Algebraic geometry codes and order domains

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References:

Mike O'Sullivan,
"New codes for the Berlekamp-Massey-Sakata algorithm,"
Finite Fields and their Applications, vol. 7, pp. 293-317, 2001.

Olav Geil and Ruud Pellikaan,
"On the structure of order domains,"
Finite Fields and their Applications, vol. 8, pp. 369-396, 2002.

http://www.win.tue.nl/ ruudp/paper/41.pdf
http://www.win.tue.nl/ ruudp/lectures/slides4-AGC-Soria.pdf

Algebraic geometry codes on an elementary foundation.

- Parameters $[n, k, d]$ of these codes
- Order bound for the minimum distance
- Generalized Hamming weights

References:

- Feng-Rao
- Høholdt-Van Lint-Pellikaan
- Heijnen-Pellikaan

Efficient decoding algorithm:

- Basic algorithm
- Majority voting of unknown syndromes
- Algorithm of Berlekamp-Massey-Sakata.

References:

- Justesen-Larsen-Jensen-Havemose-Hoeholdt
- Skorobogatov-Vladut
- Feng-Rao, Duursma
- Sakata
- O'Sullivan
- Høholdt-Van Lint-Pellikaan

Topics in this lecture:

- Theory of Gröbner bases
- Construction of new order domains, Factor ring theorem
- Converse: Presentation theorem

References:

- Heegard-Little-Saints
- O'Sullivan
- Matsumoto-Miura
- Geil-Pellikaan
- Little

Topics not treated:

- Codes on order domains
- Improved codes on order domains
- Properties of semi-groups and parameters of codes
- Decoding of codes on order domains

References:

- Geil-Høholdt
- Bras-O'Sullivan
- Campillo-Farran-Munuera
- Pellikaan-Torres

Content

- Definition of an order domain.
- Value semigroup.
- The theory of Gröbner bases for order domains.
- Factor ring theorem.
- Surprising example:
  - order domain on $\mathbb{F}[X, Y]$
  - value semigroup in $\mathbb{Q}$
- Presentation theorem.
- Dimension (order domain) = rank (value semigroup).

Notation:

- Let $(\Gamma, <)$ be a well-order.
- The minimal element is denoted by 0.
- $\Gamma_{-\infty} = \Gamma \cup \{-\infty\}$ is the order with $-\infty$ as minimal element.
- Let $\mathbb{F}$ be a field.
- Let R be an $\mathbb{F}$-algebra.

An **order function** on R is a surjective map

$$\rho : R \longrightarrow \Gamma_{-\infty},$$

such that for all $f, g, h \in R$:

- $\rho(f) = -\infty$ if and only if $f = 0$,
- $\rho(af) = \rho(f)$ for all nonzero $a \in \mathbb{F}$,
- $\rho(f + g) \leq \max\{\rho(f), \rho(g)\}$,
- and equality holds if $\rho(f) \neq \rho(g)$,
- If $\rho(f) \leq \rho(g)$, then $\rho(fh) \leq \rho(gh)$,
- If f and g are nonzero and $\rho(f) = \rho(g)$, then there exists a nonzero $a \in \mathbb{F}$ such that

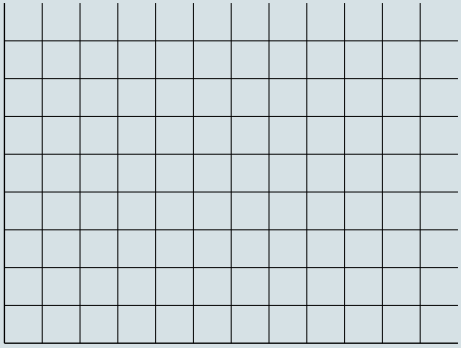
$$\rho(f - ag) < \rho(g).$$

Examples

- – $R = \mathbb{F}[X]$
  – $\rho = \deg$
- – $R = K_P(\mathcal{X})$ the ring of rational functions on an algebraic curve $\mathcal{X}$ with no poles outside the point P
  – v_P is the valuation at P and $\rho = -v_P$ the pole order at P.

