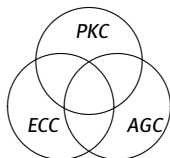


Cryptanalysis of public-key cryptosystems based on algebraic geometry codes

Ruud Pellikaan
joint work with
Irene Márquez-Corbella and Edgar Martínez-Moro

Number Theory Seminar
11 November 2011, Queens University, Kingston



- ▶ ECC = Error-correcting codes
- ▶ AGC = Algebraic geometry curves
- ▶ PKC = Public-key cryptosystems

\mathbb{F}_q finite field with q elements

Hamming distance on \mathbb{F}_q^n :

$$d(\mathbf{x}, \mathbf{y}) = |\{i \mid x_i \neq y_i\}|$$

C linear block code: \mathbb{F}_q -linear subspace of \mathbb{F}_q^n

parameters $[n, k, d]$:

n = length

$k = k(C)$ = dimension of C

$d = d(C)$ = minimum distance of C

$$d(C) = \min |\{d(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}|$$

C an \mathbb{F}_q -linear code of dimension k

G **generator matrix** of C : a $k \times n$ matrix with entries in \mathbb{F}_q and

$$C = \{ \mathbf{x}G \mid \mathbf{x} \in \mathbb{F}_q^k \}$$

Code is **nondegenerate** if G has no zero column

H **parity check matrix** of C : a $(n - k) \times n$ matrix with entries in \mathbb{F}_q and

$$C = \{ \mathbf{c} \in \mathbb{F}_q^n \mid \mathbf{c}H^T = \mathbf{0} \}$$

Katsman-Tsfasman-Vladut:

Let \mathbb{F} be a field

A **projective system** $\mathcal{P} = (P_1, \dots, P_n)$ in $\mathbb{P}^r(\mathbb{F})$ is an n -tuple of points P_j in this projective space such that not all these points lie in a hyperplane

Let $P_j = (p_{0j} : p_{1j} : \dots : p_{rj})$

Let $G_{\mathcal{P}}$ be the $(r+1) \times n$ matrix with $(p_{0j}, p_{1j}, \dots, p_{rj})^T$ as j -th column
Then $G_{\mathcal{P}}$ has rank $r+1$, since not all points lie in a hyperplane.

If \mathbb{F} is a finite field, then $G_{\mathcal{P}}$ is the generator matrix of a nondegenerate $[n, r + 1, d]$ code over \mathbb{F} where $n - d$ is the maximal number of points of \mathcal{P} that lie in a hyperplane of $\mathbb{P}^{k-1}(\mathbb{F})$

Example

Let \mathcal{X} be an irreducible projective curve over \mathbb{F}_q of degree m in \mathbb{P}^{k-1}

Let \mathcal{P} be an enumeration of n points of $\mathcal{X}(\mathbb{F}_q)$

Then $G_{\mathcal{P}}$ is the generator matrix of a code with parameters $[n, k, d]$

$$d \geq n - m$$

Conversely:

Let G be a generator matrix of a nondegenerate $[n, k, d]$ code over \mathbb{F}_q

Then G has no zero columns

Take the columns of G as homogeneous coordinates of points in $\mathbb{P}^{k-1}(\mathbb{F}_q)$

This gives the projective system \mathcal{P}_G over \mathbb{F}_q of G

One-to-one correspondence between:

generalized equivalence classes of nondegenerate $[n, k]$ codes over \mathbb{F}_q
and

equivalence classes of projective systems of n points in $\mathbb{P}^{k-1}(\mathbb{F}_q)$

$\mathbf{a} = (a_1, \dots, a_n)$ an n -tuple of **mutually distinct** elements of \mathbb{F}_q

