Integer Programming, Part 1

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Prerequisites for the course:

Theorem (Farkas' Lemma)

The system $Ax \leq b$ is infeasible if and only if the system $uA = 0, ub < 0, u \geq 0$ is feasible.

Let $P := \{x : Ax \leq b\}$ and $D := \{u : uA = c, u \geq 0\}$.

Theorem (Linear Programming Duality)

If P and D are both nonempty, then

$$\max\{cx : x \in P\} = \min\{ub : u \in D\}.$$

Theorem (Complementary slackness)

Let $x^* \in P$ and $u^* \in D$. Then x^* and u^* are both optimal if and only if

$$u_i^*(a^i x^* - b_i) = 0 \text{ for all } i$$

where a^i denotes the i-the row of matrix A.

Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda_1 \ldots, \lambda_k \in \mathbb{R}$. Then

$$x = \lambda_1 x^1 + \cdots + \lambda_k x^k$$

is a *linear combination* of $x^1, \ldots, x^k$.

- x is an *affine combination* if $\sum_i \lambda_i = 1$
- x is a *convex combination* if $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$ for all i
- x is a *nonnegative combination* if $\lambda_i \geq 0$ for all i

Let $S \subseteq \mathbb{R}^n$ be any set of vectors.

The $\left\{ \begin{array}{c} \text{linear hull} \\ \text{affine hull} \\ \text{convex hull} \\ \text{cone} \end{array} \right\}$ of S is the set of all $\left\{ \begin{array}{c} \text{linear} \\ \text{affine} \\ \text{convex} \\ \text{nonnegative} \end{array} \right\}$ combinations of $x^1, \ldots, x^k \in S$.

A set S is *affine* if $S = \text{aff}(S)$, *convex* if $S = \text{conv }(S)$, a *cone* if $S = \text{cone }(S)$.

Let $C \subseteq \mathbb{R}^n$.

- C is a *polyhedral cone* if $C = \{x : Ax \leq 0\}$ for some matrix A.
- C is *finitely generated* if $C = \text{cone}\,\{r^1, \ldots, r^q\}$ for some vectors $r^1, \ldots, r^q$

Theorem (Minkowski-Weyl for cones)

Let $C \subseteq \mathbb{R}^n$. Then C is a finitely generated cone if and only if C is a polyhedral cone.

Example

Let $C := \{x \in \mathbb{R}^2 : x_1 - x_2 \leq 0, -2x_1 + x_1 \leq 0\}$ be a polyhedral cone. Then $C = \text{cone}\,\{(1,1),(1,2)\}$, so C is also a finitely generated cone.

Example

Let $C := \text{cone}\,\{(1,5),(1,1),(4,1)\}$ be a finitely generated cone. Then $C := \{x \in \mathbb{R}^2 : x_1 - 4x_1 \leq 0. - 5x_1 + x_2 \leq 0\}$, so C is also a polyhedral cone.

- $P \subseteq \mathbb{R}^n$ is a *polyhedron* if $P = \{x : Ax \leq b\}$ for some A, b.
- $Q \subseteq \mathbb{R}^n$ is a *polytope* if $Q = \text{conv}\,\{v^1, \ldots, v^p\}$ for some vectors $v^1, \ldots, v^p$
- the *Minkowski sum* of $A, B \subseteq \mathbb{R}^n$ is $A + B := \{a + b : a \in A, b \in B\}$

Theorem (Minkowski-Weyl for polyhedra)

Let $P \subseteq \mathbb{R}^n$. Then P is a polyhedron if and only if $P = Q + C$ for some polytope Q and some finitely generated cone C.

Proof.

$\Rightarrow$: If $P = \{x : Ax \leq b\}$, consider $C_P := \{(x, y) : Ax - by \leq 0, x \in \mathbb{R}^n, y \in \mathbb{R}\}$.
$\Leftarrow$: If $P = Q + C = \text{conv}\,\{v^1, \ldots, v^p\} + \text{cone}\,\{r^1, \ldots, r^q\}$, consider

$$C_P := \text{cone}\left\{\begin{pmatrix} v^1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} v^p \\ 1 \end{pmatrix}, \begin{pmatrix} r^1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} r^q \\ 0 \end{pmatrix}\right\}$$

In either case, apply Minkowski-Weyl for cones. □

Example

Consider $P = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 2, x_1 + 2x_2 \geq 3\}$. Then $P = Q + C$, where

$$Q = \text{conv}\left\{\begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right\},\ C = \text{cone}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

Example

Consider $P = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 1\}$. Then $P = Q + C$, where

$$Q = \text{conv}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\},\ C = \text{cone}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right\}$$

Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron.

