$ is standard normally distributed, so that, with
\[
X = \sum_{l \in \mathbb{Z}; l \neq 0} \frac{s_l}{2 \pi l},
\]
we arrive at
\[
\zeta \approx \sigma X Z + \sum_{l \in \mathbb{Z}; l \neq 0} \frac{\sigma s_l}{2 \pi l} \int_0^1 e^{i 2 \pi l t} dB_t.
\]
Next, since $t \mapsto e^{i 2 \pi l t}$ is deterministic, we have that $\int_0^1 e^{i 2 \pi l t} dB_t$ has a complex normal distribution. Furthermore, since for $|k| \neq |l|$, we have
\[
\mathbb{E} \left[ \int_0^1 \cos (2 \pi l t) dB_t, \int_0^1 \cos (2 \pi k t) dB_t \right] = \int_0^1 \cos (2 \pi l t) \cos (2 \pi k t) dt = 0,
\]
we have that, with $Z_l = \sqrt{2} \int_0^1 \cos (2 \pi l t) dB_t$, the sequence $\{Z_l\}_{l=1}^\infty$ is a sequence of i.i.d. standard normal random variables, where we are using that a vector of normal random variables is independent when all covariances are equal to zero.
Similarly, we can see that, for \( l \geq 1 \), \( Z'_l = \sqrt{2} \int_0^1 \sin (2\pi lt) dB_t \) are i.i.d. standard normal variables. All these random variables are independent of \( Z = -B_1 = -\int_0^1 dB_t \), since

\[
E \left[ \int_0^1 \cos (2\pi lt) dB_t \int_0^1 1 dB_t \right] = \int_0^1 \cos (2\pi lt) dt = 0. \tag{49}
\]

Finally, also \( \{Z_l\}_{l \geq 1} \) and \( \{Z'_l\}_{l \geq 1} \) are independent. We conclude that

\[
\zeta \approx \sigma (Z X_1 + \frac{1}{2} (Y_{+,1} - Y_{-,2})) + i \sigma (Z X_2 + \frac{1}{2} (Y_{+,2} + Y_{-,1})), \tag{50}
\]

where we define, with \( s_l = R_l + i I_l \) for \( l > 0 \), and \( s_l = R'_l + i I'_l \) for \( l < 0 \),

\[
X_1 = \frac{1}{2\pi} \sum_{l=1}^{\infty} \frac{R_l + R'_l}{l}, \quad X_2 = \frac{1}{2\pi} \sum_{l=1}^{\infty} \frac{I_l + I'_l}{l}; \tag{51}
\]

\[
Y_{\pm,1} = \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{\infty} \frac{Z_l [R_l \pm R'_l]}{l}, \quad Y_{\pm,2} = -\frac{1}{\sqrt{2\pi}} \sum_{l=1}^{\infty} \frac{Z'_l [I_l \pm I'_l]}{l}. \tag{52}
\]

In the simulations presented in Section 4, we have compared the distribution of \( \zeta \) and its approximation in (50), and we see that for small \( \sigma \), the distributions are very close. See Figure 3(a)–3(c).

We note that for the bit-error probabilities (BEP), we have to look at the probability that \( \zeta \geq 1 \), which signifies the real part of the ICI, is larger than a constant, say 1. This we can do when \( \sigma \) tends to zero, to investigate the BEP when the interference decreases. We will now investigate such probabilities in more detail.

### 3.3 Tail probabilities

In this section, we investigate the tail probabilities of the random variables in Section ???. Tail probabilities are important in the ICI case since the BEP can be rewritten in terms of the tail probabilities. Therefore, a system with smaller tail probabilities performs better than a system with larger tail probabilities. In particular, our results will show that the usual Gaussian assumptions lead to an underestimation for the BEPs.

