Truncation formulas for invariant polynomials of matroids and geometric lattices

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CanaDAM, May 31 – June 3, 2011
Outline

Geometric lattices

Truncation

Truncation formulas

Application to codes
  Representation of truncated matroid
  Extended weight enumerator

Further questions
A geometric lattice $L$ is a set with partial ordering $\leq$ and some additional specifying properties.

A matroid $M$ with ground set $E$ gives rise to a geometric lattice $L(M)$, called the lattice of flats:

- **elements**: all flats of $M$
- **ordering**: $x \leq y$ if $x \subseteq y$
- **minimum**: empty set $\emptyset$
- **maximum**: whole ground set $E$
- **rank**: rank of the flat $x$ in $M$
- **atoms**: all flats of rank 1

If the matroid is *simple*, $L(M)$ is equivalent to $M$. 
Geometric lattices

Example

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & \alpha
\end{pmatrix}
\]
The Möbius function of a geometric lattice is defined for all $x \leq y$ by $\mu_L(x, x) = 1$ and

$$
\sum_{x \leq z \leq y} \mu_L(x, z) = \sum_{x \leq z \leq y} \mu_L(z, y) = 0.
$$

If $x$ and $y$ are not comparable, then $\mu_L(x, y) = 0$.

Note the function is alternating in the rank of the geometric lattice.
Truncation

Idea: “cutting off elements of highest rank”

**Truncated matroid** $T(M)$

Several equivalent descriptions:

- **independent sets**: all the independent sets of $M$ of rank $< r$
- **bases**: independent sets of rank $r - 1$ in $M$
- **rank function**: $r_{T(M)}(A) = \min\{r_M(A), r - 1\}$
Truncation

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**Truncated lattice** $T(L)$

Remove all elements of rank $r - 1$ and preserve partial ordering.
Truncation

Example

Diagram showing the concept of truncation with the set $abcdefg$ and its truncated subsets.
Various invariant polynomials are associated with matroids and geometric lattices:

- rank generating polynomial
- coboundary polynomial
- Möbius polynomial
- (spectrum polynomial)

Question: do these polynomials determine the polynomials associated with $T(M)$?

Answer: yes!
The rank generating function of a matroid is defined by

\[ R_M(X, Y) = \sum_{A \subseteq E} X^{r(E) - r(A)} Y^{\lvert A \rvert - r(A)}. \]
Truncation formulas

Rank generating function

The rank generating function of a matroid is defined by

$$R_M(X,Y) = \sum_{A \subseteq E} X^{r(E) - r(A)} Y^{|A| - r(A)}.$$ 

Theorem (Britz, 2007)

Let $M$ be a matroid. Then

$$X \cdot R_{T(M)}(X,Y) = R_M(X,Y) + (XY - 1) \cdot R_M(0,Y).$$
Coboundary polynomial

The coboundary polynomial of a geometric lattice is defined by

\[ \chi_L(S, T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu_L(x, y) \ S^{a_L(x)} \ T^{r(L) - r(y)} \]

where \( a_L(x) \) is the number of atoms \( a \) in \( L \) such that \( a \leq x \).
Truncation formulas

Coboundary polynomial

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where \( a_L(x) \) is the number of atoms \( a \) in \( L \) such that \( a \leq x \).

Theorem (Crapo, 1968)

The coboundary polynomial of a lattice of flats is determined by the rank generating function via

\[ \chi_{L(M)}(S, T) = (S - 1)^{r(M)} \cdot R_M \left( \frac{T}{S - 1}, S - 1 \right) . \]
Theorem

Let $L$ be a geometric lattice of rank $r \geq 3$. Then

$$T \cdot \chi_{T(L)}(S, T) = \chi_L(S, T) + (T - 1) \cdot \chi_L(S, 0).$$

Proof:

- use relation with rank generating function; or:
- use induction formula for Möbius function to write both sides in terms of elements with low rank.
The Möbius polynomial of a geometric lattice is defined by

\[
\mu_L(S, T) = \sum_{x \in L} \sum_{x \leq y} \mu_L(x, y) S^{r_L(x)} T^{r(L) - r_L(y)}.
\]

The Möbius polynomial is not equivalent to the rank generating function and coboundary polynomial.
**Truncation formulas**

**Möbius polynomial**

The Möbius polynomial of a geometric lattice is defined by

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The Möbius polynomial is not equivalent to the rank generating function and coboundary polynomial.