- the *lineality space* of P is $\text{lin}(P) := \{r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P \text{ and } \lambda \in \mathbb{R}\}$
- the *recession cone* of P is $\text{rec}(P) := \{r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P \text{ and } \lambda \in \mathbb{R}_+\}$

Theorem

Suppose $P = \{x : Ax \leq b\} = \textit{conv}\,\{v^1, \ldots, v^p\} + \textit{cone}\,\{r^1, \ldots, r^q\}$. *Then*

- $\textit{lin}(P) = \{x : Ax = 0\}$ *and*
- $\textit{rec}(P) = \{r : Ar \leq 0\} = \textit{cone}\,\{r^1, \ldots, r^q\}$

Example

Consider again the polyhedron $P = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 1\}$. Then

- $\text{lin}(P) = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0\}$
- $\text{rec}(P) = \{r \in \mathbb{R}^3 : r_1 \geq 0, r_2 \geq 0\} = \text{cone} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\}$

- The system $Ax \leq b$ consists of inequalities $a^i x \leq b_i$ for $i \in M$
- $a^i x \leq b_i$ is an *implicit equality* of $Ax \leq b$ if $a^i x = b_i$ for all x such that $Ax \leq b$
- $A^= x \leq b^=$ denotes the subsystem of $Ax \leq b$ containing all implicit equalities
- for any $P \subseteq \mathbb{R}^n$, $\dim(P) := \dim(\text{aff}(P))$

Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron. Then

$$aff(P) \;=\; \{x : A^= x = b^=\} \;=\; \{x : A^= x \leq b^=\}$$

Furthermore, $\dim(P) = n - rank(A^=)$.

Example

The *assignment polytope*

$$P = \left\{ x \in \mathbb{R}^{n^2} : \sum_i x_{ij} = 1 \text{ for all } j,\ \sum_j x_{ij} = 1 \text{ for all } i,\ x_{ij} \geq 0 \text{ for all } i,j \right\}$$

has dimension $n^2 - 2n + 1$

Let $P \subseteq \mathbb{R}^n$.

- An inequality $cx \leq \delta$ is *valid* for P if $cx \leq \delta$ for all $x \in P$
- A *face* of P is a set F of the form

$$F = P \cap \{x \in \mathbb{R}^n : cx = \delta\}$$

where $cx \leq \delta$ is a valid inequality for P

Let $P = \{x : a^i x \leq b_i \text{ for } i \in M\}$, and for each $I \subseteq M$ put

$$F_I := \{x \in \mathbb{R}^n : a^i x = b_i \text{ for all } i \in I, \text{ and } a^i x \leq b_i \text{ for all } i \in M \setminus I\}$$

Theorem

For each $I \subseteq M$, F_I is a face of P. Conversely, if F is a nonempty face of P, then $F = F_I$ for some $I \subseteq M$.

It follows that the number of faces of polyhedron P is finite.

Let $P = \{x : a^i x \leq b_i \text{ for } i \in M\}$.

- The inequality $a^j x \leq b_j$ is *redundant* if it is valid for

$$\{x : a^i x \leq b_i \text{ for all } i \in M \setminus \{j\}\}$$

- "$a^i x \leq b_i$ for $i \in M$" is a *minimal representation* of P if $a^i x \leq b_i$ is irredundant for $i \in M$
- A face F of P is *proper* if $F \neq \emptyset$ and $F \neq P$
- A *facet* of P is an proper face of P not strictly contained in another proper face of P

Theorem

For each facet F of P there is an inequality $a^i x \leq b_i$ so that

$$F = P \cap \{x \in \mathbb{R}^n : a^i x = b_i\}.$$

Conversely, if the inequality $a^i x \leq b_i$ is irredundant and not an implied equation, then $F = P \cap \{x \in \mathbb{R}^n : a^i x = b_i\}$ is a facet of P.

So in a minimal representation of P, each inequality is an implied equation or defines a facet.

Let P be a nonempty polyhedron.

- a set F is a *minimal face* of P if F is a face of P which does not contain a proper face

Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. The a nonempty face F of P is a minimal face if and only if $F = \{x : A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$ such that $rank(A') = rank(A)$.