We will start by computing the first two moments of the random variables appearing in (50):

\[
E[Z X_1] = E[Z Y_{\pm,1}] = 0, \tag{53}
\]

while, writing \( \text{Var}(R_1) = \sigma_R^2 \),

\[
E[(Z X_1)^2] = E[Z^2] E[X_1^2] = \sigma^2 \text{Var}(X_1) = 2 \sigma^2 \text{Var}(R_1) \sum_{l=1}^{\infty} \frac{1}{\pi^2 l^2} = 2 \sigma^2 \frac{\sigma_R^2}{6} = \frac{\sigma^2 \sigma_R^2}{3}, \tag{54}
\]

and

\[
E[Y_{\pm,1}^2] = \sigma^2 \text{Var}(R_1) \sum_{l=1}^{\infty} \frac{1}{\pi^2 l^2} = \frac{\sigma^2 \sigma_R^2}{6}. \tag{55}
\]

As a consequence, the random variable \( \xi = \sigma Z X_1 + \frac{2}{\sqrt{\pi}} (Y_{+,1} - Y_{-,2}) \), which signifies the real part of the ICI, has mean zero and variance

\[
\text{Var}(\xi) = \sigma^2 E[(Z X_1)^2] + \frac{1}{4} (E[Y_{+,1}^2] + E[Y_{-,2}^2]) = \sigma^2 \sigma_R^2 \left( \frac{1}{3} + \frac{1}{12} \right) = \frac{5 \sigma^2 \sigma_R^2}{12}. \tag{56}
\]

Therefore, the usual Gaussian assumptions lead to a tail estimate of the form

\[
P(\xi > y) \approx Q \left( \frac{y}{\sigma} \right) = \exp \left( - \frac{6 y^2}{5 \sigma^2 \sigma_R^2} (1 + o(1)) \right). \tag{57}
\]
We will now show that the tail is in fact much larger than that. In the explanation below, we will assume that \( R_l \) and \( I_l \) are \( \pm 1 \) with equal probability, for which \( \sigma^2 = 1 \).

For this, we fix an \( M \), and we investigate the probability that \( R_l = R'_l = 1 \) for \( l \leq M \), while \( \sum_{l > M} \frac{R_l + R'_l}{1} \geq 0 \) and \( Y_{+,1} + Y_{-,2} \geq 0 \). By symmetry of the random variables involved, this probability is at least \( \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \). Also, when the above is true, then

\[
X_1 \geq \sum_{l=1}^{M} \frac{1}{\pi l} = \frac{1}{\pi} \log M (1 + o(1)).
\]  

(58)

Therefore, in order for \( \sigma Z X_1 \geq y \) to hold, we only need that

\[
Z \geq \frac{\pi y}{\sigma \log M},
\]

(59)

which has probability

\[
Q \left( \frac{\pi y}{\sigma \log M} \right) \sim \exp \left( - \frac{\pi^2 y^2}{2 \sigma^2 \log^2 M} (1 + o(1)) \right).
\]

(60)

Therefore, for every \( M \geq 1 \) and \( y > 0 \),

\[
\mathbb{P}(\xi_R > y) \geq \frac{1}{4\pi} \cdot \frac{1}{4} \exp \left( - \frac{\pi^2 y^2}{2 \sigma^2 \log^2 M} (1 + o(1)) \right).
\]

(61)

For example, for \( M = \frac{y^2}{\sigma^2 \log^2 \left( \frac{1}{\sigma^2} \right)} \), we obtain that \( \frac{1}{4\pi + 1} = \exp \left( o(\frac{y^3}{\sigma^2 \log^2 \left( \frac{1}{\sigma^2} \right)}) \right) \), so that

\[
\mathbb{P}(\xi_R > y) \geq \exp \left( - \frac{\pi^2 y^2}{8 \sigma^2 \log^2 \left( \frac{y^3}{\sigma^2} \right)} (1 + o(1)) \right).
\]

(62)

The tail in (62) is much larger than the ones in (57), so that Gaussian assumptions, as formulated in (57), lead to a systematic underestimation of the tails, and therefore of the BEPs. Indeed, when \( y = 1 \) and \( \sigma \) is quite small, the exponent has become a factor \( \frac{5\pi^2}{24 \log^2 \left( \frac{1}{\sigma^2} \right)} \) smaller, which is substantial for \( \sigma \) small. This can be clearly seen in the results from simulations in Fig. 2 of Section 4, where we see thicker tails of the ICI distribution compared to a normal random variable with the same variance. A similar analysis could be carried out for the imaginary part of the ICI \( \xi_I \).

