The (extended) coset leader and list weight enumerator

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Extension codes

Weight	The number of nonzero coordinates of a vector.
Linear $[n, k]$ code	Linear subspace $C \subseteq \mathrm{GF}(q)^n$ of dimension k. Elements are called *(code)words*, n is called the *length*.
Generator matrix	The rows of this $k \times n$ matrix form a basis for C.

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Extension code	$[n, k]$ code $C \otimes \mathrm{GF}(q^m)$ over some extension field $\mathrm{GF}(q^m)$ generated by the words of C.
Generator matrix	All the extension codes of C have the same generator matrix G.

Coset leader weight enumerator

Coset Translation of the code by a vector $\mathbf{y} \in \mathrm{GF}(q)^n$.

Weight The minimum weight of all vectors in the coset.

Coset leader A vector of minimum weight in the coset.

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Extended coset leader weight enumerator

The homogeneous polynomial counting the number of cosets of a given weight "for all extension codes", notation:

$$\alpha_C(X, Y, T) = \sum_{i=0}^{n} \alpha_i(T) X^{n-i} Y^i.$$

Note that we have $\alpha_C(X, Y, q^m) = \alpha_{C \otimes \mathrm{GF}(q^m)}(X, Y)$.

List weight enumerator

Extended list weight enumerator

The homogeneous polynomial counting the number of coset leaders of a given weight "for all extension codes", notation:

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In general, we have

$$\alpha_i(T) \leq \lambda_i(T),$$

with equality iff all cosets of weight i have a unique coset leader.

Why do we study this?

The extended coset leader weight enumerator is interesting because:

- Determines the probability of correct decoding in coset leader decoding.
- Determines the average of changed symbols in *steganography* (information hiding).
- Not determined by the extended weight enumerator.

The extended list weight enumerator is interesting because:

- Determines the size of lists in list decoding.
- Determines the probability of correct decoding in list decoding.

. . . and of course because they are invariants of linear codes.

Hyperplane arrangements and projective systems

Arrangement of hyperplanes	n-tuple of hyperplanes in $\mathrm{GF}(q)^k$.
Essential arrangement	Intersection of all hyperplanes is $\{\mathbf{0}\}$, hyperplanes are in $\mathrm{PG}(k-1,q)$.
Projective system	n-tuple of points in $\mathrm{PG}(k-1,q)$.

Projective systems are the geometric duals of hyperplane arrangements. Both induce the same *geometric lattice*.

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Columns of a generator matrix G of a linear $[n,k]$ code form a hyperplane arrangement / projective system.

- One-to-one correspondence between equivalence classes.
- Independent of choice of G, so notation: $\mathcal{A}_C$ or $\mathcal{P}_C$.
- Also valid over an extension field $\mathrm{GF}(q^m)$.

Determination of coset weights

Parity check matrix	$(n-k) \times n$ matrix H such that $GH^T = 0$.
Syndrome of $\mathbf{y}$	The vector $\mathbf{s} = H\mathbf{y}^T$, zero for codewords.
Syndrome weight	Minimal number of columns whose span contains $\mathbf{s}$.

Determination of coset weights

Parity check matrix $(n-k) \times n$ matrix H such that $GH^T = 0$.

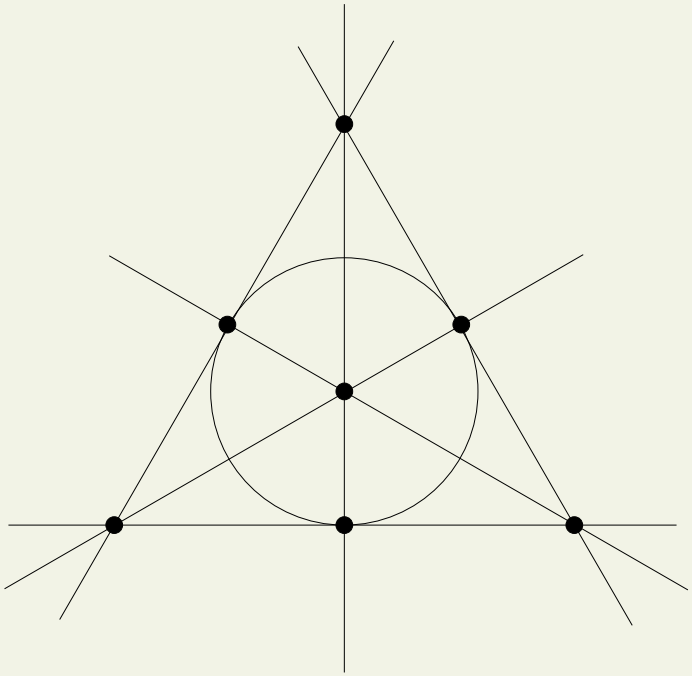
Syndrome of $\mathbf{y}$ The vector $\mathbf{s} = H\mathbf{y}^T$, zero for codewords.