**Theorem**

*Let* \( L \) *be a geometric lattice of rank* \( r > 0 \). *Then*

\[ T \cdot \mu_{T(L)}(S, T) = \mu_L(S, T) + (T - 1) \cdot \mu_L(S, 0) + S^{r - 1} T - S^r T. \]
Theorem (Brylawski, 1986)

Let $M$ be a representable matroid. Then $T(M)$ is representable over a transcendental extension field.
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Let $M$ be a representable matroid. Then $T(M)$ is representable over a transcendental extension field.

Theorem

• If $M$ is representable over an infinite field, then $T(M)$ is representable over the same field.

• If $M$ is representable over a finite field $\mathbb{F}_q$, then $T(M)$ is representable over $\mathbb{F}_{q^m}$ with $m \geq \lceil \log_q \binom{n}{r(M) - 1} \rceil + 1$.

So there are linear codes associated to truncated matroids.
Extended weight enumerator

Extension code \([n, k]\) code over some extension field \(\mathbb{F}_{q^m}\) generated by the words of \(C\), notation: \(C \otimes \mathbb{F}_{q^m}\).

Generator matrix All the extension codes of \(C\) have the same generator matrix \(G\).
Extended weight enumerator

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Extended weight enumerator

The homogeneous polynomial counting the number of words of a given weight “for all extension codes”, notation:

\[
W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.
\]

Note that with \(T = q^m\) we have \(W_C(X, Y, q^m) = W_{C \otimes \mathbb{F}_{q^m}}(X, Y)\).
Extended weight enumerator

- The rank generating function completely determines the extended weight enumerator, and vice versa.
- If $M$ is representable over multiple fields, all corresponding codes have same extended weight enumerator $W_M(X, Y, T)$. 

Theorem

Let $M$ be a matroid. Then for all codes determined by $M$ we have $T \cdot W_T(M)(X, Y, T) = W_M(X, Y, T) + (T - 1) \cdot W_M(X, Y, 0)$. 

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If $M$ is representable over multiple fields, all corresponding codes have same extended weight enumerator $W_M(X, Y, T)$.

**Theorem**

Let $M$ be a matroid. Then for all codes determined by $M$ we have

$$T \cdot W_{T(M)}(X, Y, T) = W_M(X, Y, T) + (T - 1) \cdot W_M(X, Y, 0).$$
Extended weight enumerator

\[ W_M(X, Y, T) = X^7 + 2(T - 1)X^4Y^3 + 3(T - 1)X^3Y^4 + T(T - 1)X^2Y^5 + (T - 1)(T - 2)(T - 3)Y^7 \]

\[ M \text{ represented by } \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \text{ over } \mathbb{F}_q, \ q > 2 \]
Extended weight enumerator

Example

\( M \) represented by
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & \alpha
\end{pmatrix}
\]
over \( \mathbb{F}_q, \ q > 2 \)

\[
W_{T(M)}(X, Y, T) = X^7 + (T-1)X^2Y^5 + 5(T-1)XY^6 + (T-1)(T-5)Y^7
\]
Extended weight enumerator

Example

$M$ represented by

$$\begin{bmatrix}
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over $\mathbb{F}_q$, $q > 2$

$$W_{T(M)}(X, Y, T) = X^7 + (T - 1)X^2Y^5 + 5(T - 1)XY^6 + (T - 1)(T - 5)Y^7$$

$T(M)$ represented by

$$\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 1& 1 \\
0 & 1 & 1 & 0 & 2 & 3 & 4
\end{bmatrix}$$

over $\mathbb{F}_5$
Further questions

- Better bounds for representation of $T(M)$ via codes?
- Formulas for principal truncation, Dilworth truncation
- Connections to Duursma zeta functions
- Möbius polynomial: unimodal conjecture on the Whitney numbers
- Spectrum polynomial: does it determine rank generating function?
Thank you for your attention.