### 3.4 Consequences for the BEP

To illustrate the consequences of the difference between the derived tail probabilities from our approach and from the one using the Gaussian assumptions, this section regards the BEP for the detection of the QPSK symbol \( s_{0,0} \) from \( x_{0,0} \) impaired by the ICI term \( \xi_{0,0} \). We assume here that there is no AWGN, i.e., \( n_{0,0} = 0 \), and that the CPE has been corrected for. The received signal is then given by

\[
x_{0,0} = s_{0,0} + \xi_{0,0},
\]

(63)

which can be rewritten to a separate detection for the real and imaginary part as

\[
\begin{align*}
x_R &= s_R + \xi_R, \\
x_I &= s_I + \xi_I,
\end{align*}
\]

(64)

which results in the following average probability of bit error:

\[
P_b = \frac{1}{2} (P_{b,R} + P_{b,I}),
\]

(65)
where $P_{b,R}$ and $P_{b,I}$ are the probability of bit error of the real and imaginary part, respectively. These are given by

\begin{align}
P_{b,R} &= \frac{1}{2} P(\xi_R > 1/\sqrt{2}) + \frac{1}{2} P(\xi_R < -1/\sqrt{2}) = P(\xi_R > 1/\sqrt{2}), \quad (66) \\
P_{b,I} &= \frac{1}{2} P(\xi_I > 1/\sqrt{2}) + \frac{1}{2} P(\xi_I < -1/\sqrt{2}) = P(\xi_I > 1/\sqrt{2}), \quad (67)
\end{align}

where we used the symmetry of the distribution of $\xi_R$ and $\xi_I$. This directly relates the BEP of (65) to the tail probabilities derived in Section 3.3.

This clearly illustrates the dependence of the average probability of bit error on the tail probabilities calculated above and shows that (62) provides a lowerbound on the BEP. Furthermore, it can be concluded that an underestimation of the tail probabilities by the Gaussian approximation of the ICI will directly translate into an underestimation of the BEP.

### 4 Numerical Results

In this section, we compare the analytical results of Section 3 with results from Monte-Carlo simulations. For all these results a system applying QPSK modulation is simulated.

Figure 1 depicts the experimental determined cumulative distribution function (ECDF), here denoted by $F(x)$, of the normalized real part of the ICI variable, i.e., $\Re(\xi) / \sqrt{N}$, where $\xi$ is defined by (10). $10^5$ independent experiments were performed for all values of $N$. The results are given for $\sigma^2 = 10^{-3}$. For a sample frequency of $f_s = 1/T$, this means that the corner frequency of the PN spectrum $\beta$ is given by $8.1 \cdot 10^{-2}$ times the subcarrier spacing $f_s/N$. From the results in Fig. 1 we can see the convergence in distribution for high values of $N$, which we proved in Section 3. Already for values of $N = 64$ convergence seems to be reached, since all curves lay on top of eachother. For the imaginary part of (10) the same convergence occurs.

![Figure 1: Experimental determined CDF of the real part of $\xi / \sqrt{N}$ for $N$ equals 64, 128, 256 and 512, respectively. The variance of the PN process $\sigma^2 = 10^{-3}$.](image)

Figure 2 again depicts the real part of the ICI data for $N = 512$ and $\sigma^2 = 10^{-3}$, now in a normal probability plot. This figure shows the distribution of the ICI with the corresponding normal distribution,
i.e., the normal distribution with the same mean and variance. The scaling of the plot is such that a normal distribution is depicted as a straight line. $10^5$ independent experiments were performed to obtain the results in this figure. It can be concluded from this graph that the ICI is clearly not normally distributed and that, since the curve is above and below the straight line in the left and right part of the figure, respectively, the ICI distribution has thicker tails than the normal distribution. Similar results were found for the imaginary part of the ICI. This result endorses the analytical results obtained in Section 3.

![Figure 2: Normal probability plot of the real part of $\xi/\sqrt{N}$ for $N=512$ and $\sigma^2 = 10^{-3}$.](image)

In Fig. 3 we study the limit distribution of the ICI term $\zeta$, as defined in (44), for small values of $\sigma_\varepsilon$. This figure compares the ECDF results from simulations of the normalized real part of the ICI term, where the ICI is defined in (10), with the approximation of the limit distribution for small $\sigma_\varepsilon$ as given in (50). The results are given in Fig. 3(a), Fig. 3(b) and Fig. 3(c) for $\sigma^2 = 10^{-2}, 10^{-3}$ and $10^{-4}$, respectively. This corresponds to a -3 dB oscillator bandwidth $\beta$ of $8.1 \cdot 10^{-1}$, $8.1 \cdot 10^{-2}$ and $8.1 \cdot 10^{-3}$ times the subcarrier spacing $f_s$, respectively. For all results $10^5$ independent experiments were performed.