- a set F is a *vertex* of P if F is a face of P of dimension 0, i.e. $F = \{x\}$
- P is *pointed* if P has a vertex

Theorem

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. If P is pointed, then equivalent:

- *$\{\overline{x}\}$ is a vertex*
- *$\overline{x}$ satisfies n linearly independent inequalities from $Ax \leq b$ with equality*
- *there are no $x', x'' \in P \setminus \{\overline{x}\}$ so that $\overline{x}$ is a convex combination of x', x''.*

Let P be any nonempty polyhedron. Put

- $t := \dim(\text{lin}(P))$
- let $F_1, \ldots, F_p$ be the minimal faces of P; let $v^i \in F_1$
- let $R_1, \ldots, R_q$ be the $(t+1)$-dimensional faces of $\text{rec}(P)$; let $r^i \in R_i \setminus \text{lin}(P)$

Theorem

$P = conv\,\{v^1, \ldots, v^p\} \;+\; cone\,\{r^1, \ldots, r^p\} \;+\; lin(P)$

Proof.

The special case where $t = 0$ reduces to Minkowski-Weyl.
If $t > 0$, let $a^1, \ldots, a^t$ be a basis of $\text{lin}(P)$. Then $P = Q + \text{lin}(P)$ where

$$Q := \{x \in P : a^i x = 0 \text{ for } i = 1, \ldots t\}$$

We have $\dim(\text{lin}(Q)) = 0$, and hence $Q = \text{conv}\,\{v^1, \ldots, v^p\} \;+\; \text{cone}\,\{r^1, \ldots, r^p\}$. □

- Sections 3.1, 3.2, 3.3, 3.14 are prerequisite knowledge for the course. Review these sections if necessary, and verify that you are able to make exercises 3.1, 3.2, 3.30.
- Read sections 3.4 — 3.12.
- Make exercises 3.7, 3.10, 3.16, 3.33.

A polyhedron $P \subseteq \mathbb{R}^n$ is *integral* if $P = \text{conv}\,(P \cap \mathbb{Z}^n)$.

If P is integral, then $\max\{cx : x \in P \cap \mathbb{Z}^n\} = \max\{cx : x \in P\}$ is an ordinary linear program.

We want to write combinatorial optimization problems as optimization over integral polyhedra.

We next develop some tools for proving that a polyhedron P is integral.

Theorem

$P \subseteq \mathbb{R}^n$ a rational polyhedron. Then equivalent:

1. $P = conv\,(P \cap \mathbb{Z}^n)$
2. $F \cap \mathbb{Z}^n \neq \emptyset$ *for every minimal face* F *of* P
3. $\max\{cx : x \in P\}$ *attained by integral vector* $\bar{x} \in P \cap \mathbb{Z}^n$ *for each* $c \in \mathbb{R}^n$*, if attained at all*
4. $\max\{cx : x \in P\} \in \mathbb{Z} \cup \{\infty\}$ *for each* $c \in \mathbb{Z}^n$

Proof.

Some of the implications are straightforward... □

Theorem

$P \subseteq \mathbb{R}^n$ a rational polyhedron. Then equivalent:

1. $P = conv\ (P \cap \mathbb{Z}^n)$
2. $F \cap \mathbb{Z}^n \neq \emptyset$ *for every minimal face F of P*
3. $\max\{cx : x \in P\}$ *attained by integral vector* $\bar{x} \in P \cap \mathbb{Z}^n$ *for each* $c \in \mathbb{R}^n$, *if attained at all*
4. $\max\{cx : x \in P\} \in \mathbb{Z} \cup \{\infty\}$ *for each* $c \in \mathbb{Z}^n$

Proof.

(4) $\Rightarrow$ (2): We may assume $P = \{x : Ax \leq b\}$ for some A, b with integer entries

- Consider a minimal face F. $F = \{x : A^F x = b^F\}$ for a subsystem $A^F x \leq b^F$ of $Ax \leq b$
- Suppose $F \cap \mathbb{Z}^n = \emptyset$; then $uA \in \mathbb{Z}^n$ and $ub \notin \mathbb{Z}$ for some u. W.l.o.g. $u \geq 0$
- With $c := uA$, $z := ub$, we have $cx = uAx \leq ub = z$ for all $x \in P$, with equality if $x \in F$.
- Then $\max\{cx : x \in P\} = z \notin \mathbb{Z}$

□

Theorem

$P \subseteq \mathbb{R}^n$ a rational polyhedron. Then equivalent:

1. $P = conv\,(P \cap \mathbb{Z}^n)$
2. $F \cap \mathbb{Z}^n \neq \emptyset$ *for every minimal face* F *of* P
3. $\max\{cx : x \in P\}$ *attained by integral vector* $\bar{x} \in P \cap \mathbb{Z}^n$ *for each* $c \in \mathbb{R}^n$, *if attained at all*
4. $\max\{cx : x \in P\} \in \mathbb{Z} \cup \{\infty\}$ *for each* $c \in \mathbb{Z}^n$

Proof.

