Arrangements, matroids and codes

third lecture

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Content third lecture

1. Extended weight enumerator
2. Formulas for puncturing and shortening
3. Graph theory and colorings
4. Exercises

Extended weight enumerator

Extension code

Let G be the generator matrix of a linear $[n, k]$ code C over $\mathbb{F}_q$

$\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^m}$

Consider the code $C \otimes \mathbb{F}_{q^m}$ over $\mathbb{F}_{q^m}$
by taking all $\mathbb{F}_{q^m}$-linear combinations of the codewords in C
This is called the extension code of C over $\mathbb{F}_{q^m}$

G is also a generator matrix for the extension code $C \otimes \mathbb{F}_{q^m}$
Hence $C \otimes \mathbb{F}_{q^m}$ has dimension k over $\mathbb{F}_{q^m}$

A formalism for the weight enumerator

Remember:

Definition

For a subset J of $[n] := \{1, 2, \dots, n\}$ define

$$\begin{aligned} C(J) &= \{\mathbf{c} \in C : c_j = 0 \text{ for all } j \in J\} \\ l(J) &= \dim C(J) \end{aligned}$$

Lemma

Let C be a linear code with generator matrix G
Let $J \subseteq [n]$ and $|J| = t$
G_J is the $k \times t$ submatrix of G existing of the columns of G indexed by J
Let $r(J)$ be the rank of G_J
Then $l(J) = k - r(J)$

Dimension of C(J) under extension

$l(J) = k - r(J)$ by a previous lemma
$r(J)$ is independent of the extension field $\mathbb{F}_{q^m}$

Therefore

$$\dim_{\mathbb{F}_q} C(J) = \dim_{\mathbb{F}_{q^m}} (C \otimes \mathbb{F}_{q^m})(J)$$

This motivates the usage of T as a variable for q^m in the next definition

Extension of definition B_t

Remember:
Let C be a linear code over $\mathbb{F}_q$

$$\begin{aligned} B_J &= q^{l(J)} - 1 \\ B_t &= \sum_{|J|=t} B_J \end{aligned}$$

Extend: Definition

$$\begin{aligned} B_J(T) &= T^{l(J)} - 1 \\ B_t(T) &= \sum_{|J|=t} B_J(T) \end{aligned}$$

Note that $B_J(q^m)$ is the number of nonzero codewords in $(C \otimes \mathbb{F}_{q^m})(J)$

Proposition on $B_t(T)$

Proposition

Let C be an $\mathbb{F}_q$-linear code of dimension k
Let d and $d^\perp$ be the minimum distance of C and $C^\perp$, respectively
Let $J \subseteq [n]$ and $|J| = t$
Then

$$B_t(T) = \begin{cases} \binom{n}{t}(T^{k-t} - 1) & \text{for all } t < d^\perp \\ 0 & \text{for all } t > n - d \end{cases}$$

Proof

Follows directly from the lemma on $l(J)$

Extension of definition $W_C(X, Y)$

Remember:

$$W_C(X, Y) = X^n + \sum_{t=0}^{n} B_t (X - Y)^t Y^{n-t}$$

Define the extended weight enumerator by

$$W_C(X, Y, T) = X^n + \sum_{t=0}^{n} B_t(T)(X - Y)^t Y^{n-t}$$

Relating $A_w(T)$ and $B_t(T)$'s

Theorem

The following holds:

$$W_C(X, Y, T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w$$

$$A_0(T) = 1, \text{ and } A_w(T) = \sum_{t=n-w}^{n} (-1)^{n+w+t} \binom{t}{n-w} B_t(T)$$

for $0 < w \leq n$ and

$$B_t(T) = \sum_{w=d}^{n-t} \binom{n-w}{t} A_w(T)$$

Proof is similar to the proof relating the A_w's and B_t's

$A_w(T)$ of MDS codes

Proposition

The weight distribution of an MDS code of length n and dimension k is given by

$$A_w(T) = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left(T^{w-d+1-j} - 1 \right)$$