$\mathbf{b} = (b_1, \dots, b_n)$ an n -tuple of **nonzero** elements of \mathbb{F}_q

$$GRS_k(\mathbf{a}, \mathbf{b}) = \{ (f(a_1)b_1, \dots, f(a_n)b_n) \mid f(X) \in \mathbb{F}_q[X], \deg(f(X)) < k \}$$

parameters: $[n, k, n - k + 1]$ if $k \leq n$

generator matrix:

$$G_k(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} b_1 & \cdots & b_j & \cdots & b_n \\ a_1 b_1 & \cdots & a_j b_j & \cdots & a_n b_n \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_1^{k-1} b_1 & \cdots & a_j^{k-1} b_j & \cdots & a_n^{k-1} b_n \end{pmatrix}$$

The projective system of the the code $GRS_k(\mathbf{a}, \mathbf{b})$ with generator matrix $G_k(\mathbf{a}, \mathbf{b})$ is

$$\mathcal{P}_k(\mathbf{a}) = ((1 : a_j : \dots : a_j^i : \dots : a_j^{k-1}) \mid j = 1, \dots, n)$$

Consider the **embedding** $\mathbb{P}^1 \rightarrow \mathbb{P}^r$ by the degree r map given by

$$(y_0 : y_1) \mapsto (y_0^r : y_0^{r-1}y_1 : \dots : y_0^{r-i}y_1^i : \dots : y_0y_1^{r-1} : y_1^r)$$

The image of this map in \mathbb{P}^r is the **NRC (normal rational curve)** \mathcal{X}_r

Every hyperplane intersects \mathcal{X}_r in at most r points and

$$\mathcal{P}_k(\mathbf{a}) \subseteq \mathcal{X}_{k-1}(\mathbb{F}_q)$$

The **vanishing ideal** $I(\mathcal{X}_r)$ of \mathcal{X}_r is generated by the **quadratic polynomials**:

$$X_i X_{r-i} - X_j X_{r-j}, \text{ for } 0 \leq i < j \leq r$$

that is the **determinantal ideal** of the 2×2 minors of the $2 \times r$ matrix

$$\begin{pmatrix} X_0 & X_1 & \cdots & X_i & \cdots & X_{r-1} \\ X_1 & X_2 & \cdots & X_{i+1} & \cdots & X_r \end{pmatrix}$$

since the rows of the matrix

$$\begin{pmatrix} 1 & y & \cdots & y^i & \cdots & y^{r-1} \\ y & y^2 & \cdots & y^{i+1} & \cdots & y^r \end{pmatrix}$$

are dependent for all y and

Let \mathcal{X} be an **algebraic variety** over \mathbb{F}_q
with a subset \mathcal{P} of $\mathcal{X}(\mathbb{F}_q)$ enumerated by P_1, \dots, P_n

Suppose that we have a vector space L over \mathbb{F}_q
of functions on \mathcal{X} with values in \mathbb{F}_q

So $f(P_i) \in \mathbb{F}_q$ for all i and $f \in L$

In this way we have an **evaluation map**

$$ev_{\mathcal{P}} : L \longrightarrow \mathbb{F}_q^n$$

defined by $ev_{\mathcal{P}}(f) = (f(P_1), \dots, f(P_n))$

This evaluation map is linear, so its image is a linear code

The classical example:

Generalized Reed-Solomon codes

The geometric object \mathcal{X} is the **affine line** over \mathbb{F}_q

The points are n distinct elements of \mathbb{F}_q

L is the vector space of polynomials of degree at most $k - 1$
and with coefficients in \mathbb{F}_q

This vector space has dimension k

Such polynomials have **at most $k - 1$ zeros**

so nonzero codewords have at least $n - k + 1$ nonzeros

This code has parameters $[n, k, n - k + 1]$ if $k \leq n$

K function field over \mathbb{F}_q

$[K : \mathbb{F}_q(X)] < \infty$

discrete valuation

$\mathbb{F}[X]$ regular functions

Zeta function

RHS is proved

block code $C \subseteq \mathbb{F}_q^n$

K number field

$[K : \mathbb{Q}] < \infty$

valuation

\mathcal{O} ring of integers

Zeta function

RHS conjectured

lattice $\Gamma \subseteq \mathbb{R}^n$

Let \mathcal{X} be an algebraic curve over \mathbb{F}_q of genus g

$\mathbb{F}_q(\mathcal{X})$ is the function field of the curve \mathcal{X} with field of constants \mathbb{F}_q