(2) $\Rightarrow$ (1): We may again assume $P = \{x : Ax \leq b\}$ for some A, b with integer entries

- Let $F^1, \ldots, F^p$ be the minimal faces of P, and pick any $v^i \in F^i \cap \mathbb{Z}^n$ for each i.
- Then $P = \text{conv}\,\{v^1, \ldots, v^p\} + \text{rec}(P)$
- $\text{rec}(P) = \{r : Ar \leq 0\} = \text{cone}\,\{r^1, \ldots, r^q\}$ for some rational r^i, since A is rational.
- Scaling the r^j, we may assume $r^j \in \mathbb{Z}^n$ for all j.
- So $P = \text{conv}\,\{v^1, \ldots, v^p\} + \text{cone}\,\{r^1, \ldots, r^q\}$ with $v^i, r^j \in \mathbb{Z}^n$, hence $P = \text{conv}\,(P \cap \mathbb{Z}^n)$

□

We next consider the question:

Which matrices A are such that $\{x : Ax \leq b, x \geq 0\}$ *is integral for all* $b \in \mathbb{Z}^n$*?*

A matrix A is *totally unimodular (TU)* if $\det(B) \in \{-1, 0, 1\}$ for every square submatrix B of A

Lemma

If B is an integer matrix such that $\det(B) \in \{-1, 1\}$*, then* B^{-1} *is an integer matrix.*

Theorem (Hoffman and Kruskal)

A *is TU* $\Leftrightarrow \{x : Ax \leq b, x \geq 0\}$ *is integral for each* $b \in \mathbb{Z}^n$*.*

If A is a matrix, then an *equitable bicoloring of A* is a partiton of the columns into 'red' and 'blue' columns so that

$$\left(\sum_{j \text{ red}} A_j\right) - \left(\sum_{j \text{ blue}} A_j\right) \in \{-1, 0, 1\}^m$$

Here A_j is the j-th column of A.

Theorem

A TU $\iff$ every column submatrix admits an equitable bicoloring.

Proof.

$\Longrightarrow$:

Let B be a column submatrix of A. Since A is TU, so is B. The polyhedron

$$P := \{x : \lfloor \frac{1}{2} B\mathbf{1} \rfloor \leq Bx \leq \lceil \frac{1}{2} B\mathbf{1} \rceil, 0 \leq x \leq \mathbf{1}\}$$

contains an $x \in \mathbb{Z}^n$. Let the i-th column be red if $x_i = 0$, blue otherwise. □

If A is a matrix, then an *equitable bicoloring of A* is a partiton of the columns into 'red' and 'blue' columns so that

$$\left(\sum_{j \text{ red}} A_j\right) - \left(\sum_{j \text{ blue}} A_j\right) \in \{0,1\}^m$$

Here A_j is the j-th column of A.

Theorem

A TU $\iff$ every column submatrix admits an equitable bicoloring.

Proof.

$\Longleftarrow$:

Suppose not. Let B be a smallest submatrix of A with $\delta := \det(B) \notin \{-1,0,1\}$. Then δB^{-1} is a $0,\pm1$-matrix. Let d be the first column of δB^{-1}. Then $Bd = \delta e_1$. If δ is even, then $\delta = 0$. If δ is odd, then $\delta = 1$. □

If A is a matrix, then an *equitable bicoloring of A* is a partiton of the columns into 'red' and 'blue' columns so that

$$\left(\sum_{j \text{ red}} A_j\right) - \left(\sum_{j \text{ blue}} A_j\right) \in \{0,1\}^m$$

Here A_j is the j-th column of A.

Theorem

A TU $\iff$ every column submatrix admits an equitable bicoloring.

