Arrangements, matroids and codes

fifth lecture

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joint work with
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ACAGM summer school
Leuven Belgium, 22 July 2011

Content fifth lecture

1. Posets
2. Geometric lattices
3. Characteristic polynomial
4. Overview of relations between polynomials

Posets

Posets

Let L be a set and $\leq$ a relation on L such that:

(PO.1) $x \leq x$, for all x in L (reflexive)

(PO.2) If $x \leq y$ and $y \leq x$, then $x = y$, for all $x, y \in L$ (anti-symmetric)

(PO.3) If $x \leq y$ and $y \leq z$, then $x \leq z$, for all x, y and z in L (transitive)

The pair $(L, \leq)$ or just L is called a poset with partial order $\leq$ on the set L
Define $x < y$ if $x \leq y$ and $x \neq y$

The elements x and y in L are comparable if $x \leq y$ or $y \leq x$
A poset L is called a linear order if every two elements are comparable

Define $L_x = \{y \in L : x \leq y\}$ and $L^x = \{y \in L : y \leq x\}$
$[x, y] = \{z \in L : x \leq z \leq y\}$ is the interval between x and y
Notice that $[x, y] = L_x \cap L^y$

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Chains

Let $(L, \leq)$ be a poset

A chain of length r from x to y in L

is a sequence of elements $x_0, x_1, \ldots, x_r$ in L such that

$$x = x_0 < x_1 < \cdots < x_r = y$$

Let $r \geq 0$ be an integer and $x, y \in L$

Then $c_r(x, y)$ denotes the number of chains of length r from x to y

Now $c_r(x, y)$ is finite if L is finite

The poset is called locally finite if $c_r(x, y)$ is finite

for all $x, y \in L$ and every integer $r \geq 0$

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Proposition on chains

Let L be a locally finite poset
Then for all $x \leq y$ in L:

(N.1) $c_0(x, y) = 0$ if x and y are not comparable.

(N.2) $c_0(x, x) = 1$, $c_r(x, x) = 0$ for all $r > 0$ and $c_0(x, y) = 0$ if $x < y$.

(N.3) $c_{r+1}(x, y) = \sum_{x \leq z < y} c_r(x, z) = \sum_{x < z \leq y} c_r(z, y)$.

Proof Statements (N.1) and (N.2) are trivial
Let $z < y$ and $x = x_0 < x_1 < \cdots < x_r = z$ a chain of length r from x to z
then $x = x_0 < x_1 < \cdots < x_r < x_{r+1} = y$ is a chain
of length $r + 1$ from x to y
and every chain of length $r + 1$ from x to y is
obtained uniquely in this way
Hence $c_{r+1}(x, y) = \sum_{x \leq z < y} c_r(x, z)$
The last equality is proved similarly

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The Möbius function

Let L be a locally finite poset

The Möbius function of L denoted by μ_L or μ is defined by

$$\mu(x, y) = \sum_{r=0}^{\infty} (-1)^r c_r(x, y)$$

Proposition For all $x, y \in L$:

(M.1) $\mu(x, y) = 0$ if x and y are not comparable

(M.2) $\mu(x, x) = 1$

(M.3) If $x < y$, then $\sum_{x \le z \le y} \mu(x, z) = \sum_{x \le z \le y} \mu(z, y) = 0$

(M.4) If $x < y$, then

$$\mu(x, y) = -\sum_{x \le z < y} \mu(x, z) = -\sum_{x < z \le y} \mu(z, y)$$

Sum functions

Let L be a finite poset
Suppose that L has 0_L and 1_L as minimum and maximum, respectively
That is: $0_L \leq x \leq 1_L$ for all $x \in L$
Define $\mu(x) = \mu(0_L, x)$ and $\mu(L) = \mu(0_L, 1_L)$

Let A be an abelian group and $f : L \to A$ a map from L to A
The sum function $\hat{f}$ of f is defined by

$$\hat{f}(x) = \sum_{y \leq x} f(y)$$

Define similarly the sum function $\check{f}$ of f by

$$\check{f}(x) = \sum_{x \leq y} f(y)$$

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Möbius inversion formula

Theorem

Let L be a finite poset with a minimum 0_L and maximum 1_L

Then

$$f(x) = \sum_{y \leq x} \mu(y, x) \hat{f}(y)$$

Similarly

$$f(x) = \sum_{x \leq y} \mu(x, y) \check{f}(y)$$

Geometric lattices

Lattices

Let L be a finite poset and $x, y \in L$
Then x and y have a least upper bound if
there is a $z \in L$ such that $x \leq z$ and $y \leq z$
and if $x \leq w$ and $y \leq w$, then $z \leq w$ for all $w \in L$