Example

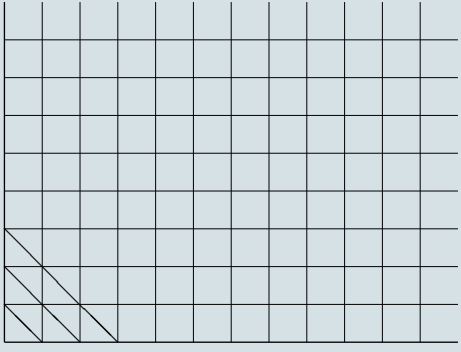
- $R = \mathbb{F}[X, Y]$
- $\Gamma = \mathbb{N}_0^2$ with lexicographic order $<_{lex}$.
  - $(\alpha_1, \alpha_2) <_{lex} (\beta_1, \beta_2)$ iff
  - $\alpha_1 < \beta_1$, or $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.
- Multi-index notation: $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2}$
- – Let $f = \sum_{\alpha \leq \gamma} a_\alpha X^\alpha$
  - with $a_\alpha \in \mathbb{F}$ and $a_\gamma \neq 0$.
- Then $\rho(f) = \gamma$.



$$
\begin{array}{ccccccccc}
 & (0,0) & < & (0,1) & < & \cdots & < & (0,n) & < & \cdots \\
< & (1,0) & < & (1,1) & < & \cdots & < & (1,n) & < & \cdots \\
 & & & & & \vdots & & & & \\
< & (m,0) & < & (m,1) & & \cdots & < & (m,n) & < & \cdots
\end{array}
$$

Example

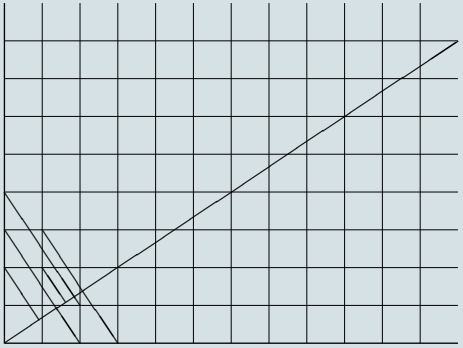
- $R = \mathbb{F}[X, Y]$
- $\Gamma = \mathbb{N}_0^2$
  - with total degree lexicographic order $<_{dlex}$.
  - $(\alpha_1, \alpha_2) <_{dlex} (\beta_1, \beta_2)$ iff
  - $\alpha_1 + \alpha_2 < \beta_1 + \beta_2$, or
  - $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ and $\alpha <_{lex} \beta$.



$$
\begin{array}{l}
(0,0) \;<\\
(0,1) \;<\; (1,0) \;<\\
(0,2) \;<\; (1,1) \;<\; (2,0) \;<\\
(0,3) \;<\; (1,2) \;<\; (2,1) \;<\; (3,0) \;<\; \cdots
\end{array}
$$

Example

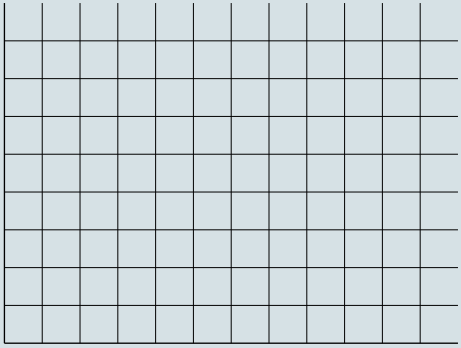
- $R = \mathbb{F}[X, Y]$
- $\Gamma = \mathbb{N}_0^2$
- $\mathbf{w} = (w_1, w_2) \in \mathbb{N}^2$
  - with total weighted degree lexicographic order $<_{wlex}$.
  - $(\alpha_1, \alpha_2) <_{wlex} (\beta_1, \beta_2)$ iff
  - $\alpha_1 w_1 + \alpha_2 w_2 < \beta_1 w_1 + \beta_2 w_2$, or
  - $\alpha_1 w_1 + \alpha_2 w_2 = \beta_1 w_1 + \beta_2 w_2$
  - and $\alpha <_{wlex} \beta$.