It can be concluded from Fig. 3 that the resemblance of the two distributions increases with decreasing $\beta$. Already for $\sigma^2 = 10^{-3}$ reasonable agreement seems to be achieved.

The empirical determined cumulative distribution function of the ICI for $N=512$ and $\sigma^2 = 10^{-4}$ is depicted on a logarithmic scale in Fig. 4, which enables us to study the tails of the distributions. The figure, again, depicts the simulated real part of the normalized ICI, the real part of the limit distribution of the ICI term $\zeta$, but now also the corresponding normal distribution, i.e., with the same mean and variance as the other variables. Results are given for $N=512$ and $\sigma^2 = 10^{-4}$ and $10^6$ experiments are carried out for each result.

It can be concluded from Fig. 4 that the limit distribution well approximates the ICI, even in the tails of the distribution. Furthermore, it is found that the Gaussian distribution shows lower tail probabilities, and has a faster fall off. These results clearly confirm the results of the analytical study in Section 3.3.
Figure 3: Experimental determined CDF of the real part of $\xi/\sqrt{N}$ (dashed lines) and the approximation of the limit distribution for small $\sigma_\varepsilon$ (solid lines) and $N = 512$.

Figure 4: Experimental determined CDF of the real part of $\xi/\sqrt{N}$ (dotted line) and the approximation of the limit distribution $\zeta$ (solid line) and the normal distribution (dashed line) with the same mean and variance. For $N = 512$ and $\sigma_\varepsilon^2 = 10^{-4}$.
5 Conclusions

The wide application of orthogonal frequency division multiplexing (OFDM) in wireless systems justifies a careful investigation of its performance limiting factors. The influence of imperfect oscillators, i.e., phase noise (PN), is identified as one of the major impairments of OFDM, especially when low-cost and high-frequency systems are considered. Therefore, the distribution of the inter-carrier-interference (ICI) due to PN is studied in this paper.

In most of the previous contributions, the ICI was presumed to be distributed according to a zero-mean complex Gaussian distribution. In this paper, however, it was shown that this assumption is not valid and the limit distribution of the ICI term for large number of subcarriers is derived. This distribution is shown to exhibit thicker tails than the Gaussian distribution with the same mean and variance. In an analysis of the tail probabilities these finding were confirmed and it was shown that bit error probabilities are severely underestimated when the Gaussian approximation for the ICI term is used.

Results from a numerical study show the validity of the limit distribution, obtained under the assumption of a large number of subcarries, already for a modest number of subcarriers. Furthermore, they show that for small values of the PN variance, the limit distribution very well resembles the ICI distribution. Finally, it is shown that the tail probabilities are severely underestimated by the corresponding Gaussian distribution.

The result in this paper can be used by system designers to better specify the level of tolerable PN, for an OFDM system to achieve a certain bit-error-probability. This paper shows that applying the Gaussian approximation for this would lead to a serious under specification of the LO.

A Proof of (26), (33), (39) and (40)

In this appendix, we will prove the technical results (26), (33), (39) and (40). We will make frequent use of the fact that for a complex random variable with $\mathbb{E}[X_n] = 0$, and $\mathbb{E}[|X_n|^2] \rightarrow 0$, we have $X_N \xrightarrow{P} 0$. For convenience, we will prove the statements in the order (39), (26), (33) and (40).

Proof of (39). We bound the second moment of the summands with $|l| > K$, for any $K$, by

$$
\mathbb{E} \left[ \left| \sum_{K \leq |l| \leq N/2} \frac{i s_l}{N} e^{\frac{j 2 \pi}{N} l} \sum_{n=0}^{N-1} \frac{\epsilon'_{n+1} e^{i \sum_{j=0}^{n} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right|^2 \right] = \sigma_s^2 \sum_{K \leq |l| \leq N/2} \frac{1}{N^2 \frac{|l|}{N} - 1} \mathbb{E} \left[ \left| \sum_{n=0}^{N-1} \frac{\epsilon'_{n+1} e^{i \sum_{j=0}^{n} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right|^2 \right],
$$

