Intermezzo - Week 2

May 9, 2017

Question 1. *Suppose there are two envelopes, with two different amounts of money (say $a \geq 0$ and $b \geq 0$ euros). You choose one of the envelopes (without knowing the content) and are allowed to open it. After that you are allowed to keep the money or swap envelopes and take the money in the other envelope. Without loss of generality assume $a < b$.*

At first glance, it seems there is no hope of taking the envelope with the most money more often than the other one. So the expected amount of money you'll end up with seems to be $(a+b)/2$. However, you can do better than that... The trick is to use a randomized strategy!

Suppose you are able to generate a continuous random variable X, independent from the way you choose an envelope. The strategy is as follows: (i) choose either one of the envelopes with probability 1/2 and examine the contents; (ii) Generate a sample from X (call it x); (iii) if the amount in the envelope is larger than x keep it, otherwise switch envelopes.

(a) Using this strategy, what is the probability you will end up with b euros?

(b) What is the expected amount of money you will get by using this strategy? Compare this with $(a+b)/2$.

(c) What is a good choice for the distribution of X? Decide this before continuing to read the question.

(d) Suppose $a = 100$ and $b = 175$ euros. Using your choice of X compute the probability you will end up of 175 euros, and your expected winnings.

Question 2. (A paradox) *Suppose you are given a choice of two envelopes, and you are told that one of the envelopes contains twice as much money as the other (the amount* 0 *is not allowed). After you chose one of the envelopes you are given a chance to swap it with the other envelope, but you must do so before opening it.*

Given the set-up it seems switching won't bring you any advantage. However, one of your colleagues comes up with the following reasoning that makes you wonder... Let a *denote the amount in the envelope you chose (which is unknown to you).*

- *The probability that* a *is the smaller amount is 1/2 and the probability that it is the largest amount is also 1/2.*
- *The other envelope therefore has either* $a/2$ *or* $2a$ *euros.*
- *Thus the other envelope contains* $a/2$ *euros with probability 1/2 and* $2a$ *euros with probability 1/2.*
- *So the* ***expected*** *amount of money in the other envelope is*

$$\frac{1}{2}2a + \frac{1}{2}\frac{a}{2} \;=\; \frac{5}{4}a \; .$$

- *This is more than* a*, so on average I gain by swapping envelopes.*

(a) *Something doesn't seem quite right in this argument... Where's the catch?*

(b) *What if you are allowed to see the amount in the first envelop you took? What if in addition you are told the envelops contain at least 1 euro and at most 200 euros?*

Question 3. ***The secretary problem (a.k.a. the marriage problem)****: imagine an administrator willing to hire the best secretary out of n rankable applicants for a position (assume there are no ties). The applicants are interviewed one-by-one in random order. A decision about each particular applicant is to be made immediately after the interview. Once rejected, an applicant cannot be recalled. During the interview, the administrator can rank the applicant among all applicants interviewed so far, but is unaware of the quality of yet unseen applicants. The question is to find an optimal strategy to maximize the probability of selecting the best applicant.*

At first glance it seems the probability we'll hire the best secretary cannot be too high. However, there is a (simple) strategy that guarantees you'll hire the best one with probability approximately 0.37 !!! This is actually the best you can hope for...

The strategy is a sort of exploration/exploitation approach. You begin by examining the first $r-1$ candidates and rejecting all of them. However, you take note of the score of the best candidate out of those $r-1$. In the next stage you continue to examine the remaining candidates until you encounter one that is better than any of the $r-1$ candidates you've seen in the initial stage. If you run out of candidates then simply hire the last candidate you see.

(a) *Let P_r denote the probability the selected candidate is the best out of the n applicants. Show that*

$$P_r = \frac{r-1}{n} \sum_{i=r}^{n} \frac{1}{i-1} \ .$$

(b) *We cannot evaluate the above summation exactly, but can approximate it well by an integral. Actually one can show that*

$$\sum_{i=r}^{n} \frac{1}{i-1} \geq \int_{r}^{n+1} \frac{1}{t-1} dt \ .$$

Use this fact to show that

$$P_r \geq \frac{r-1}{n} \log \frac{n}{r-1} \ .$$

(c) *Optimize the lower bound on P_r with respect to r and show that with this strategy you should take $r \approx \frac{n}{e} - 1$ (rounding is needed to ensure r is an integer). This choice ensures $P_r \approx 1/e \approx 0.37$.*

Note: *There is a connection between this problem (with $n = 2$) and variations of the envelopes problem. There are also game-theoretical interpretations of this problem, leading to interesting (and challenging) results with applications in clinical trials and security.*