$$
\begin{array}{lllllllll}
(0,0) & < & & & & & & & \\
(0,1) & < & (1,0) & < & & & & & \\
(0,2) & < & (1,1) & < & (0,3) & < & (2,0) & < & \\
 & < & (1,2) & < & (0,4) & < & (2,1) & < & \\
 & < & (1,3) & < & (3,0) & & & &
\end{array}
$$

Example

- $R = \mathbb{F}[X, Y]$
- $\Gamma = \mathbb{N}_0^2$
  - with order $<$.
  - $(\alpha_1, \alpha_2) < (\beta_1, \beta_2)$ iff
  - $\alpha_1\sqrt{2} + \beta_1 < \alpha_2\sqrt{2} + \beta_2$.



$$
\begin{array}{llllll}
(0,0) & < & & & & \\
(0,1) & < & (1,0) & < & & \\
(0,2) & < & (1,1) & < & (2,0) & < \\
(0,3) & < & (1,2) & < & (2,1) & < \\
(0,4) & < & (3,0) & < & (1,3) & < \quad (2,2) \quad < \cdots
\end{array}
$$

- The admissible well-orders on $\mathbb{N}_0^r$ are all classified (Robbiano).
- Let $\mathbf{a}.\mathbf{b}$ be the standard innerproduct.
- – There exist $\mathbf{a}_1, \ldots, \mathbf{a}_s$ in $\mathbb{R}_+^r$ such that
  – for $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^r$: $\mathbf{a} < \mathbf{b}$
  – if and only if there exists a t such that
  – $\mathbf{a}.\mathbf{a}_i = \mathbf{b}.\mathbf{a}_i$ for all $i < t$ and $\mathbf{a}.\mathbf{a}_t < \mathbf{b}.\mathbf{a}_t$.
- The smallest s is called the **height** of the order.

- The order $<$ is isomorphic to
  - $(\mathbb{N}_0, <)$ iff the coordinates of $\mathbf{a}_1$ are all positive.
  - $<_{lex}$ iff $s = r$ and $\mathbf{a}_1 = \mathbf{e}_1, \ldots, \mathbf{a}_r = \mathbf{e}_r$ the standard basis.
  - $<_{dlex}$ iff $s = r$ and $\mathbf{a}_1 = (1, 1, \ldots, 1)$ and $\mathbf{a}_2 = \mathbf{e}_2, \ldots, \mathbf{a}_r = \mathbf{e}_r$.
  - $<_{wlex}$ iff $s = r$ and $\mathbf{a}_1 = (w_1, w_2, \ldots, w_r)$ and $\mathbf{a}_2 = \mathbf{e}_2, \ldots, \mathbf{a}_r = \mathbf{e}_r$.

- **Propostion**:
  If there exists an order function on R,
  then R is an integral domain
- (R, ρ, Γ) is called an **order structure** over $\mathbb{F}$ if
  - $\mathbb{F}$ is a field
  - R is an $\mathbb{F}$-algebra
  - Γ a well-order
  - $\rho : R \to \Gamma_{-\infty}$ an order function
- Then R is called an **order domain**.
- **Question**:
  Which rings are order domains and how do we construct them ?

Well-behaving bases

- Let R be an $\mathbb{F}$-algebra.
- Let Γ be a well-order.
- Let $(f_\alpha \mid \alpha \in \Gamma)$ be a basis of R over $\mathbb{F}$.
- R_γ be the subspace of R generated by $(f_\alpha \mid \alpha \leq \gamma)$.
- Define
$$l(\alpha, \beta) = \min\{\, \gamma \in \Gamma \mid f_\alpha f_\beta \in R_\gamma \,\}.$$
- The basis is called **well-behaving**, if
$$\alpha < \beta \;\Rightarrow\; l(\alpha, \gamma) < l(\beta, \gamma)$$
for all $\alpha, \beta, \gamma \in \Gamma$.