for $w \geq d = n - k + 1$

Proof

Similar to the proof for A_w

Weight enumerator of extension code

Proposition

Let C be a linear $[n, k]$ code over $\mathbb{F}_q$

Then

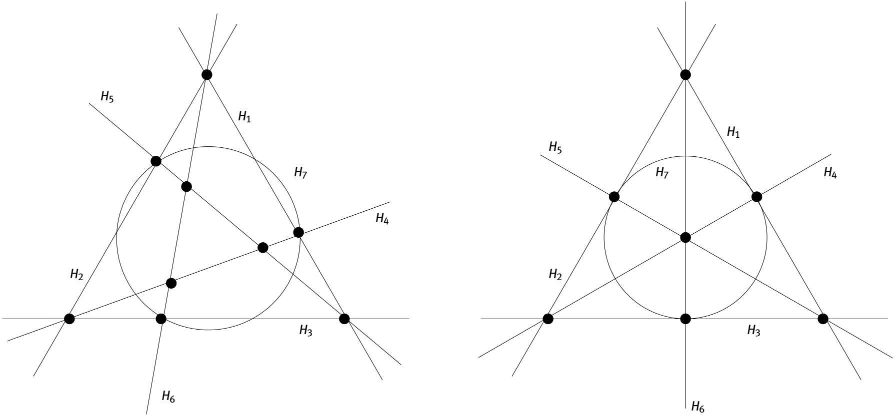
$$W_C(X, Y, q^m) = W_{C \otimes \mathbb{F}_{q^m}}(X, Y)$$

The number of codewords in $C \otimes \mathbb{F}_{q^m}$ of weight w is equal to $A_w(q^m)$

Proof

Substituting $T = q^m$ in $B_t(T)$ gives $B_t(q^m)$ which is equal to the B_t of $C \otimes \mathbb{F}_{q^m}$

Example



$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Example, q even

Let C be the code with generator matrix G
We have seen for q even that

$$W_C(X,Y) = X^7 + 7(q-1)X^3Y^4 + 7(q-1)(q-2)XY^6 + (q-1)(q-2)(q-4)Y^7$$

So

$$W_{C\otimes\mathbb{F}_{q^m}}(X,Y) =$$

$$X^7 + 7(q^m-1)X^3Y^4 + 7(q^m-1)(q^m-2)XY^6 + (q^m-1)(q^m-2)(q^m-4)Y^7$$

Therefore

$$W_C(X,Y,T) =$$

$$X^7 + 7(T-1)X^3Y^4 + 7(T-1)(T-2)XY^6 + (T-1)(T-2)(T-4)Y^7$$

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Example, q odd

For q odd we have seen that

$$W_C(X,Y) = X^7 + 6(q-1)X^3Y^4 +$$
$$3(q-1)X^2Y^5 + (q-1)(7q-17)XY^6 + (q-1)(q-3)^2Y^7$$

So

$$W_{C\otimes\mathbb{F}_{q^m}}(X,Y) = X^7 + 6(q^m-1)X^3Y^4 +$$
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MacWilliams identity of ext. wt. enum.

Theorem

Let C be an $[n, k]$ code over $\mathbb{F}_q$

Then

$$W_{C^\perp}(X, Y, T) = T^{-k} W_C(X + (q-1)Y, X - Y, T)$$

Proof

Substituting $T = q^m$ gives the MacWilliams identity for $C \otimes \mathbb{F}_{q^m}$

$$W_{C^\perp}(X, Y, q^m) = q^{-mk} W_C(X + (q-1)Y, X - Y, q^m)$$

which holds for all m

Now $A_w(T)$ is a polynomial in T with coefficient in $\mathbb{Z}$

Giving infinitely many identities for the weight distributions of $C \otimes \mathbb{F}_{q^m}$ and $C^\perp \otimes \mathbb{F}_{q^m} = (C \otimes \mathbb{F}_{q^m})^\perp$

Another formula for ext. wt. enum.

Proposition

Let C be a linear $[n, k]$ code over $\mathbb{F}_q$
The following formula will be useful later in identifying the extended weight enumerator with the Tutte polynomial

$$W_C(X, Y, T) = \sum_{t=0}^{n} \sum_{|J|=t} T^{l(J)} (X - Y)^t Y^{n-t}$$

Proof

Use the description of $W_C(X, Y, T)$ in terms of the $B_t(T)$ and the definition of $B_t(T)$ in terms of the $l(J)$

Puncturing and shortening

Puncturing and shortening

There are several ways to get new codes from existing ones

Puncturing and shortening codes give
an alternative algorithm for finding the extended weight enumerator

This algorithm is based on the Tutte-Grothendieck decomposition
of matrices introduced by Brylawski

Greene used this for the determination of the weight enumerator

Puncturing and shortening

Let C be a linear $[n, k]$ code and let $J \subseteq [n]$

Then the code C punctured by J is obtained
by deleting all the coordinates indexed by J from the codewords of C
$G_{[n]\setminus J}$ is a generator matrix of this punctured code

The length of this punctured code is $n - |J|$
and its dimension is at most k

If we puncture the code $C(J)$ by J, we get the code C shortened by J
The length of this shortened code is $n - |J|$ and its dimension is $l(J)$

The operations of puncturing and shortening a code are each others dual
puncturing a code C by J and then taking the dual
gives the same code as shortening $C^{\perp}$ by J

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Generalizing definition $l(J)$

