Intermezzo - Week 4

May 16, 2017

Question 1. ***(It's all about the money!)*** *Suppose you have $M_0 = 10000$ euros to invest in a certain financial stock. You can only buy the stock on Mondays and can only sell it on Fridays. Based on historical data you've seen that it is very reasonable to assume each week the stock increases 80% with probability 1/2 and decreases 60% with probability 1/2. Furthermore, the stock behaves independently in each week. For notational convenience let $a = 0.8$ and $b = -0.6$ denote the profit and loss rates, respectively.*

(a) Say you can only invest on this stock in one week. Is it a good idea to invest?

(b) Now say you have an horizon of one year ($n = 52$ weeks). Say you invest $M_0 = 10000$ euros on the first week. At the end of the week you will have either 18000 or 4000 euros. Let that amount be denoted by M_1. In the next week you'll invest the amount you have again and so on. Let's do a small simulation to see how much money will you have after one year? Is this a good way to invest in such a way?

(c) Now let's think of a different investment strategy. Each week you invest $100c\%$ of you assets in this stock, where we want to choose c in a wise way. Let's do a simulation of this strategy for various values c and see what we observe.

(d) Say your friend John Kelly[1] told you to take

$$c = c^* = \frac{a+b}{2|ab|} \ .$$

Using this choice of c repeat the above simulation an see what happens.

(e) Let's try to see why the above choice is so sensible. Let $X_i \in \{0, 1\}$, $i \in 1, \ldots, n$ indicate the outcome of the stock each week, were $X_i = 1$ corresponds to a profit and $X_i = 0$ corresponds to a loss. Note that X_i are independent Bernoulli random variables with parameter $p = 1/2$. Say you invest a proportion c of your assets each week. Show that the amount of money you'll have in the end of the n rounds is

$$M_n = M_0 \prod_{i=1}^{n} (1+ca)^{X_i}(1+cb)^{1-X_i} = M_0(1+ca)^Y(1+cb)^{n-Y} \ ,$$

where $Y = \sum_{i=1}^{n} X_i$.

[1] `https://en.wikipedia.org/wiki/John_Larry_Kelly,_Jr.`

(f) *Note that Y is a Binomial random variable with parameters n and $p = 1/2$. By the law of large numbers we know that $Y/n \to 1/2$. So, the amount of money after n rounds is approximately*[2]

$$M_n = M_0 \left((1+ca)^{Y/n} (1+cb)^{1-Y/n} \right)^n \approx \left(\sqrt{(1+ca)(1+cb)} \right)^n .$$

Figure out what is the value c that maximizes the right-hand-side of the above expression.

(g) *How will the result change if the profit probability is $p \in (0,1)$? Show that the optimal allocation will be*

$$c^* = \min \left(\max \left(\frac{pa + (1-p)b}{|ab|}, 1 \right), 0 \right) .$$

(h) *Now let's make things a bit more realistic, and much more complicated. Suppose there is a cost for buying or selling stock. Namely, if you buy or sell x euros of stock you'll have to pay a transaction fee of λx, where $\lambda \in [0,1]$. Let's say you still want to use the same strategy and that at every round you want to re-invest a proportion c of your assets, meaning you'll have to buy or sell some stock (in the financial jargon, you'll have to re-balance your portfolio). Show that the amount of money you'll have after n rounds is*

$$M(1-\lambda c) \left(\frac{1 + ca(1-\lambda) - \lambda c}{1 - \lambda c} \right)^Y \left(\frac{1 + cb(1+\lambda) + \lambda c}{1 + \lambda c} \right)^{n-Y} .$$

(i) *Using the same reasoning as in part (f) find the value of c that maximizes your "profit" when $\lambda = 0.1$ (which is a very hefty commission on your trades). You'll likely have to solve the optimization problem numerically.*

[2] Here I'm sweeping some important stuff under the rug - Note that while Y/n is getting close to a constant, the "speed" at which this happens will influence the quality/validity of the approximation, as we are computing a power with exponent n, which is bigger and bigger.