Proposition:

- Let (R, ρ, Γ) be an order structure.
- Let $(f_\alpha \mid \alpha \in \Gamma)$ be a sequence of elements in R such that $\rho(f_\alpha) = \alpha$ for all $\alpha \in \Gamma$.
- Then this is a well-behaving basis.

Proposition:

- Let $(\Gamma, <)$ be a well-order.
- Let $(f_\alpha \mid \alpha \in \Gamma)$ be a well-behaving basis.
- Define $\rho(f) = -\infty$ if $f = 0$.
- Otherwise $\rho(f) = \gamma$, where γ is the minimal element of Γ such that $f \in R_\gamma$.
- Then (R, ρ, Γ) is an order structure.

Proposition:

- Let $\mathbb{F}$ be a subfield of $\mathbb{G}$.
- Let (R, ρ, Γ) be an order structure over $\mathbb{F}$.
- Then $(R \otimes_{\mathbb{F}} \mathbb{G}, \rho, \Gamma)$ is an order structure over $\mathbb{G}$.

The value semigroup

- Let (R, ρ, Γ) be an order structure.
- The function $l(\alpha, \beta)$ does not depend on the chosen well-behaving basis, since
  - $R_\gamma = \{\ f \in R \mid \rho(f) \leq \gamma\ \}$.
  - $l(\alpha, \beta) = \min\{\ \gamma \mid R_\alpha R_\beta \subseteq R_\gamma\ \}$.
- Define the operation $+$ on Γ by

$$\alpha + \beta = l(\alpha, \beta).$$

- The minimum element of Γ is denoted by

$$0$$

.

- Let (R, ρ, Γ) be an order structure.
- Then $(\Gamma, +, 0, <)$ is a well-ordered semigroup.
- That means that the operation $+$ is
  - associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$,
  - commutative: $\alpha + \beta = \beta + \alpha$,
  - cancellative: $\alpha + \gamma = \beta + \gamma \;\Rightarrow\; \alpha = \beta$,
  - 0 is the neutral element: $\alpha + 0 = \alpha$;
- and the well-order $<$ is admissible:

$$0 \leq \alpha \text{ and } \alpha < \beta \;\Rightarrow\; \alpha + \gamma < \beta + \gamma.$$

- Furthermore $+$ is inverse free:
  $\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$.

- Let (R, ρ, Γ) be an order structure.
- From now on the well-order Γ is a well-ordered semigroup.
- Thus $\rho(fg) = \rho(f) + \rho(g)$ for all $f, g \in R$.

Example

- Let $(\Gamma, +, 0, <)$ be a well-ordered semigroup.
- The **semigroup algebra** $\mathbb{F}[\Gamma]$ has as basis $(X^\alpha \mid \alpha \in \Gamma)$.
- The multiplication is defined by

$$X^\alpha X^\beta = X^{\alpha+\beta}$$

  and extended linearly.
- Then the basis $(X^\alpha \mid \alpha \in \Gamma)$ is well-behaving, and
- $\mathbb{F}[\Gamma]$ is an order domain.

- Let (R, ρ, Γ) be an order structure.
- If Γ is a finitely generated semigroup,
  then R is a finitely generated algebra over $\mathbb{F}$:

$$R \cong \mathbb{F}[X_1, \ldots, X_m]/I.$$

- The converse is not true.
- The order structure is called
  finitely generated or **Noetherian**
  if $(\Gamma, +)$ is a finitely generated semigroup.

- Let $(\Gamma, +, 0)$ be a semigroup.
- Let $D(\Gamma)$ be the **group of differences** of Γ.
- The **rank** of Γ is defined by

$$\dim_{\mathbb{Q}} D(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

- If Γ is finitely generated semigroup,
  then $D(\Gamma)$ is a finitely generated abelian group.
- Hence

$$D(\Gamma) \cong \mathbb{Z}^r \oplus T,$$

  - where r is the rank of Γ,
  - and T a (finite) torsion group.