The *signed incidence matrix* of a directed graph $D = (V, A)$ is the $V \times A$ matrix A_D with entries $a_{va} \in \{-1, 0, 1\}$ such that

$$a_{va} = \begin{cases} -1 & \text{if } v \text{ is the tail of } a \\ 1 & \text{if } v \text{ is the head of } a \\ 0 & \text{otherwise} \end{cases}$$

Corollary

The incidence matrix of any directed graph $D = (V, A)$ is totally unimodular.

- Read 4.1, 4.2, and first two pages of 4.3. Try to fully understand all calls to other theorems in the proof of Thm 4.1.
- Make exercises 4.3, 4.4, 4.5, 4.6.

A rational system $Ax \leq b$ is *totally dual integral (TDI)* if

$$\min\{yb : yA = c, y \geq 0\}$$

either has an integral optimal solution or is infeasible for each $c \in \mathbb{Z}^n$.

Theorem

If $Ax \leq b$ is TDI and b integral, then $\{x : Ax \leq b\}$ is integral.

Proof.

If $Ax \leq b$ is TDI and b integral, then $\min\{yb : yA = c, y \geq 0\} \in \mathbb{Z} \cup \{\infty\}$ for each $c \in \mathbb{Z}^n$.
Hence $\max\{cx : Ax \leq b\} \in \mathbb{Z} \cup \{\infty\}$ for each $c \in \mathbb{Z}^n$, so that $\{x : Ax \leq b\}$ is integral. □

We next show:

Theorem

If P is rational, then there exists a TDI system $Ax \leq b$ so that $P = \{x : Ax \leq b\}$.

Theorem

If P is (rational and) integral, then there exists a TDI system $Ax \leq b$ so that $P = \{x : Ax \leq b\}$ and b integral.

Theorem

If P is rational, then there exists a TDI system $Ax \leq b$ so that $P = \{x : Ax \leq b\}$.

Proof.

- if $P = \emptyset$, then $P = \{x : 0x \leq -1\}$
- P is rational, so we may assume $P = \{x : Mx \leq d\}$ for integer M, d
- Put $C := \{c \in \mathbb{Z}^n : c = \lambda M, 0 \leq \lambda \leq 1\}$, $\delta_c = \max\{cx : x \in P\}$ for all $c \in C$
- let $Ax \leq b$ be the system of all inequalities $cx \leq \delta_c$ for $c \in C$
- $P \subseteq \{x : Ax \leq b\} \subseteq \{x : Mx \leq d\} = P$, so we have equality throughout
- to show that $Ax \leq b$ is TDI, consider a $c \in \mathbb{Z}^n$ so that $\max\{cx : Ax \leq b\}$ is finite
- $\max\{cx : Ax \leq b\} = \max\{cx : Mx \leq d\} = \min\{yd : yM = c, y \geq 0\}$ has an optimal solution y^*; put $\lambda = y^* - \lfloor y^* \rfloor$, $c' := \lambda M, c'' := \lfloor y^* \rfloor M$;
- $\min\{yd : yM = c, y \geq 0\} = \min\{yd : yM = c', y \geq 0\} + \min\{yd : yM = c", y \geq 0\}$, so $\lambda, \lfloor y^* \rfloor$ are optimal solutions for the latter two problems
- $\min\{yd : yM = c', y \geq 0\} = \max\{c'x : x \in P\} = \delta_{c'}$, and $c'x \leq \delta_{c'}$ is a row of $Ax \leq b$
- $\min\{yb : yA = c, y \geq 0\}$ has an integer optimal solution obtained from $\lfloor y^* \rfloor$ and c'

Theorem

If P is integral, then there exists a TDI system $Ax \leq b$ so that $P = \{x : Ax \leq b\}$ and b integral.

Proof.

If P is integer, then in the previous proof $\delta_c := \max\{cx : x \in P\} \in \mathbb{Z}$ for each $c \in C \subseteq \mathbb{Z}^n$ □

Theorem

Let A, G be rational matrices, b a rational vector. Put

$$P := \{(x, y) : Ax + Gy \leq b\},\ S := \{(x, y) : Ax + Gy \leq b, x \in \mathbb{Z}^n\}.$$

Then:

1. $\exists A', G', b'$ *rational so that* $\text{conv}\,(S) = \{(x, y) : A'x + G'y \leq b'\}$.
2. *if* $S \neq \emptyset$, *then the recession cones of* P *and* $\text{conv}\,(S)$ *coincide.*

Proof.