Let f be a nonzero rational function on the curve

The divisor of zeros and poles of f is denoted by (f)

Let E be a divisor of \mathcal{X} of degree m

Then

$$L(E) = \{ f \in \mathbb{F}_q(\mathcal{X}) \mid f = 0 \text{ or } (f) \geq -E \}$$

The dimension of the space $L(E)$ is denoted by $l(E)$

Then $l(E) \geq m + 1 - g$ and equality holds if $m > 2g - 2$

by the Theorem of Riemann-Roch

Let $\mathcal{P} = (P_1, \dots, P_n)$ an n -tuple of mutual distinct points of $\mathcal{X}(\mathbb{F}_q)$ with divisor $D = P_1 + \dots + P_n$

If the support of E is disjoint from D , then the **evaluation map**

$$\text{ev}_{\mathcal{P}} : L(E) \rightarrow \mathbb{F}_q^n$$

where $\text{ev}_{\mathcal{P}}(f) = (f(P_1), \dots, f(P_n))$, is well defined.

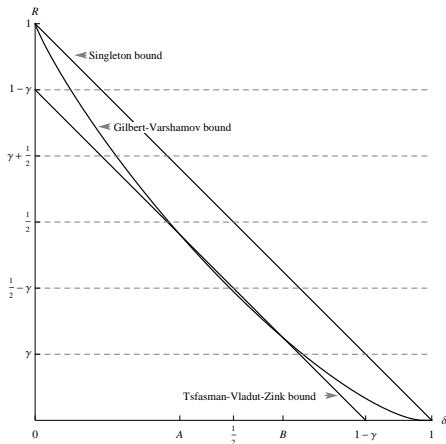
The **algebraic geometry code** $C_L(\mathcal{X}, \mathcal{P}, E)$

is the image of $L(E)$ under the evaluation map $\text{ev}_{\mathcal{P}}$

If $m < n$, then $C_L(\mathcal{X}, \mathcal{P}, E)$ is an $[n, k, d]$ code with

$$k \geq m + 1 - g \text{ and } d \geq n - m$$

Information rate	$R = k/n$
Relative minimum distance	$\delta = d/n$
Singleton	$R + \delta \leq 1$
Gilbert-Varshamov	$R \geq 1 - H_q(\delta)$
q-ary entropy function	H_q
Goppa for AG codes	$R + \delta \geq 1 - \gamma$
Relative genus	$\gamma = g/n$
Ihara-Tsfasman-Vladut-Zink	$\gamma = \frac{1}{\sqrt{q}-1}$



Figuur: Bounds on R as a function of δ for $q = 49$ and $\gamma = \frac{1}{6}$.

Let ω be a **differential form** with a simple pole at P_j with residue 1 for all $j = 1, \dots, n$

Let K be the **canonical divisor** of ω
Let m be the degree of the divisor E on \mathcal{X} with disjoint support from \mathcal{P}

Let $E^\perp = D - E + K$ and $m^\perp = \deg(E^\perp)$
Then $m^\perp = 2g - 2 - m + n$ and

$$C_L(\mathcal{X}, \mathcal{P}, E)^\perp = C_L(\mathcal{X}, \mathcal{P}, E^\perp)$$