(68)

where we use the independence of $s_l$ for different $l$, and we write $\sigma_s^2 = \mathbb{E}[|s_l|^2]$. We write out

$$
\mathbb{E} \left[ \left| \sum_{n=0}^{N-1} \frac{\epsilon'_{n+1} e^{i \sum_{j=0}^{n} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right|^2 \right] = \frac{N-1}{n_1=0} \sum_{n_2=0}^{N-1} \mathbb{E} \left[ \frac{\epsilon'_{n_1+1} e^{i \sum_{j=0}^{n_1} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \frac{\epsilon'_{n_2+1} e^{i \sum_{j=0}^{n_2} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right].
$$

(69)

When $n_1 < n_2$, we have that $\epsilon'_{n_2+1}$ is independent of $\epsilon'_{n_1+1} e^{i \sum_{j=0}^{n_1} \epsilon_j} e^{-i \sum_{j=0}^{n_2} \epsilon_j}$, so that the expected value equals zero. Therefore, we arrive at

$$
\mathbb{E} \left[ \left| \sum_{n=0}^{N-1} \frac{\epsilon'_{n+1} e^{i \sum_{j=0}^{n} \epsilon_j}}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right|^2 \right] = \sum_{n=0}^{N-1} \mathbb{E} \left[ \frac{\epsilon'_{n+1}^2}{N^{\frac{1}{2}} \frac{|l|}{N} - 1} \right] = \sigma_s^2.
$$

(70)
Therefore, we obtain
\[
\begin{align*}
E & \left[ \left| \sum_{K \leq |l| \leq N/2} \frac{i s_l}{N[e^{i \frac{2\pi l}{N}} - 1]} \sum_{n=0}^{N-1} \varepsilon'_n e^{i \sum_{j=0}^n \varepsilon'_j [e^{i \frac{2\pi i n}{N}} - 1]} \right|^2 \right] \\
& = \sigma^2 \sigma^2 \sum_{K \leq |l| \leq N/2} \frac{1}{N^2 |e^{i \frac{2\pi l}{N}} - 1|^2}. \quad (71)
\end{align*}
\]

It is not hard to see that when $|l| \geq \frac{N}{4}$, we have that
\[
|e^{i \frac{2\pi l}{N}} - 1| \geq \left| 1 - \cos \left( \frac{2\pi l}{N} \right) \right| \geq 1.
\]

Furthermore, using the fact that $|\sin(x)| \geq \frac{2|x|}{x}$ for all $|x| \leq \frac{\pi}{2}$, we obtain that for $|l| \leq \frac{N}{4}$
\[
|e^{i \frac{2\pi l}{N}} - 1| \geq \left| \sin \left( \frac{2\pi l}{N} \right) \right| \geq \frac{4|l|}{N}.
\]

We conclude that
\[
|e^{i \frac{2\pi l}{N}} - 1| \geq \min \left\{ \frac{|l|}{N}, 1 \right\}. \quad (74)
\]

As a consequence, we obtain that
\[
\begin{align*}
E & \left[ \left| \sum_{K \leq |l| \leq N/2} \frac{i s_l}{N[e^{i \frac{2\pi l}{N}} - 1]} \sum_{n=0}^{N-1} \varepsilon'_n e^{i \sum_{j=0}^n \varepsilon'_j [e^{i \frac{2\pi i n}{N}} - 1]} \right|^2 \right] \\
& \leq \sigma^2 \sigma^2 \sum_{K \leq |l| \leq N/2} \frac{1}{\min\{||l|^2, N^2\}}, \quad (75)
\end{align*}
\]

so that, uniformly in $N$, the variance of the summands with $|l| > K$ is bounded by a constant times $K^{-1}$. Therefore, when $K = K_N$ tends to infinity, the variance of this term tends to zero, so that this term tends to zero in probability. An identical computation shows that
\[
\begin{align*}
E & \left[ \left| \sum_{K \leq |l| \leq \infty} \frac{s_l}{2\pi l} \sum_{n=0}^{N-1} \varepsilon'_n e^{i \sum_{j=0}^n \varepsilon'_j [e^{i \frac{2\pi i n}{N}} - 1]} \right|^2 \right] \to 0, \quad (76)
\end{align*}
\]

which implies that the contribution due to $|l| \geq K_N$ in $\zeta$ converge to 0 in probability. We are left to investigate the contribution due to $0 < |l| \leq K_N$, since we know that
\[
\delta_N - \delta'_N \xrightarrow{P} 0, \quad (77)
\]

where
\[
\delta'_N = \sum_{1 < |l| \leq K_N} s_l \left[ \frac{i}{N[e^{i \frac{2\pi l}{N}} - 1]} - \frac{1}{2\pi l} \sum_{n=0}^{N-1} \varepsilon'_n e^{i \sum_{j=0}^n \varepsilon'_j [e^{i \frac{2\pi i n}{N}} - 1]} \right]. \quad (78)
\]