- $P = \text{conv}\,\{v^1, \ldots, v^t\} + \text{cone}\,\{r^1, \ldots, r^q\}$ for some rational v^i, integral r^j.
- Consider $T := \{\sum_i \lambda_i v^i + \sum_j \mu_j r^j : \sum \lambda_i = 1, \lambda \geq 0, 0 \leq \mu \leq \mathbf{1}\}$
- Put $T_I := \{(x, y) \in T : x \in \mathbb{Z}^n\}$, $R_I := \{\sum_j \mu_j r^j : \mu \in \mathbb{Z}^q_+\}$
- $S = T_I + R_I$; hence $\text{conv}\,(S) = \text{conv}\,(T_I) + \text{conv}\,(R_I)$
- $\text{conv}\,(T_I)$ is a rational polyhedron,
- $\text{rec}(\text{conv}\,(S)) = \text{conv}\,(R_I) = \text{cone}\,\{r^1, \ldots, r^q\} = \text{rec}(P)$ is a rational cone

□

- Read sections 4.6, 4.8.
- Make exercises 4.9, 4.11, 4.18, 4.20.

A *matching* in an undirected graph $G = (V, E)$ is a set of pairwise disjoint edges $M \subseteq E$.

IP (Maximum cardinality matching)

$$\begin{aligned} \max \quad & \textstyle\sum_{e \in E} x_e \\ & \textstyle\sum_{e \in \delta(v)} x_e \leq 1 \quad v \in V \\ & x \in \{0,1\}^E \end{aligned}$$

Let

$$P(G) := \text{conv } \{x \in \{0,1\}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V\}$$

and put

$$Q(G) := \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V, x \geq 0\}$$

Is $Q(G)$ integral for all undirected graphs $G = (V, E)$, i.e. is $P(G) = Q(G)$?

The *incidence matrix* of an undirected graph $G = (V, E)$ is the $V \times E$ matrix A_G with entries $a_{ve} \in \{0, 1\}$ such that

$$a_{ve} = \begin{cases} 1 & \text{if } v \text{ is incident with } e \\ 0 & \text{otherwise} \end{cases}$$

Clearly

$$Q(G) = \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V, x \geq 0\} = \{x \in \mathbb{R}^E : A_G x \leq \mathbf{1}, x \geq 0\}$$

Lemma

The incidence matrix of G is totally unimodular if and only if G is bipartite.

Theorem

If G is bipartite, then P(G) is integral.

$$P(G) := \text{conv } \{x \in \{0,1\}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V\}$$

Lemma

If $G = (V, E)$ then for any $U \subseteq V$ with $|U|$ odd the inequality

$$\sum_{e \in E[U]} x_e \leq \frac{|U| - 1}{2}$$

is valid for $P(G)$.

Is

$$P(G) = \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V, \ \sum_{e \in E[U]} x_e \leq \frac{|U|-1}{2} \text{ for all odd } U \subseteq V, \ x \geq 0\}$$

??

Theorem

Let $G = (V, E)$ be a graph. The polyhedron

$$\{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e = 1 \text{ for all } v \in V, \ \sum_{e \in \delta(U)} x_e \geq 1 \text{ for all odd } U \subseteq V, \ x \geq 0\}$$

is integral.

Proof.

Suppose not.

- let $G = (V, E)$ be a counterexample to the theorem with $|V| + |E|$ as small as possible.
- let $\bar{x}$ be a fractional vertex of the polyhedron; there is some odd U so that $\sum_{e \in \delta(U)} \bar{x}_e = 1$
- $\bar{x}$ is a convex combination of integer vectors in the polyhedron, contradiction

□

Applying the theorem to a 'doubled' version of G, it then follows that

$$P(G) = \{x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \leq 1 \text{ for all } v \in V, \ \sum_{e \in E[U]} x_e \leq \frac{|U| - 1}{2} \text{ for all odd } U \subseteq V, \ x \geq 0\}$$

A *spanning tree* of $G = (V, E)$ is a set $T \subseteq E$ so that (V, T) is acyclic and connected.

IP (Maximum weight spanning tree)

$$
\begin{aligned}
\max \quad & \textstyle\sum_{e \in E} w_e x_e \\
& \textstyle\sum_{e \in E[S]} x_e \leq |S| - 1 \quad S \subseteq V, S \neq \emptyset \\
& \textstyle\sum_{e \in E} x_e = |V| - 1 \\
& x \in \{0,1\}^E
\end{aligned}
$$

$$P(G) := \{x \in \mathbb{R}^E : \sum_{e \in E[S]} x_e \leq |S| - 1 \text{ for all nonempty } S \subseteq V,\ \sum_{e \in E} x_e = |V| - 1,\ x \geq 0\}$$

Theorem

$P(G)$ is integral.