Syndrome weight Minimal number of columns whose span contains $\mathbf{s}$.

- Isomorphism between cosets and syndromes, because $H(\mathbf{y}+\mathbf{c})^T = H\mathbf{y}^T + H\mathbf{c}^T = H\mathbf{y}^T$.
- Syndrome weight is equal to corresponding coset weight (weight of coset leader).
- α_i is the number of vectors that are in the span of i columns of H but not in the span of $i-1$ columns of H.

Determination of coset weights

Example



The $[7,4]$ binary Hamming code has parity check matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

This is a projective system of seven points in the projective plane.

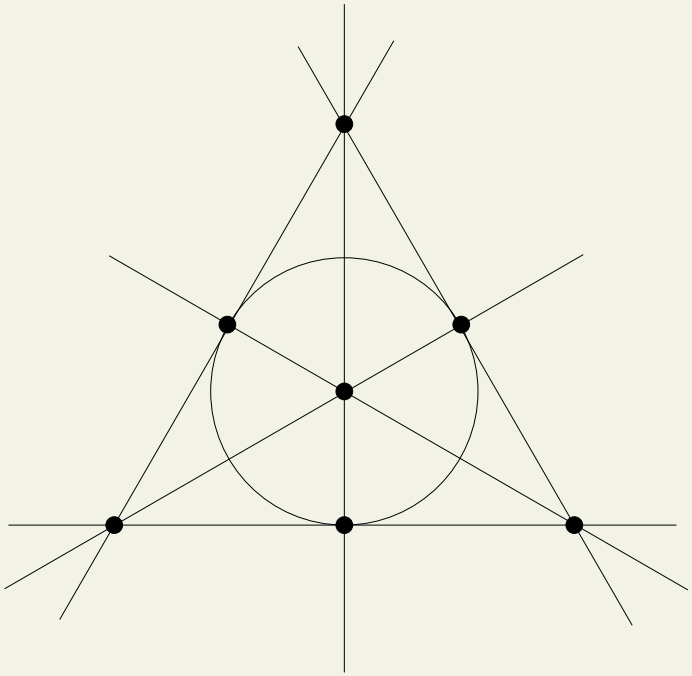
The extended coset leader weights are given by

$$\alpha_0(T) = 1$$

The code itself.

Determination of coset weights

Example



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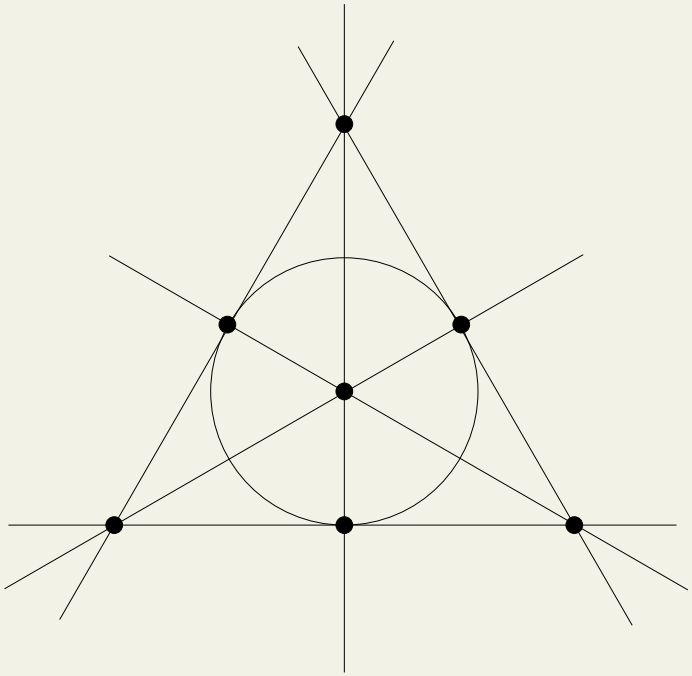
$$\alpha_1(T) = 7(T-1)$$