Proposition:

- Let $(\Gamma, +, 0)$ be an inverse-free semigroup.
- Let Γ be finitely generated and of rank r.
- Then there exists an embedding

$$\varphi : \Gamma \to \mathbb{N}_0^r$$

of semigroups such that

$$D(\varphi(\Gamma)) = \mathbb{Z}^r.$$

- Furthermore r is the smallest integer such that there exist an embedding of Γ in $\mathbb{N}_0^r$.

- Let (R, ρ, Γ) be a finitely generated order structure of rank r.
- Then Γ is a finitely generated inverse-free semigroup.
- Hence Γ can be viewed as a sub semigroup of $\mathbb{N}_0^r$ such that

$$D(\Gamma) = D(\mathbb{N}_0^r).$$

- The well-order has an extension to an admissible well-order on $\mathbb{N}_0^r$.
- Hence $\rho : R \to \mathbb{N}_0^r \cup \{-\infty\}$ is a **weight function** of rank r:
  - it satisfies the conditions of an order function,
  - and $\rho(fg) = \rho(f) + \rho(g)$ for all $f, g \in R$.

- Let R and S be affine $\mathbb{F}$-algebras.
- Then $R = \mathbb{F}[X_1, \ldots, X_m]/I$ and $S = \mathbb{F}[Y_1, \ldots, Y_n]/J$ are the coordinate rings of the varieties $\mathcal{X}$ and $\mathcal{Y}$.
- The tensor product $R \otimes_{\mathbb{F}} S$ is given by

$$R \otimes_{\mathbb{F}} S = \mathbb{F}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(I + J).$$

- It is the coordinate ring of $\mathcal{X} \times \mathcal{Y}$.
- – Let R be an order domains with value semigroup Γ.
  – Let S be an order domains with value semigroup Λ.
  – Then $R \otimes_{\mathbb{F}} S$ is an order domain with $\Gamma \oplus \Lambda$ as value semigroup.

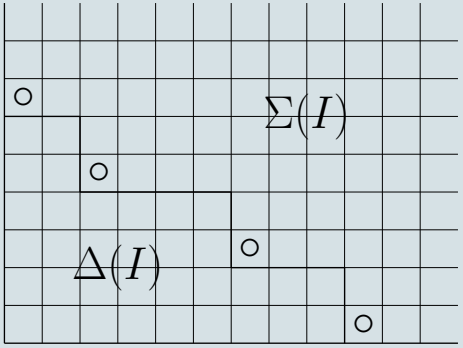
- Let $(\Gamma, +, 0, <)$ be a well-ordered semigroup.
- – The partial order $\leq_p$ is defined by
  – $\alpha \leq_p \beta$ if and only if
  – $\beta = \alpha + \gamma$ for some $\gamma \in \Gamma$.
- – Let Σ be a subset of Γ.
  – Define

$$\min \Sigma = \{\ \alpha \in \Sigma \mid \beta \in \Sigma, \beta \leq_p \alpha \ \Rightarrow\ \beta = \alpha\ \},$$

  the set of **minimal elements** of Σ.
- For every $\beta \in \Sigma$ there exists an $\alpha \in \min \Sigma$ such that $\alpha \leq_p \beta$.

- Let (R, ρ, Γ) be an order structure.
- Let I be an ideal in R.
- – Define $\Sigma(I) = \{\, \rho(f) \mid f \in I,\ f \neq 0 \,\}$.
  – and $\sigma(I) = \min \Sigma(I)$.
  – If Γ is finitely generated, then $\sigma(I)$ is finite.
- Define the **footprint** or **delta set** of I by

$$\Delta(I) = \Gamma \setminus \Sigma(I).$$



The circles ○ denote the positions of the elements of $\sigma(I)$.

- Let $\mathcal{G}$ be a subset of I.
- – $\mathcal{G}$ is called a **Gröbner basis** of I
  – if $\sigma(I) \subseteq \{ \rho(g) \mid g \in \mathcal{G} \}$.
- – $\mathcal{G}$ is called a **minimal Gröbner basis** of I
  – if for every $\gamma \in \sigma(I)$ there is exactly one $g \in \mathcal{G}$ such that $\gamma = \rho(g)$.