Embedding of \mathcal{X} in **linear system** of E of degree m

Let f_1, f_2, \dots, f_k be a basis of $L(E)$

$$\varphi : \mathcal{X} \longrightarrow \mathbb{P}^{k-1}$$

$$P \mapsto (f_1(P), f_2(P), \dots, f_k(P))$$

$\mathcal{Y} = \varphi(\mathcal{X})$ is a curve of degree m in \mathbb{P}^{k-1}

$\mathcal{Q} = (\varphi(P_1), \dots, \varphi(P_n))$ **projective system**

$$G_{\mathcal{Q}} = \begin{pmatrix} f_1(P_1) & \cdots & f_1(P_j) & \cdots & f_1(P_n) \\ f_2(P_1) & \cdots & f_2(P_j) & \cdots & f_2(P_n) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ f_k(P_1) & \cdots & f_k(P_j) & \cdots & f_k(P_n) \end{pmatrix} \text{generator matrix}$$

minimum distance $\geq n - m$

Decoding problem

Input: (G, \mathbf{y})

where G is a $k \times n$ matrix G over \mathbb{F}_q of rank k , and \mathbf{y} in \mathbb{F}_q^n

Output: A closest codeword \mathbf{c}

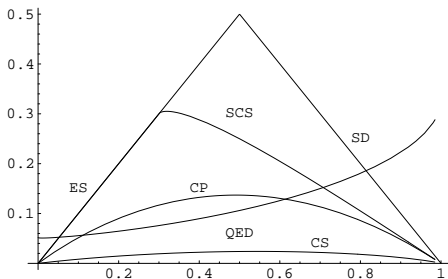
so $d(\mathbf{c}, \mathbf{y})$ is minimal for all \mathbf{c} in the code C with generator matrix G

This problem is **NP-hard**

Berlekamp-McEliece-Van Tilborg

Decoding arbitrary linear codes

Exponential complexity $\approx q^{e(R)n}$



x-axis: information rate $R = k/n$

y-axis: complexity exponent $e(R)$

Efficient decoding algorithms up to half the minimum distance for:

- Generalized Reed-Solomon codes
- Goppa codes
- Algebraic geometry codes

Polynomial complexity $\mathcal{O}(n^3)$

- Peterson, Arimoto 1960
- Berlekamp-Massey 1963
- Justesen-Larsen-Havemose-Jensen-Hoeholdt 1989
- Skorobogatov-Vladut 1990
- Sakata 1990
- Feng-Rao, Duursma 1993
- Sudan, Guruswami 1997

PKC systems use **trapdoor one-way functions**

by mathematical problems that are (supposedly) **hard**

RSA, **factoring integers**: given $n = pq$ find (p, q)

Diffie-Hellman, **discrete-log problem** in \mathbb{F}_q : given $b = a^n$ find n

Elliptic curve PKC, **addition on elliptic curve**: given $Q = nP$, find n

Code based PKC systems: **decoding of codes**

McEliece (Goppa codes)

Niederreiter (Generalized Reed-Solomon codes)

Janwa-Moreno (Algebraic geometry codes)

McEliece:

Let \mathcal{C} be a class of codes that have efficient decoding algorithms correcting t errors with $t \leq (d - 1)/2$

Secret key: (S, G, P)

S an invertible $k \times k$ matrix

G a $k \times n$ generator matrix of a code C in \mathcal{C} .

P an $n \times n$ permutation matrix

Public key: $G' = SG P$

Message: m in \mathbb{F}_q^k

Encryption: $y = mG' + e$ with random chosen e in \mathbb{F}_q^n of weight t

Decryption: $yP^{-1} = mSG + eP^{-1}$ and eP^{-1} has weight t

Decoder gives $c = mSG$ as closest codeword

Binary Goppa codes with parameters $[n, k, d]$ where

$$n = 2^m, \quad k \geq 2^m - mr, \quad d \geq 2r + 1$$

Are subfield subcodes of GRS codes over \mathbb{F}_{2^m} with parameters

$$[2^m, 2^m - mr, r + 1]$$

Testcase: binary Goppa code with $m = 10, r = 50$

$$[1024, \geq 524, \geq 101]$$

Corrects 50 errors

- McEliece 1978
- Brickell-Lee 1988
- Leon 1988
- van Tilburg 1988
- Stern 1989
- Canteaut-Chabaud-Sendrier 1998
- Bernstein-Lange-Peters 2008