Seven projective points.

Determination of coset weights

Example



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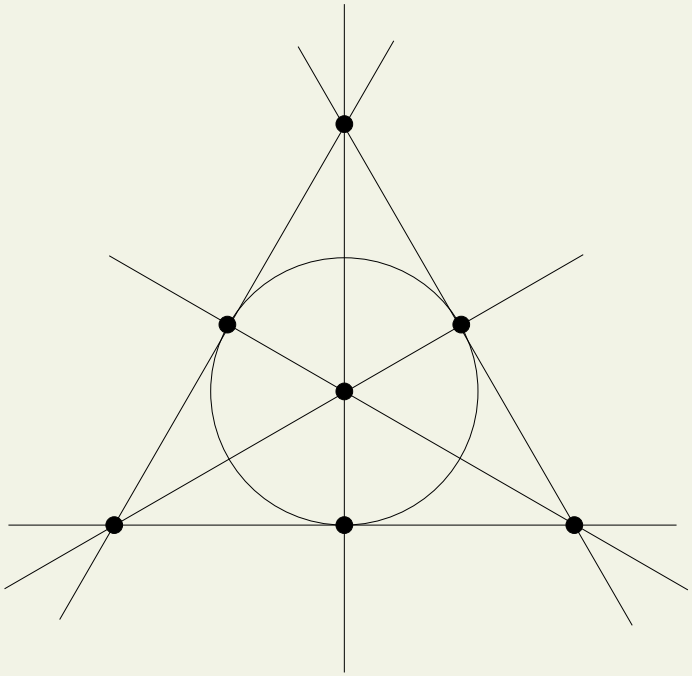
The extended coset leader weights are given by

$$\alpha_2(T) = 7(T-1)(T-2)$$

$(T+1)-3$ extra points on 7 projective lines.

Determination of coset weights

Example



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This is a projective system of seven points in the projective plane.