Proposition:

- Let I be an ideal in the order domain R.
- Let $\mathcal{G}$ be a Gröbner basis of I.
- Then $\Sigma(I)$ is an ideal in Γ.
- Let $(f_\alpha \mid \alpha \in \Gamma)$ be a well-behaving basis.
- $\mathcal{G}$ generates I in R.
- Every $f \in R$ is equivalent modulo I to

$$\sum_{\alpha \in \Delta(I)} a_\alpha f_\alpha$$

- This expression is unique.
- $\dim_{\mathbb{F}}(R/I) = \#\Delta(I)$

Factor ring theorem:

- Let $w : \Gamma \to \Lambda$ be a surjective morphism of well-ordered semigroups.
- Let (R, ρ, Γ) be an order structure with $(f_\alpha \mid \alpha \in \Gamma)$ as well-behaving basis.
- I is an ideal in R with Gröbner bases $\mathcal{G}$.
- Suppose that $w : \Delta(I) \to \Lambda$ is injective.
- Suppose that every element g of $\mathcal{G}$:

$$g = af_\alpha + bf_\beta + \sum_{w(\gamma)<\delta} c_\gamma f_\gamma,$$

  where $\alpha, \beta \in \mathbb{F}$, $\alpha \neq \beta$ and $\delta := w(\alpha) = w(\beta)$.
- Then $(R/I, \sigma, \Lambda)$ is an order structure.

- Let $w : \mathbb{N}_0^m \to \mathbb{N}_0^r$ be a morphism of semigroups.
- Let $<$ be an admissible order on $\mathbb{N}_0^r$.
- Let $\prec$ be an admissible order on $\mathbb{N}_0^m$.
- Define
  - $\prec_w$ by $\alpha \prec_w \beta$ iff
  - $w(\alpha) < w(\beta)$, or $w(\alpha) = w(\beta)$ and $\alpha \prec \beta$.
- Then
  - $\prec_w$ is an admissible well-order on $\mathbb{N}_0^m$
  - and w is a morphism of ordered semigroups.

- The map w induces a map on the monomials by $w(X^\alpha) = w(\alpha)$
- and on the nonzero polynomials, by

$$w\left(\sum a_\alpha X^\alpha\right) = \max\{w(\alpha) \mid a_\alpha \neq 0\},$$

where the maximum is taken with respect to $<$.

Example

$$w\left(X^{\alpha}Y^{\beta}\right) = w(\alpha,\beta) = 4\alpha + 5\beta.$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$(0,6)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$
$(0,5)$	$(1,5)$	$\cdot$	$\cdot$	$\cdot$	$\cdots$
$(0,4)$	$(1,4)$	$(2,4)$	$\cdot$	$\cdot$	$\cdots$
$(0,3)$	$(1,3)$	$(2,3)$	$(3,3)$	$\cdot$	$\cdots$
$(0,2)$	$(1,2)$	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2)$
$(0,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,0)$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
30	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$
25	29	$\cdot$	$\cdot$	$\cdot$	$\cdots$
20	24	28	$\cdot$	$\cdot$	$\cdots$
15	19	23	27	$\cdot$	$\cdots$
10	14	18	22	26	30
5	9	13	17	21	25
0	4	8	12	16	20

$$w\left(X^5\right) = w\left(Y^4\right).$$

Example

- $R = \mathbb{F}[X, Y]$ and $\Gamma = \mathbb{N}_0^2$ with $<_{lex}$.
- $\Lambda = \mathbb{N}_0\langle 4, 5\rangle$.
- – $w : \mathbb{N}_0^2 \to \Lambda$ with
  – $w(\alpha, \beta) = 4\alpha + 5\beta$.
- – Let $G = X^5 - Y^4 - Y$, $I = (X^5 - Y^4 - Y)$.
  – Then $\mathcal{G} = \{G\}$ is a Gröbner basis for I.
- – Then $S = \mathbb{F}[X, Y]/(X^5 - Y^4 - Y)$ is an order domain,
  – with Λ as value semigroup,
  – with $(x^\alpha y^\beta \mid \alpha < 5)$ as well-behaving basis.