If x and y have a least upper bound, then such an element is unique
and it is called the join of x and y and denoted by $x \vee y$

Similarly the greatest lower bound of x and y is defined
If it exists, then it is unique and it is called the meet
of x and y and denoted by $x \wedge y$

A poset L is called a lattice if
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Lattice of integers dividing n

Let N be a positie integer
Let L be the set of positive integers dividing N
with the divisibility relation as partial order
Then $0_L = 1$ is the minimum of L and $1_L = N$ is the maximum
Now $m \vee n = \operatorname{lcm}(m, n)$ and $m \wedge n = \gcd(m, n)$
Hence L is a lattice
Furthermore

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ (-1)^r & \text{if } n \text{ is the product of } r \text{ mutually distinct primes;} \\ 0 & \text{if } n \text{ is divisible by the square of a prime.} \end{cases}$$

Hence $\mu(n)$ is the classical Möbius function
with

$$\mu(d, n) = \mu(\frac{n}{d}) \text{ if } d|n$$

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$$\varphi(n) = |\{i \in \mathbb{N} : \gcd(i,n) = 1\}| \quad \text{and}$$

$$V_d = \{i \in [n] : \gcd(i,n) = \tfrac{n}{d}\} \quad \text{for } d|n$$

Then

$$\{\, i \cdot \tfrac{n}{d} : i \in [d]\,, \gcd(i,d) = 1 \,\} = V_d$$

so $|V_d| = \varphi(d)$

Now $[n]$ is the disjoint union of the subsets V_d with $d|n$

Hence the sum function of $\varphi(n)$ is given by

$$\hat{\varphi}(n) = \sum_{d|n} \varphi(d) = n$$

Therefore by Möbius inversion

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Example: t-subsets of $[n]$ with $t \leq k$

Let k and n be nonnegative integers such that $k \leq n$
Let $L_k = \{[n]\}$ and L_t the collection of all t-subsets of $[n]$ for $t < k$
Let $L_{n,k}$ be the union of all L_t, $t \leq k$
with partial order given by the inclusion

Then $L_{n,k}$ is a poset with $0 = \emptyset$ as minimum and $1 = [n]$ as maximum
Now $I \vee J = I \cup J$ and $I \wedge J = I \cap J$
Hence L is a lattice

Furthermore

$$\mu(I, J) = (-1)^{|J|-|I|}$$

if $I \leq J$ and $|J| < k$ and

$$\mu(I, \mathcal{X}) = -\sum_{I \leq J < \mathcal{X}} (-1)^{|J|-|I|}$$

for all $I < 1$

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Let L the poset of subsets of $[n]$
Let $A_1, \ldots, A_n$ be a collection of subsets of a finite set A
Define for a subset J of $\mathcal{X}$

$$A_J = \bigcap_{j \in J} A_j \text{ and } f(J) = |A_J \setminus (\cup_{I<J} A_I)|$$

Then A_J is the disjoint union of the subsets $A_I \setminus (\cup_{K<I} A_K)$ for all $I \leq J$
Hence the sum function is equal to

$$\hat{f}(J) = \sum_{I \leq J} f(I) = \sum_{I \leq J} |A_I \setminus (\cup_{K<I} A_K)| = |A_J|$$

Möbius inversion gives that

$$|A_J \setminus (\cup_{I<J} A_I)| = \sum_{I \leq J} (-1)^{|J|-|I|} |A_I|$$

Principle of inclusion/exlusion

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Geometric lattices

Let L be a finite lattice with minimum 0 and maximum 1
y is a cover of x if $x < y$ and there is no z with $x < z < y$
An atom is an element $a \in L$ that is a cover of 0
A lattice is called atomic if for every $x > 0$ in L there exist atoms $a_1, \ldots, a_r$ such that $x = a_1 \vee \cdots \vee a_r$
The minimum length of a chain from 0 to x
is called the rank of x and is denoted by $r_L(x)$ or $r(x)$ for short

A lattice is called semimodular if for all mutually distinct $x, y \in L$
$x \vee y$ covers x and y if there exists a z such that x and y cover z

A finite lattice L is called a geometric lattice if
it is atomic and semimodular
If L is a geometric lattice L, then it has a minimum and a maximum
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Proposition on geometric lattices

Let L be a finite geometric lattice
Then for all $x, y \in L$:

(GL.1) If $x < y$, then $r(x) < r(y)$ (strictly monotone)

(GL.2) $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$ (semimodular inequality)

(GL.3) If $x \leq y$, then every chain from x to y can be extended to a maximal chain with the same end points and all such maximal chains have the same length $r(y) - r(x)$ (Jordan-Hölder property)