Let $\mathbb{F}$ be a field
Let G be a $k \times n$ matrix over $\mathbb{F}$
possibly of rank smaller than k and/or with zero columns
Define for $J \subseteq [n]$:

$$l(J) = l(J, G) = k - r(G_J).$$

Define the extended weight enumerator of G by

$$W_G(X, Y, T) = \sum_{t=0}^{n} \sum_{|J|=t} T^{l(J,G)} (X - Y)^t Y^{n-t}$$

This coincides with $W_C(X, Y, T)$ if G is a generator matrix of C

Proposition

(i) $W_G(X, Y, T)$ is invariant under row-equivalence of matrices

(ii) Let G' be a $l \times n$ matrix with the same row-space as G then $W_G(X, Y, T) = T^{k-l} W_{G'}(X, Y, T)$

(iii) $W_G(X, Y, T)$ is invariant under permutation of the columns of G

(iv) $W_G(X, Y, T)$ is invariant under multiplying a column of G with an element of $\mathbb{F}^*$

(v) If G is the direct sum of G_1 and G_2, that is of the form

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

then $W_G(X, Y, T) = W_{G_1}(X, Y, T) \cdot W_{G_2}(X, Y, T)$

Reduced matrix

Let G be a $k \times n$ matrix with entries in $\mathbb{F}$
Suppose that the j-th column is not the zero vector
Then there exists a matrix G' row-equivalent to G
such that the j-th column is of the form $(1, 0, \ldots, 0)^T$
Such a matrix is called reduced at the j-th column

$$G' = \left(\begin{array}{c|ccc} 1 & g'_{12} & \cdots & g'_{1n} \\ 0 & g'_{22} & \cdots & g'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & g'_{k2} & \cdots & g'_{kn} \end{array} \right)$$

In general, this reduction is not unique

$G \setminus a$ and G/a

Let G be a matrix that is reduced at the j-th column a
$G \setminus a$ is the $k \times (n-1)$ matrix with column a removed from G
G/a is the $(k-1) \times (n-1)$ with first row removed from $G \setminus a$

$$G = \left(\begin{array}{c|c} 1 & \\ 0 & G \setminus a \\ \vdots & \\ 0 & \end{array} \right) \text{ and } G \setminus a = \left(\begin{array}{ccc} g_{12} & \cdots & g_{1n} \\ \hline & & \\ & G/a & \\ & & \end{array} \right)$$

View $G \setminus a$ as G punctured by a
and G/a as G shortened by a

Proposition

Let G be a $k \times n$ matrix that is reduced at the j-th column a
Then

$$W_G = (X - Y)W_{G/a} + YW_{G\backslash a}$$

The (X, Y, T) part in $W_G(X, Y, T)$ is omitted for clarity

Proof See notes

Graph theory and colorings

Definition

A graph Γ is a pair (V, E)
where V is a non-empty set and E is a set disjoint from V
The elements of V are vertices
and members of E are edges

Edges are incident to one or two vertices
the ends of the edge

If u and v are vertices that are incident with an edge
then they are called neighbors or adjacent

Suppose that $V' \subseteq V$ and $E' \subseteq E$ and
all the endpoints of e' in E' are in V'
Then $\Gamma' = (V', E')$ is called a subgraph of Γ

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A loop is an edge that is incident with exactly one vertex
Two edges are called parallel if they are incident with the same vertices

The graph is called simple if it has no loops and no parallel edges

Deleting loops and parallel edges from a graph gives a simple graph
There is a choice in the process of deleting parallel edges
but the resulting graphs are all isomorphic

This simple graph is called the simplification $\bar{\Gamma}$ of the graph Γ

Graph colorings

Let $\Gamma = (V, E)$ be a graph
Let K be a finite set and $k = |K|$
The elements of K are called colors

A k-coloring of Γ is a map $\gamma : V \to K$
such that $\gamma(u) \neq \gamma(v)$ for all distinct adjacent vertices u and v in V

So vertex u has color $\gamma(u)$ and
all other adjacent vertices have a color distinct from $\gamma(u)$

Let $P_\Gamma(k)$ be the number of k-colorings of Γ
Then P_Γ is called the chromatic polynomial of Γ

Example: disconnected graph

If the graph Γ has no edges and v vertices
then

$$P_\Gamma(k) = k^v$$

since it is equal to the number of all maps from V to K

In particular there is no map from V to an empty set
in case V is nonempty

So the number of 0-colorings is zero for every graph.