The extended coset leader weights are given by

$$\alpha_3(T) = (T-1)(T-2)(T-4)$$

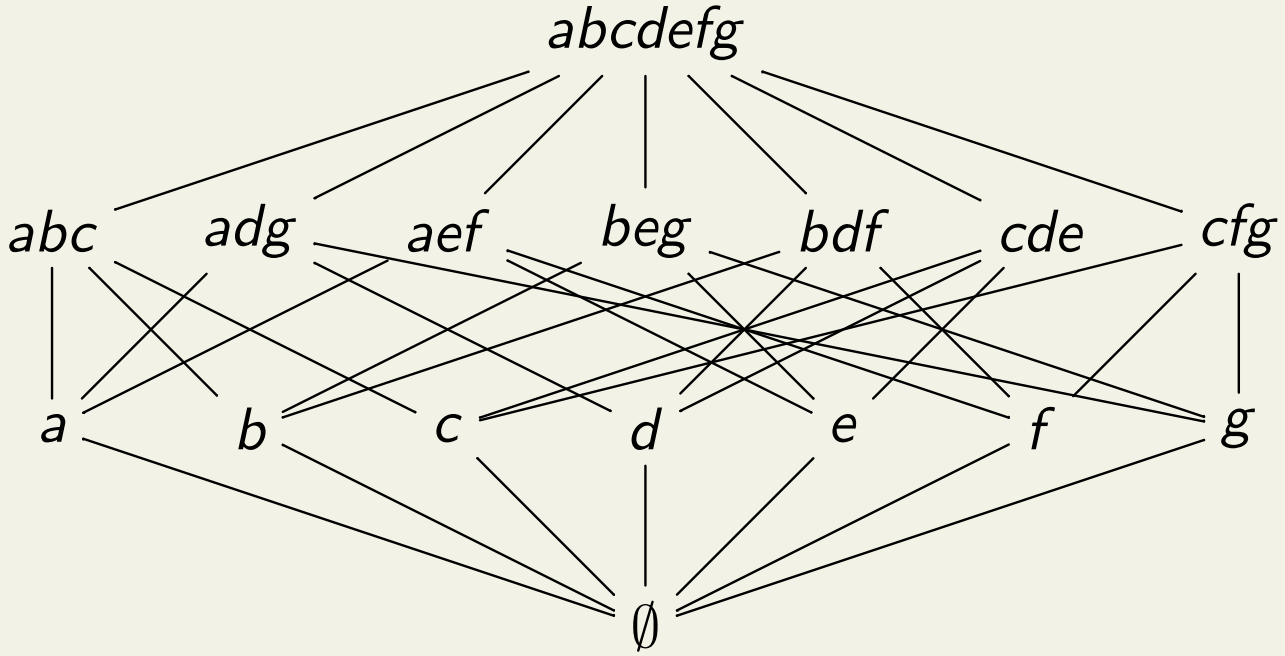
$\alpha_0(T) + \alpha_1(T) + \alpha_2(T) + \alpha_3(T) = T^3$ total number of cosets.

Determination of coset weights

This kind of inclusion/exclusion counting is formalized by the *geometric lattice* associated to a projective system and its *Möbius function*.

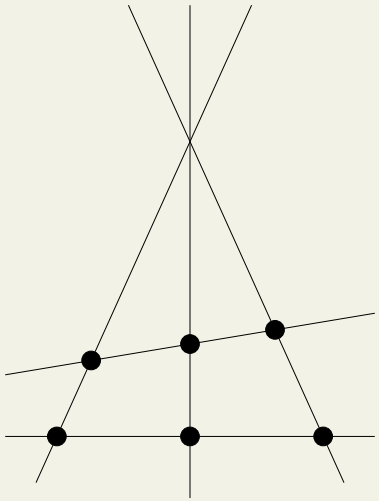
Example (continued)

The geometric lattice associated to the $[7, 4]$ binary Hamming code is visualized by

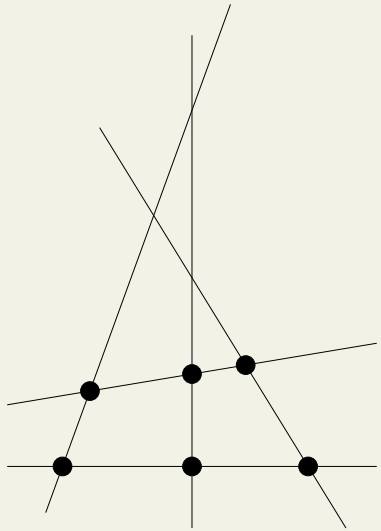


Determination of coset weights

Example



1 coset with
3 coset leaders



3 cosets with
2 leaders each

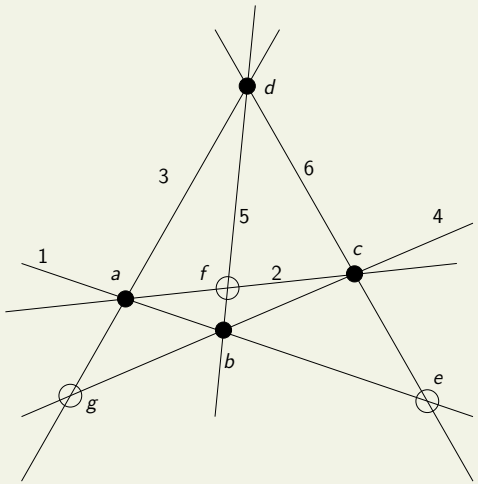
Projective systems with equal geometric lattices may have different coset leader weight enumerators!