McEliece PKC system using binary Goppa [1024, 524, 101]
can be broken in:

- 1400 days by a single CPU or
- 7 days by a cluster of 200 CPU's

Suppose \mathcal{C} is the class of Generalized Reed-Solomon codes

A GRS code of length n and dimension $k = r + 1$ gives a projective system of n points in general position on a NRC of degree r in projective space of dimension r

Special case: $k = 3$ and $r = 2$:

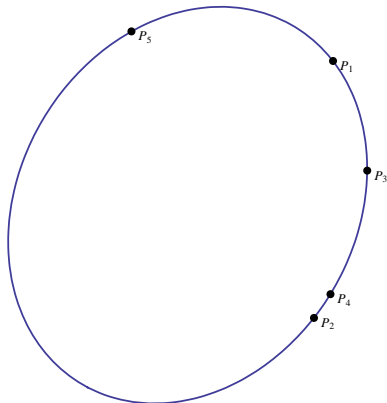
a NRC of degree 2 in the projective plane is a **conic**

5 points in general position determine this conic

Steiner: parametrization of this conic in the plane given these 5 points

Algorithm of **Sidelnikov-Shestakov** for arbitrary k

Complexity: linear algebra $\mathcal{O}(n^3)$



Veronese 1882, Bordiga 1885, Castelnuovo 1885:

Let \mathcal{P} be a collection of $r + 3$ points in general position in \mathbb{P}^r
Then there is a unique NRC of degree r passing through the points of \mathcal{P}

Twisted cubic, $r=3$:

(spiraal)

Berger-Loidreau: use **subcodes** of GRS codes

Attack: **Wieschebrink, Marquez-Martinez-P**

Projection of NRC to linear subspace

Singular rational curve in projective space

Interpolation problem

Goppa codes are specific subfield subcodes of GRS codes

Janwa-Moreno: use algebraic geometry codes

Problem for the attacker:

Input: a generator matrix of an AG code $C_L(\mathcal{X}, \mathcal{P}, E)$

Output: $(\mathcal{X}, \mathcal{P}, E)$ if this triple is unique or $(\mathcal{X}', \mathcal{P}', E')$ otherwise

This system was broken for codes on curves of genus $g \leq 2$
by **Faure-Minder**

A code C over \mathbb{F} is called **weakly algebraic-geometric (WAG)** if $C = C_L(\mathcal{X}, \mathcal{P}, E)$ for some triple $(\mathcal{X}, \mathcal{P}, E)$ where:

- \mathcal{X} is an algebraic curve over \mathbb{F}_q
- \mathcal{P} is an n -tuple of mutually distinct points of $\mathcal{X}(\mathbb{F}_q)$
- E is divisor of degree m on \mathcal{X}

Then $(\mathcal{X}, \mathcal{P}, E)$ is called a **WAG representation** of C

If $m < n$, then it is called **AG**

If $2g - 2 < m < n$, then it is called **strongly algebraic-geometric (SAG)**

Theorem[P-Shen-van Wee]: Every code has a WAG representation

Two representations $(\mathcal{X}, \mathcal{P}, E)$ and $(\mathcal{Y}, \mathcal{Q}, F)$ are called **equivalent** or **isomorphic** if there is an isomorphism of curves $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ such that $\varphi(\mathcal{P}) = \mathcal{Q}$ and $\varphi(E) \equiv F$

They are called **strict equivalent** or **strict isomorphic** if moreover $\varphi(E) \equiv_{\mathcal{Q}} F$

Proposition

Let $(\mathcal{X}, \mathcal{P}, E)$ and $(\mathcal{Y}, \mathcal{Q}, F)$ be WAG representations of C and D , resp. Then:

- (1) If $(\mathcal{X}, \mathcal{P}, E)$ and $(\mathcal{Y}, \mathcal{Q}, F)$ are equivalent, then $C \equiv D$
- (2) If $(\mathcal{X}, \mathcal{P}, E)$ and $(\mathcal{Y}, \mathcal{Q}, F)$ are strict equivalent, then $C = D$

Theorem[Munuera-P]:

Let \mathcal{X} be a curve of genus g and $D = P_1 + \dots + P_n$

Let E and F be divisors of degree m with $2g - 1 < m < n - 1$

Then

$$C_L(\mathcal{X}, \mathcal{P}, E) = C_L(\mathcal{X}, \mathcal{P}, F) \text{ if and only if } E \equiv_{\mathcal{P}} F$$

Let $(\mathcal{X}, \mathcal{P}, E)$ be a WAG representation of C such that $m > 2g$

Let $r = l(E) - 1$ and $\{f_0, \dots, f_r\}$ be a basis of $L(E)$

Consider the following map:

$$\varphi_E : \mathcal{X} \longrightarrow \mathbb{P}^r$$

defined by $\varphi_E(P) = (f_0(P), \dots, f_r(P))$

If $m > 2g$, then $r = m - g$ and φ_E defines an **embedding** of the curve \mathcal{X} in \mathbb{P}^r of degree m with image $\mathcal{Y} = \varphi_E(\mathcal{X})$

Let $Q_j = \varphi_E(P_j)$ and $\mathcal{Q} = (Q_1, \dots, Q_n)$ then $\varphi_E(E) = \mathcal{X} \cdot H = F$ for some hyperplane H of \mathbb{P}^r that is disjoint from \mathcal{Q}

Furthermore $(\mathcal{Y}, \mathcal{Q}, F)$ is also a WAG representation of the code C that is strict isomorphic with $(\mathcal{X}, \mathcal{P}, E)$

Normal rational normal curve is defined by quadratic equations.

The canonical model of a non-hyperelliptic projective curve of genus at least three is the intersection of quadrics and cubics, and of quadrics only except in case of a trigonal curve and a plane quintic

[Enriques](#) 1919, [Petri](#) 1923 and [Babbage](#) 1939

This result for the canonical divisor was generalized for arbitrary divisors E under certain constraints on the degree

[Mumford](#) 1970, [Saint-Donat](#) 1972 and [Arbarello](#) 1978

Let \mathcal{X} be an absolutely irreducible and nonsingular curve of genus g over the perfect field \mathbb{F}

Let E be a divisor on \mathcal{X} of degree m

If $m \geq 2g + 1$

then φ_E gives an embedding of \mathcal{X} onto $\mathcal{Y} = \varphi_E(\mathcal{X})$ which is a normal curve in \mathbb{P}^{m-g}

If $m \geq 2g + 2$, then \mathcal{Y} is an **intersection of quadrics**

More precisely:

$I(\mathcal{Y})$ is generated by $I_2(\mathcal{Y})$ the set of homogeneous elements of degree two in $I(\mathcal{Y})$

Let \mathcal{Y} be a curve embedded in projective r -space of degree m

Let $I(\mathcal{Y})$ be the vanishing ideal of \mathcal{Y}

Let \mathcal{Q} be a subset of \mathcal{Y} of n points

Then

$$I(\mathcal{Y}) \subseteq I(\mathcal{Q})$$

Hence

$$I(\mathcal{Y})_2 \subseteq I_2(\mathcal{Q})$$

Suppose $I(\mathcal{Y})$ is generated by $I_2(\mathcal{Y})$

If $n > 2m$, then $I(\mathcal{Y}) = I_2(\mathcal{Q})$

By Bézout's Theorem

Let \mathcal{Q} be an n -tuple of points in \mathbb{P}^{k-1} over \mathbb{F} not in a hyperplane