Example

- $R = \mathbb{F}[X, Y, Z]$ and $\Gamma = \mathbb{N}_0^3$ with $<_{lex}$.
- $\Lambda = \mathbb{N}_0\langle(4,0),(5,0),(1,1)\rangle$.
- – $w : \mathbb{N}_0^3 \to \Lambda$ with
  – $w(\alpha, \beta, \gamma) = \alpha(4,0) + \beta(5,0) + \gamma(1,1)$.
- – Let $G = X^5 - Y^4 - Z$, $I = (X^5 - Y^4 - Z)$.
  – Then $\mathcal{G} = \{G\}$ is a Gröbner basis for I.
- – Then
$$S = \mathbb{F}[X, Y, Z]/(X^5 - Y^4 - Z) \cong \mathbb{F}[X, Y]$$
  is an order domain,
  – with Λ as value semigroup.

Example (O'Sullivan, Geil-Pellikaan)

- Let $R = \mathbb{F}[X_1, X_2, \ldots, X_n, \ldots]$, the polynomial ring with infinitely many variables.
- Take the lexicographic order on the monomials with

$$1 < X_1 < X_2 < \cdots < X_n < \cdots$$

- Let $\Gamma = \mathbb{N}_0^{(\infty)}$ the set of all sequences of nonnegative integers such that at most finitely many of them are nonzero.
- So $R = \mathbb{F}[\mathbb{N}_0^{(\infty)}]$.

- Let $\lambda_i = \frac{1}{3}\left(2^i + 2^{-i+1}\right)$ for all $i \in \mathbb{N}$.
- Let $\Lambda = \mathbb{N}_0\langle \lambda_i \mid i \in \mathbb{N}\rangle$.
- Then: $2\lambda_{i+1} = 3\lambda_i + \sum_{j=0}^{i-1} \lambda_j$.
- Define $w : \mathbb{N}_0^{(\infty)} \to \Lambda$ by $w(\alpha) = \sum_i \alpha_i \lambda_i$.
- Let $G_i = X_{i+1}^2 - X_i^3 \prod_{j \leq i-1} X_j + X_{i+2}$.
- Then

$$w(X_{i+1}^2) = 2\lambda_{i+1} = 3\lambda_i + \sum_{j=0}^{i-1} \lambda_j =$$

$$w\left(X_i^3 \prod_{j \leq i-1} X_j\right) > w(X_{i+2}).$$

- Let the $G_i = X_{i+1}^2 - X_i^3 \prod_{j \leq i-1} X_j + X_{i+2}$ generate the ideal I.
- Then $\mathcal{G} = \{G_i \mid i \in \mathbb{N}\}$ is a Gröbner basis for I.
- and
$$S = \mathbb{F}[X_1, X_2, \ldots, X_n, \ldots]/I \cong \mathbb{F}[X_1, X_2]$$
is an order domain of dimension 2,
- with Λ as value semigroup.
- $\Lambda \subset \mathbb{Q}$, so Λ has rank 1.

- Let (R, ρ, Γ) be a finitely generated order structure.
- May assume that Γ is embedded in $\mathbb{N}_0^r$.
- Let $\mathcal{A} = \{\gamma_1, \ldots, \gamma_m\}$ be a set of generators of Γ.
- Choose $x_1, \ldots, x_m \in R$ such that $\rho(x_i) = \gamma_i$.
- Then $R = \mathbb{F}[x_1, \ldots, x_m]$.
- Identify R with $\mathbb{F}[X_1, \ldots, X_m]/I$.
- Define the map $w : \mathbb{N}_0^m \to \mathbb{N}_0^r$ by

$$w(\alpha) = \sum \alpha_i \gamma_i.$$

The presentation theorem:

- $\Delta(\mathcal{A}) := \{\alpha \in \mathbb{N}_0^m | w(\alpha) \leq w(\beta) \Rightarrow \alpha \preceq_w \beta\}$.
- Then $\Delta(I) = \Delta(\mathcal{A})$.
- Let $\sigma(\mathcal{A}) = \min\ \mathbb{N}_0^m \setminus \Delta(\mathcal{A})$.
- For each $\alpha \in \sigma(\mathcal{A})$ there is a unique $\alpha' \in \Delta(\mathcal{A})$ such that $w(\alpha') = w(\alpha)$ and we can write x^α in a unique way as

$$x^{\alpha'} + \sum_{\gamma \in \Delta(\mathcal{A}),\ w(\gamma) < w(\alpha)} c_{\alpha\gamma} x^\gamma.$$

- $G_\alpha := X^\alpha - X^{\alpha'} - \sum_{\gamma \in \Delta(\mathcal{A}),\ w(\gamma) < w(\alpha)} c_{\alpha\gamma} X^\gamma$.
- Then $\{G_\alpha \mid \alpha \in \sigma(\mathcal{A})\}$ is a Gröbner basis of I.

- Let R be an affine domain over a field $\mathbb{F}$.
- Let $\mathbb{Q}(R)$ be the field of fractions of R.
- The dimension of R is the transcendence degree of $\mathbb{Q}(R)$ over $\mathbb{F}$.
- The dimension of an order domain is at least the rank of the value semigroup.
- Not always equal.
- **Theorem**:
  - Let (R, ρ, Γ) be a finitely generated order structure.
  - Then the dimension of R is equal to the rank of the semigroup.

Proof:

- Assume for simplicity $(\Gamma, <) \cong (\mathbb{N}_0, <)$.
- Let r be the rank of $(\Gamma, +)$.
- Then $(\Gamma, +)$ is a sub semigroup of $(\mathbb{N}_0^r, +)$.
- The well-order $<$ on Γ is induced by a well-order on $\mathbb{N}_0^r$.
- – These are all classified.
  – We can find $\mathbf{a}_1, \ldots, \mathbf{a}_s$ in $\mathbb{R}_+^r$
  – such that for $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^r$: $\mathbf{a} < \mathbf{b}$ iff
  – there exists a t such that $\mathbf{a}.\mathbf{a}_t < \mathbf{b}.\mathbf{a}_t$
  – and $\mathbf{a}.\mathbf{a}_i = \mathbf{b}.\mathbf{a}_i$ for all $i < t$.

- Now the coordinates of $\mathbf{a}_1$ are all positive, since $(\Gamma, <) \cong (\mathbb{N}_0, <)$.
- Define
$$R(n) = \{f \in R \mid \rho(f).\mathbf{a}_1 \leq n\}.$$
- Then $R(n)$ is a vector space over $\mathbb{F}$.
- Its dimension is at most equal to the number of integral points $\mathbf{b} \in \mathbb{N}_0^r$ such that $\mathbf{b}.\mathbf{a}_1 \leq n$.
- Suppose that $\dim R = s$.

- Then there exists a transcendence basis $x_1, \ldots, x_s$ of R.
- $b := \max\{\, \rho(x_i).\mathbf{a}_1 \mid \text{ for all } i = 1, \ldots, s \,\}$.
- Then $R(b)$ contains $x_1, \ldots, x_s$ and
- $R(bn)$ contains all monomials in $x_1, \ldots, x_s$ of degree at most n.
- These are independent and the number of these is

$$p(n) := \binom{n+s}{s} = \frac{n^s}{s!} + \text{ L.O.T. in } n$$

- This is a polynomial in n of degree s.
- $\deg \dim R(bn) \geq \deg p(n) = s$.

- Let c be the minimum of all the coordinates of $\mathbf{a}_1$.
- Then $c > 0$.
- $\{\mathbf{b} \in \mathbb{N}_0^r | \mathbf{b}.\mathbf{a}_1 \leq bn\} \subseteq \{\mathbf{b} \in \mathbb{N}_0^r | b_i \leq bn/c \text{ for all } i\}$.
- $\dim_{\mathbb{F}} R(bn) \leq (1 + bn/c)^r$.
- Therefore $s \leq r$.
- Always $s \geq r$.