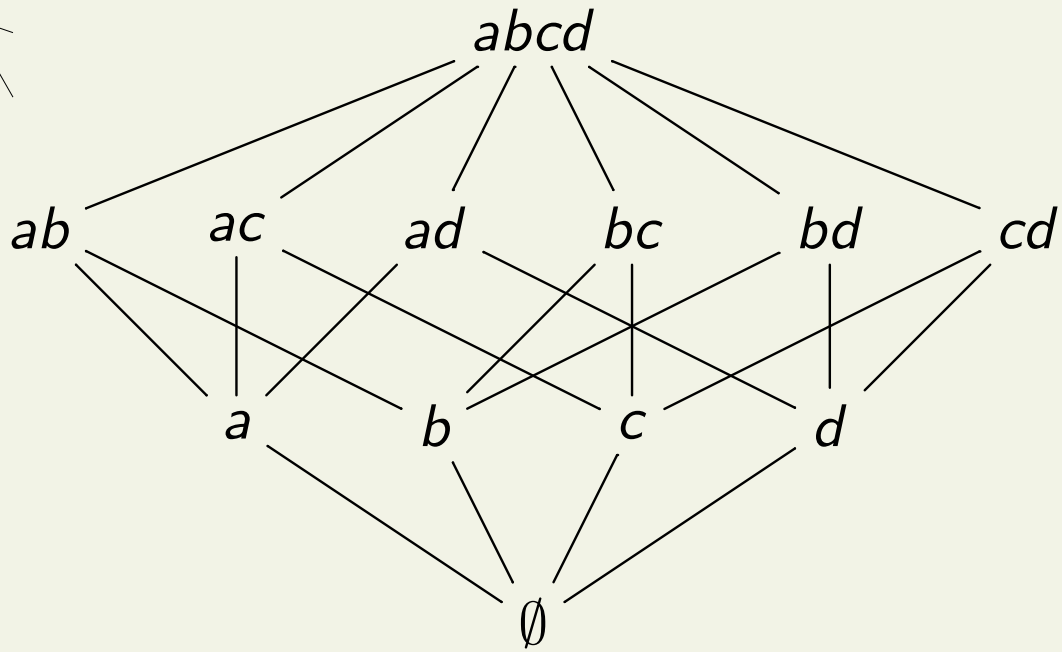
Derived code

- Start with $[n, k]$ code.
- Consider the projective system $\mathcal{P}_C$.
- Look at all hyperplanes spanned by $k-1$ points of $\mathcal{P}_C$.
  (Ignore $k-1$ points that span spaces of lower dimension.)
- Remove (multiple) copies of hyperplanes.
- These hyperplanes form an arrangement $\mathcal{A}$.
- The *derived code* $D(C)$ is the code such that $\mathcal{A} = \mathcal{A}_{D(C)}$.

Derived code

Example





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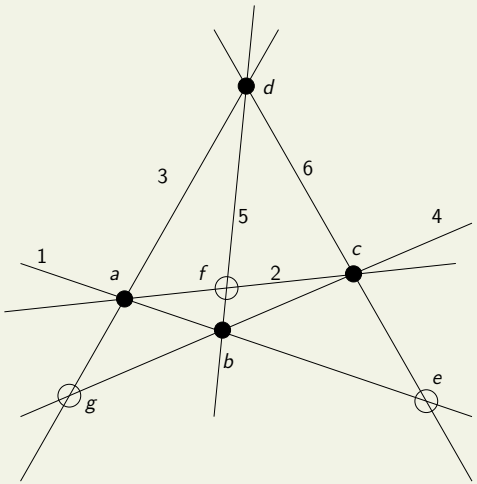