Let $G_{\mathcal{Q}}$ be the $k \times n$ matrix associated to \mathcal{Q} with basis $\mathbf{g}_1, \dots, \mathbf{g}_k$

Denote the **second symmetric power** of C by $S^2(C)$

If $x_i = \mathbf{g}_i$ then $S^2(C)$ has basis $\{x_i x_j \mid 1 \leq i \leq j \leq n\}$

and dimension $\binom{k+1}{2}$

Denote by $\langle C * C \rangle$ or $C^{(2)}$ the **square** of C

that is, the linear subspace in \mathbb{F}^n generated by $\{\mathbf{a} * \mathbf{b} \mid \mathbf{a}, \mathbf{b} \in C\}$

Consider the linear map $\sigma : S^2(C) \longrightarrow C^{(2)}$

where the element $x_i x_j$ is mapped to $\mathbf{g}_i * \mathbf{g}_j$

The kernel of this map will be denoted by $K^2(C)$

Then

$$0 \longrightarrow K^2(C) \longrightarrow S^2(C) \longrightarrow C^{(2)} \longrightarrow 0$$

is an exact sequence and

$$I_2(\mathcal{Q}) = \left\{ \sum_{1 \leq i < j \leq k} a_{ij} X_i X_j \mid \sum_{1 \leq i < j \leq k} a_{ij} x_i x_j \in K^2(C) \right\}$$

Proposition

Let \mathcal{Q} be an n -tuple of points in \mathbb{P}^r over \mathbb{F} not in a hyperplane

Then the complexity of the computation of $I_2(\mathcal{Q})$ is at most $\mathcal{O}(n^2 \binom{r}{2})$

A code C over \mathbb{F}_q is called **very strong algebraic-geometric (VSAG)** if $C = C_L(\mathcal{X}, \mathcal{P}, E)$ and the curve \mathcal{X} over \mathbb{F}_q has genus g \mathcal{P} consists of n points and E has degree m such that

$$2g + 2 < m < \frac{1}{2}n \quad \text{or} \quad \frac{1}{2}n + 2g - 2 < m < n - 4$$

The dimension of a such a code is $k = m + 1 - g$
Thus the dimension satisfies the following bound

$$g + 3 < k < \frac{1}{2}n - g + 1 \quad \text{or} \quad \frac{1}{2}n + g - 1 < k < n - g - 3.$$

Note that the dual of a VSAG code is again VSAG

Let C be a VSAG code

Then a VSAG representation can be obtained efficiently
from its generator matrix

Moreover all VSAG representations of C are strict isomorphic

The dimension of a VSAG code satisfies the following bound

$$g + 3 < k < \frac{1}{2}n - g + 1 \quad \text{or} \quad \frac{1}{2}n + g - 1 < k < n - g - 3.$$

Let $R = k/n$ and $\gamma = g/n$

Then for $n \rightarrow \infty$ and $\gamma \leq \frac{1}{4}$ (so $q \geq 25$):

$$\gamma \leq R \leq \frac{1}{2} - \gamma \quad \text{or} \quad \frac{1}{2} + \gamma \leq R \leq 1 - \gamma$$

Proposition

If $\gamma \leq \frac{1}{4}$ and C is a SAG code in the range

$$\gamma \leq R \leq 1 - 3\gamma \quad \text{or} \quad 3\gamma \leq R \leq 1 - \gamma$$

then by shortening C sufficiently many times one gets a VSAG code

SAG codes are **not secure** for a code based PKC

1. If $\frac{1}{6} \leq \gamma \leq \frac{1}{4}$ in the range:

$$\gamma \leq R \leq 1 - 3\gamma \quad \text{or} \quad 3\gamma \leq R \leq 1 - \gamma$$

2. If $\gamma \leq \frac{1}{6}$ in the range:

$$\gamma \leq R \leq 1 - \gamma$$