Let N be a finite set. A function $f : 2^N \to \mathbb{R}$ is *submodular* if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$$

Example

- Let $G = (V, E)$ be an undirected graph and let $f : 2^V \to \mathbb{R}$ be defined by $f(S) := |\delta(S)|$. Then f is submodular.
- Let $G = (V, E)$ be an undirected graph and let $f : 2^E \to \mathbb{R}$ be defined by

$$f(S) := |V| - \#\text{ of components of } (V, S)$$

  Then f is submodular.

The *submodular polyhedron* is

$$P(f) := \{x \in \mathbb{R}^N : \sum_{j \in S} x_j \leq f(S) \text{ for all } S \subseteq N\}$$

The *submodular polyhedron* is

$$P(f) := \{x \in \mathbb{R}^N : \sum_{j \in S} x_j \leq f(S) \text{ for all } S \subseteq N\}$$

Theorem

Suppose $f : 2^N \to \mathbb{Z}$ is submodular and $f(\emptyset) = 0$. Then $P(f)$ is integral, and the system $\sum_{j \in S} x_j \leq f(S)$ for all $S \subseteq N$ is TDI.

Proof.

Let $c \in \mathbb{Z}^n$. Consider $\max\{cx : \sum_{j \in S} x_j \leq f(S) \text{ for all } S \subseteq N\}$. W.l.o.g. $c_1 \geq c_2 \geq \ldots \geq c_n$. A feasible solution $\bar{x}$ is obtained by setting $\bar{x}_j := f(S_j) - f(S_{j-1})$, where $S_j := \{1, \ldots, j\}$. The dual is

$$\min\{\sum_{S \subseteq N} f(S) y_S : \sum_{j \in S} y_S = c_j \text{ for all } j \in N, y \geq 0\}$$

We construct an integral optimal solution $\bar{y}$ for this problem so that $c\bar{x} = \sum_{S \subseteq N} f(S) y_S$ proving optimality of both integral solutions. □

Theorem (Nash-Williams' orientation theorem)

A $2k$-connected undirected graph G can be oriented so as to become a k-connected directed graph.

Proof.

Let $D = (V, A)$ arise from G by arbitrarily orienting the edges.

- if there is an $x \in \mathbb{Z}^A$ so that

$$0 \leq x \leq \mathbf{1},\ x[\delta^-(U)] - x[\delta^+(U)] \leq |\delta^-(U)| - k \ \text{ for all } U \subseteq V \qquad (*)$$

  then we are done by reversing the arcs a so that $x_a = 1$.
- We consider the polyhedron $P := \{x \in \mathbb{R}^A : (*)\}$. Then $x = \frac{1}{2}\mathbf{1} \in P$, so $P \neq \emptyset$
- We will prove TDI-ness of $(*)$. Then $P_I = P \neq \emptyset$, hence $P \cap \mathbb{Z}^A \neq \emptyset$.

□

Proof.

We prove TDI-ness of

$$0 \leq x \leq \mathbf{1},\ x[\delta^-(U)] - x[\delta^+(U)] \leq |\delta^-(U)| - k \ \text{ for all } U \subseteq V \qquad (*)$$

- Consider the LP dual to maximizing cx over $(*)$.
- Let z be an optimal dual solution so that $\sum_U z_U |U| \cdot |V \setminus U|$ minimal.
- Then $\mathcal{U} := \{U \mid z_U > 0\}$ is *cross-free*, i.e. if $T, U \in \mathcal{U}$, then

$$U \subseteq T,\ T \subseteq U,\ U \cap T = \emptyset \text{ or } U \cup T = V$$

- The restricted system

$$0 \leq x \leq \mathbf{1},\ x[\delta^-(U)] - x[\delta^+(U)] \leq |\delta^-(U)| - k \text{ for all } U \in \mathcal{U}$$

  is totally unimodular.
- So the dual optimum is determined by this TU system, and hence is integral.

□

- Read sections 4.3, 4.4, 4.5, 4.7.
- Make exercises 4.9, 4.12, 4.14, 4.18.