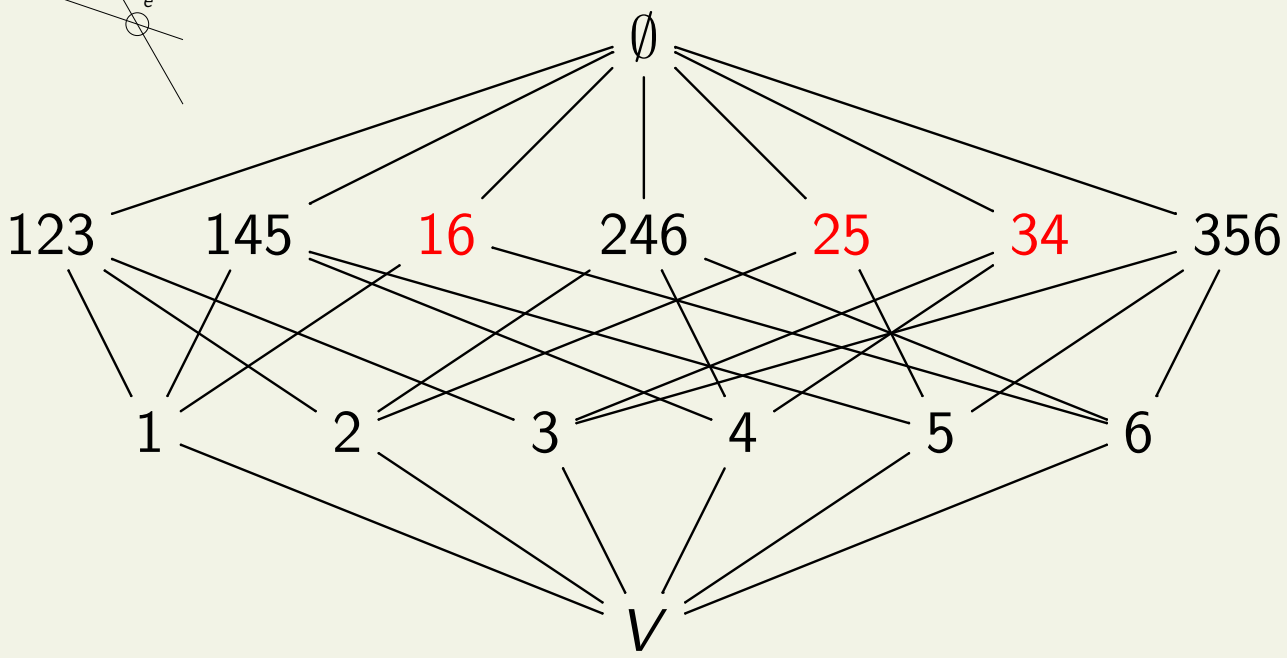

Extended coset leader weight enumerator

- The lattice of $\mathcal{P}_C$, upside-down, is contained in the lattice of $\mathcal{A}_{D(C)}$.
- This gives an injection $\psi : L(\mathcal{P}_C) \hookrightarrow L(\mathcal{A}_{D(C)})$.
- All elements that are not in the image $\psi(L(\mathcal{P}_C))$ should be counted similar to the largest element below it that is in $\psi(L(\mathcal{P}_C))$.
- Therefore, define $r^*(x) = \max\{r(y) : y \in \psi(L(\mathcal{P}_C)), y \leq x\}$.

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Theorem

The extended coset leader weight enumerator is equal to

$$\alpha_C(X, Y, T) = \sum_{x,y \in L(\mathcal{A}_{D(C)})} \mu(x, y) T^{n-k-r(y)} X^{k+r^*(x)} Y^{n-k-r^*(x)}.$$

Summary

- The extended coset leader weight enumerator is an important invariant of linear codes.
- Determining coset weights is equivalent to counting points in spans of points.
- Counting points can be formalized by using the geometric lattice of the derived code.

Further questions

- Does the extended coset leader weight enumerator determine the extended weight enumerator?
- Can we define a *derived lattice*?
- Taking $D(D(D(\cdots(C)\cdots)))$ eventually gives all hyperplanes in $\mathrm{PG}(k-1,q)$. How fast?
- Dependencies between dependencies are known as *second order syzygies* in computational geometry. Can this interpretation help?
- Can we determine $\alpha_C(X,Y,T)$ for concrete classes of codes? (For example: generalized Reed-Solomon codes)
- Can we classify codes using their coset leader weight enumerator?
- ...

Thank you for your attention.