Therefore, it suffices to prove that
\[
\delta'_N \xrightarrow{P} 0. \quad (79)
\]

For this, we note that
\[
\begin{align*}
\delta'_N & = \sum_{1 < |l| \leq K_N} s_l \frac{2\pi i l - N[e^{i \frac{2\pi l}{N}} - 1]}{N[e^{i \frac{2\pi l}{N}} - 1](2\pi l)} \sum_{n=0}^{N-1} \varepsilon'_n e^{i \sum_{j=0}^n \varepsilon'_j [e^{i \frac{2\pi i n}{N}} - 1]}.
\end{align*}
\]

(80)
Again we have that $E[\delta_n^2] = 0$, and we can compute similarly to (71) that

$$E[|\delta_n'|^2] = \sigma_n^2 \sum_{0 < |i| \leq K_N} \frac{|2\pi i l - N[e^{i\frac{2\pi i}{N}}] - 1|^2}{N^2 |e^{i\frac{2\pi i}{N}}| - 1^2 (2\pi l)^2}.$$  \hspace{1cm} (81)

A Taylor expansion yields that

$$|2\pi i l - N[e^{i\frac{2\pi i}{N}}]| \leq C\frac{l^2}{N},$$  \hspace{1cm} (82)

Therefore, also using (73), we arrive at

$$E[|\delta_n'|^2] \leq C\sigma_n^2 \sum_{0 < |i| \leq K_N} \frac{1}{N^2} \leq \frac{cK}{N}.$$  \hspace{1cm} (83)

This converges to zero for every $K_N = o(N)$, for example for $K_N = \sqrt{N}$. This proves the convergence in (39). \hspace{1cm} $\square$

**Proof of (26).** Recall that

$$\eta_N = \frac{i}{N} \sum_{0 < |i| \leq N/2} \frac{S_i}{e^{i\frac{2\pi i}{N}} - 1} \left[ e^{i\epsilon^{n+1}_n} - 1 - i\epsilon^{n+1}_n \right] e^{i\sum_{j=0}^{n} \epsilon_j} \left[ e^{i\frac{2\pi i}{N}} - 1 \right].$$  \hspace{1cm} (84)

We will show that $E[|\eta_n|^2]$ converges to zero, which, together with the fact that

$$E[\eta_n] = 0,$$  \hspace{1cm} (85)

implies that $\eta_n$ converges to zero in probability. To prove that the second moment of $\eta_n$ converges to zero, we use the computations in the proof of (39) to obtain that

$$E[|\eta_n|^2] = \sigma_n^2 \sum_{0 < |i| \leq N/2} \frac{1}{N^2 |e^{i\frac{2\pi i}{N}}| - 1^2} E\left[ \sum_{n=0}^{N-1} \left[ e^{i\epsilon^{n+1}_n} - 1 - i\epsilon^{n+1}_n \right] e^{i\sum_{j=0}^{n} \epsilon_j} \right]^2.$$  \hspace{1cm} (86)

Similarly to (70), we obtain that

$$E\left[ \sum_{n=0}^{N-1} \left[ e^{i\epsilon^{n+1}_n} - 1 - i\epsilon^{n+1}_n \right] e^{i\sum_{j=0}^{n} \epsilon_j} \right]^2 = \sum_{n=0}^{N-1} E\left[ |e^{i\epsilon^{n+1}_n} - 1 - i\epsilon^{n+1}_n|^2 \right] \leq \sum_{n=0}^{N-1} E\left[ \epsilon^{n+1}_n \right] = \frac{3\sigma^4}{N},$$  \hspace{1cm} (87)

where we use that $E[\epsilon^{n+1}_n] = \frac{3\sigma^4}{N}$. We arrive at

$$E[|\eta_n|^2] \leq \frac{3\sigma^4 \sigma_n^2}{N} \sum_{0 < |i| \leq N/2} \frac{1}{N^2 |e^{i\frac{2\pi i}{N}}| - 1^2} = O(N^{-1}),$$  \hspace{1cm} (88)

using (75) with $K = 1$. This proves that the approximation in (26) is valid, and even shows that the error in the approximation is, with high probability, smaller than $N^{-\frac{1}{2}+\delta}$ and any $\delta > 0$. \hspace{1cm} $\square$