Example: complete graph

Let K_n be the complete graph on n vertices in which
every pair of two distinct vertices is connected by exactly one edge

Then there is no k coloring if $k < n$
Now let $k \geq n$
Take an enumeration of the vertices
Then there are k possible choices of a color of the first vertex
Now suppose by induction that we have a coloring of the first i vertices
then there are $k - i$ possibilities to color the next vertex
since the $(i + 1)$-th vertex is connected to the first i vertices
Hence

$$P_{K_n}(k) = k(k-1)\cdots(k-n+1)$$

So $P_{K_n}(k)$ is a polynomial in k of degree n

$P_\Gamma(k)$ is a polynomial

Proposition

Let $\Gamma = (V, E)$ be a graph

Then $P_\Gamma(k)$ is a polynomial in k

Proof

Let $\gamma : V \to K$ be a k-coloring of Γ with exactly i colors

Let σ be a permutation of K

Then $\sigma \circ \gamma$ is also a k-coloring of Γ with exactly i colors

Two such colorings are called equivalent

Then $k(k-1)\cdots(k-i+1)$ is the number of colorings

in the equivalence class of a given k-coloring of Γ with exactly i colors

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Proof: $P_\Gamma(k)$ is a polynomial

Let m_i be the number of equivalence classes of colorings with exactly i colors of the set K

Let $v = |V|$
Then $P_\Gamma(k)$ is equal to

$$m_1 k + \ldots + m_i k(k-1)\cdots(k-i+1) + \ldots + m_v k(k-1)\cdots(k-v+1)$$

(Complete) bipartite graph

A graph $\Gamma = (V, E)$ is called bipartite
if V is the disjoint union of two nonempty sets M and N
such that the ends of an edge are in M and in N

Hence no two points in M are adjacent and no two points in N are adjacent. Let m and n be integers such that $1 \leq m \leq n$

The complete bipartite graph $K_{m,n}$ is the graph
on a set of vertices V that is the disjoint union of two sets
M and N with $|M| = m$ and $|N| = n$ and such that
every vertex in M is connected with every vertex in N by a unique edge

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Deletion and contraction

Let $\Gamma = (V, E)$ be a graph
Let e be an edge that is incident to the vertices u and v

Then the deletion $\Gamma \setminus e$ is the graph
with vertices V and edges $E \setminus \{e\}$

The contraction Γ/e is the graph
obtained by identifying u and v and deleting e

Notice that the number of k-colorings of Γ does not change
by deleting loops and a parallel edge
Hence the chromatic polynomial of Γ and $\bar{\Gamma}$ are the same

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Proposition

Let $\Gamma = (V, E)$ be a simple graph
Let e be an edge of Γ
Then

$$P_\Gamma(k) = P_{\Gamma \setminus e}(k) - P_{\Gamma / e}(k)$$

for all positive integers k

Proof

Define Γ / e formally as follows
Let $\tilde{u} = \tilde{v} = \{u, v\}$, and $\tilde{w} = \{w\}$ if $w \neq u$ and $w \neq v$
Let $\tilde{V} = \{\tilde{w} : w \in V\}$
Then Γ / e is the graph $(\tilde{V}, E \setminus \{e\})$
where an edge $f \neq e$ is incident with $\tilde{w}$ in Γ / e
if f is incident with w in Γ

Proof of deletion-contraction formula

Let u and v be the vertices of e
Then $u \neq v$, since the graph is simple

Let γ be a k-coloring of $\Gamma \setminus e$
Then γ is also a coloring of Γ if and only if $\gamma(u) \neq \gamma(v)$

If $\gamma(u) = \gamma(v)$, then consider the induced map $\tilde{\gamma}$ on $\tilde{V}$
defined by $\tilde{\gamma}(\tilde{u}) = \gamma(u)$ and $\tilde{\gamma}(\tilde{w}) = \gamma(w)$ if $w \neq u$ and $w \neq v$
The map $\tilde{\gamma}$ gives a k-coloring of Γ/e

Conversely, every k-coloring of Γ/e gives a k-coloring γ
of $\Gamma \setminus e$ such that $\gamma(u) = \gamma(v)$
Therefore

$$P_{\Gamma \setminus e}(k) = P_{\Gamma}(k) + P_{\Gamma/e}(k)$$

Exercises

Exercises third lecture

1. Compute the extended weight enumerator of the binary simplex code $\mathcal{S}_3(2)$ and its dual Hamming code $\mathcal{H}_3(2)$
2. Compute the extended weight enumerators of the ternary simplex code $\mathcal{S}_3(3)$ and its dual the ternary Hamming code $\mathcal{H}_3(3)$
3. Compute the extended weight enumerators of the n-fold repetition code and its dual
4. Compute the extended weight enumerators of all codes of length at most 5 using the puncturing-shortening formula
5. Give the complexity of the computation of the extended weight enumerator of code by means of the puncturing-shortening formula as a function of the length n and dimension k of the code
6. Compute the chromatic polynomial of $K_{3,3}$
7. Compute the chromatic polynomials of all simple graphs on at most 4 points by using the deletion-contraction formula