Geometric lattices of an arrangement

Let $\mathbb{F}$ be a field
Let $\mathcal{A} = (H_1, \dots, H_n)$ be an essential arrangement over $\mathbb{F}$ in $V = \mathbb{F}^k$
Let $L(\mathcal{A})$ be the collection of all intersections of elements of $\mathcal{A}$
By definition $\mathbb{F}^k$ is the empty intersection
Define the partial order $\leq$ by

$$x \leq y \text{ if and only if } y \subseteq x$$

Then V is the minimum element and $\{0\}$ is the maximum element
Furthermore

$$x \vee y = x \cap y \quad \text{and} \quad x \wedge y = \bigcap \{z : x \cup y \subseteq z\}$$

Hence $L(\mathcal{A})$ is a lattice
In fact it is a geometric lattice

Geometric lattices of an arrangement

Let $\mathbb{F}$ be a field
Let $\mathcal{A} = (H_1, \ldots, H_n)$ be an essential arrangement over $\mathbb{F}$ in $V = \mathbb{F}^k$
Let $L(\mathcal{A})$ be the collection of all intersections of elements of $\mathcal{A}$
By definition $\mathbb{F}^k$ is the empty intersection
Define the partial order $\leq$ by

$$x \leq y \text{ if and only if } y \subseteq x$$

Then V is the minimum element and $\{0\}$ is the maximum element
Furthermore

$$x \vee y = x \cap y \text{ and } x \wedge y = \bigcap \{z : x \cup y \subseteq z\}$$

Hence $L(\mathcal{A})$ is a lattice
In fact it is a geometric lattice

Geometric lattices and matroid vice versa

Let L be a finite geometric lattice
Let $M(L)$ be the set of all atoms of L

Let $\mathcal{I}(L)$ be the collection of all subsets I of $M(L)$ such that
$r(a_1 \vee \cdots \vee a_r) = r$ if $I = \{a_1, \ldots, a_r\}$ is a collection of r atoms of L
Then $(M(L), \mathcal{I}(L))$ is a simple matroid

Let $M = (E, \mathcal{I})$ be a matroid
A k-flat of M is a maximal subset of E of rank k

Let $L(M)$ be the collection of all flats of M
it is called the lattice of flats of M

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Cryptomorphic

Example

$$M(L_{n,k}) = U_{n,k} \text{ and } L(U_{n,k}) = L_{n,k}$$

Let L be a finite geometric lattice
Then L is isomorphic with $L(M(L))$

Let M be a matroid
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Characteristic polynomial

The characteristic and coboundary polynomial

Let L be a finite geometric lattice

The characteristic polynomial $\chi_L(T)$ is defined by:

$$\chi_L(T) = \sum_{x \in L} \mu_L(x) T^{r(L)-r(x)}$$

The two variable characteristic polynomial or coboundary polynomial is defined by

$$\chi_L(S,T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu(x,y) S^{a(x)} T^{r(L)-r(y)}$$

where $a(x)$ is the number of atoms a in L such that $a \leq x$

Then

$$\chi_L(S,T) = \sum_{i=0}^{r} S^i \chi_i(T) \text{ with } \chi_i(T) = \sum_{x \in L_i} \chi_{L_x}(T)$$

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Technische Universiteit Eindhoven University of Technology

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Rota's Crosscut Theorem

Let L be a finite geometric lattice
Let $M(L) = (E, \mathscr{I})$ be the matroid associated with L
Then

$$\chi_L(T) = \sum_{J \subseteq E} (-1)^{|J|} T^{r(L)-r(J)}$$

Therefore the characteristic polynomials of L
is a function of the Tutte polynomial of $M(L)$

$$\chi_L(T) = (-1)^{r(L)} t_{M(L)}(1 - T, 0)$$

More generally

$$\chi_L(S, T) = (S - 1)^{r(L)} R_{M(L)} \left(\frac{T}{S - 1}, S - 1 \right)$$

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The characteristic polynomial of a graph

Let $\Gamma = (V, E)$ be a finite simple graph
Let χ_Γ be the characteristic polynomial of the geometric lattice $L(M_\Gamma)$
Then for all positive integers k:

$$P_\Gamma(k) = \chi_\Gamma(k)$$

So the chromatic polynomial of a graph is the prime example of a characteristic polynomial and the two variable characteristic polynomial of a graph is also called the dichromatic polynomial of the graph

Let γ be a coloring of Γ
Then an edge is called bad if if joins two vertices with the same color
The i-defect polynomial $\chi_i(T)$ counts up to a factor of T the number of ways of coloring Γ with i bad edges

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The characteristic polynomial of a code

Let C be a nondegenerate code

Let $\chi(S, T)$ be the coboundary polynomial of the geometric lattice $L(M_C)$

Then

$$A_{n-i}(T) = \chi_i(T)$$

for all i

Overview relations between polynomials