**Proof of (33).** This again follows from a second moment calculation. We can interchange the order of summation when we have a finite sum, but for an infinite sum that is not so clear. However, by a similar computation as in the proof of (39), we see that

$$\sigma \int_0^1 e^{i\sigma t} (S(t) - S_{\leq K_N}(t)) dB_t \overset{P}{\to} 0,$$  \hspace{1cm} (89)

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where

\[ S_{\leq K_N}(t) = \sum_{0 < |l| \leq K_N} \frac{s_l}{2\pi l} \left[ e^{i2\pi l t} - 1 \right]. \tag{90} \]

Also, it is not hard to see that

\[ \sum_{|l| > K_N} \frac{\sigma s_l}{2\pi l} \int_0^1 e^{i\sigma_B t} [e^{i2\pi l t} - 1] dB_t \xrightarrow{P} 0. \tag{91} \]

Noting that

\[ \zeta = \sigma \int_0^1 e^{i\sigma_B t} S(t) dB_t + \sum_{|l| > K_N} \frac{\sigma s_l}{2\pi l} \int_0^1 e^{i\sigma_B t} [e^{i2\pi l t} - 1] dB_t - \sigma \int_0^1 e^{i\sigma_B t} (S(t) - S_{\leq K_N}(t)) dB_t \tag{92} \]

then completes the proof.

**Proof of (40).** We use that

\[ \varepsilon'_j = \sigma \left[ B_{\frac{j+1}{N}} - B_{\frac{j}{N}} \right], \]

where \( \{B_t\}_{t \geq 0} \) is a standard Brownian motion. Then,

\[ \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i\sum_{j=0}^{n} \varepsilon'_j} S \left( \frac{n}{N} \right) = \sigma \sum_{n=0}^{N-1} e^{i\sigma_B \frac{n+1}{N}} S \left( \frac{n}{N} \right) \left[ B_{\frac{n+2}{N}} - B_{\frac{n+1}{N}} \right]. \tag{93} \]

Clearly, we may replace \( S \left( \frac{n}{N} \right) \) by \( S \left( \frac{n+1}{N} \right) \), since

\[ S \left( \frac{n+1}{N} \right) - S \left( \frac{n}{N} \right) = \sum_{l \in \mathbb{Z}, l \neq 0} \left[ e^{i2\pi l/N} - 1 \right] \frac{s_l}{2\pi l} e^{i2\pi l/N}, \tag{94} \]

and a simple second moment computation as in the proof of (39) proves that the contribution due to this change is small. Therefore, we arrive at the statement that

\[ \sum_{n=0}^{N-1} \varepsilon'_{n+1} e^{i\sum_{j=0}^{n} \varepsilon'_j} S \left( \frac{n}{N} \right) = \sigma \sum_{n=0}^{N-1} e^{i\sigma_B \frac{n+1}{N}} S \left( \frac{n+1}{N} \right) \left[ B_{\frac{n+2}{N}} - B_{\frac{n+1}{N}} \right] + o_P(1) \]

\[ = \sigma \sum_{n=1}^{N} e^{i\sigma_B \frac{n}{N}} S \left( \frac{n}{N} \right) \left[ B_{\frac{n+1}{N}} - B_{\frac{n}{N}} \right] + o_P(1), \tag{95} \]

where \( o_P(1) \) is a random variable that converges to 0 in probability. The above is the usual approximation to the stochastic integral

\[ \sigma \int_0^1 e^{i\sigma_B t} S(t) dB_t. \tag{96} \]

Since for every \( t \), the integrand \( e^{i\sigma_B t} S(t) \) has a finite second moment, and since the process \( t \mapsto e^{i\sigma_B t} S(t) \) is predictable, the sum in (95) converges in probability to the integral in (96). See, e.g., [22, Section 13.8] for convergence results of stochastic integrals. \( \square \)

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