Code Based Cryptology at TU/e

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Content

Coding

- correct transmission of data
- error-correction
- no secrecy involved
- communication: internet, telephone, ...
- fault tolerant computing
- memory: computer compact disc, DVD, USB stick ...

Crypto

- private transmission of data
- secrecy involved
- privacy
- eaves dropping
- insert false messages
- authentication
- electronic signature
- identity fraud

Security

- secure transmission of data
- secrecy involved
- electronic voting
- electronic commerce
- money transfer
- databases of patients

Public-key cryptography (PKC)

- Diffie and Hellman 1976 in the public domain in
- Ellis in 1970 for secret service, not made public until 1997
- advantage with respect to symmetric-key cryptography
- no exchange of secret key between sender and receiver

One-way function

- At the heart of any public-key cryptosystem is a
- one-way function
- a function $y = f(x)$ that is
- easy to evaluate but
- for which it is computationally infeasible **(one hopes)**
- to find the inverse $x = f^{-1}(y)$

Examples of one-way function

- Example 1
- differentiation a function is easy
- integrating a function is difficult

- Example 2
- checking whether a given proof is correct is easy
- finding the proof of a proposition is difficult

Integer factorization

- $x = (p, q)$ is a pair of distinct prime numbers
- $y = pq$ is its product
- proposed by Cocks in 1973 in secret service
- Rivest-Shamir-Adleman (RSA) in 1978 in public domain
- based on the hardness of factorizing integers

Discrete logarithm

- G is a group (written multiplicatively)
- with $a \in G$ and x an integer
- $y = a^x$
- Diffie-Hellman in 1974 and 1976 in public domain
- proposed by Williamson in 1974 in secret service
- based on difficulty of finding discrete logarithms in a finite field

Elliptic curve discrete logarithm

- G is an elliptic curve group (written additively) over a finite field
- P is a point on the curve
- $x = k$ a positive integer k
- $y = kP$ is another point on the curve
- obtained by the multiplication of P with a positive integer k
- proposed by Koblitz and Miller in 1985
- based on the difficulty of inverting this function in G

Code based cryptography

- H is a given $r \times n$ matrix with entries in $\mathbb{F}_q$
- $\mathbf{x}$ is in $\mathbb{F}_q^n$ of weight at most t
- $\mathbf{y} = \mathbf{x}H^T$
- proposed by McEliece in 1978 and later by Niederreiter
- based on the difficulty of decoding error-correcting codes
- it is NP complete

NP complete problems

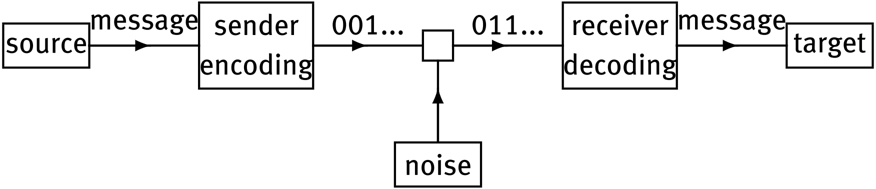
- NP = nondeterministic polynomial time
- given a problem with yes/no answer
- if answer is yes and the solution is given
- then one can check it in polynomial time

- Input: integer n
- Query: can one factorize n in $n = pq$ with p and $q > 1$?
- if answer is yes and someone gives p and q
- then one easily checks that $n = pq$
- otherwise it is difficult to find p and q

Abstract

- error-correcting codes
- error-correcting pairs correct errors efficiently
- applies to many known codes
- prime example Generalized Reed-Solomon codes
- can be explained in a short time
- is a distinguisher of certain classes of codes
- McEliece public-key cryptosystem
- polynomial attack if algebraic geometry codes are used
- ECP map is a one-way function

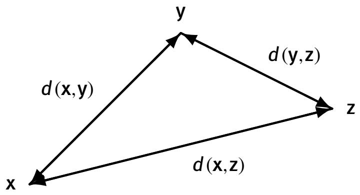
Information theory: Shannon



Block diagram of a communication system

Q alphabet of q elements
Hamming distance between
$\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in Q^n

$$d(\mathbf{x}, \mathbf{y}) = \min |\{i : x_i \neq y_i\}|$$



Triangle inequality

Block codes

C block code is a subset of Q^n

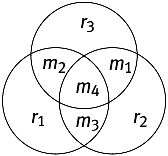
$$d(C) = \min |\{d(\mathbf{x}, \mathbf{y}) \ : \ \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}|$$

minimum distance of C

$$t(C) = \lfloor \frac{d(C) - 1}{2} \rfloor$$

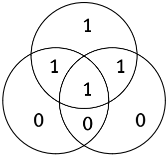
error-correcting capacity of C

Hamming code - 1



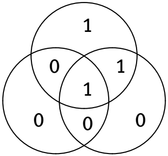
Venn diagram of the Hamming code

Hamming code - 2



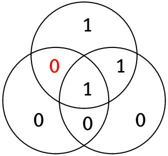
Venn diagram of a code word sent

Hamming code - 3



Venn diagram of a received word

Hamming code - 4



Correction of one error

Linear codes and their parameters

$\mathbb{F}_q$ the finite field with q elements, $q = p^e$ and p prime
$\mathbb{F}_q^n$ is an $\mathbb{F}_q$-linear vector space of dimension n

C linear code is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$

parameters $[n, k, d]_q$ or $[n, k, d]$

q = size finite field
n = length of C
k = dimension of C
d = minimum distance of C

Encoding linear code

Let C a linear code in $\mathbb{F}_q^n$ of dimension k
It has a basis $\mathbf{g}_1, \ldots, \mathbf{g}_k$
Let G be the $k \times n$ matrix with rows $\mathbf{g}_1, \ldots, \mathbf{g}_k$
Then G is called a generator matrix of C

The encoding

$$\mathcal{E} : \mathbb{F}_q^k \longrightarrow \mathbb{F}_q^n$$

of C s given by $\mathcal{E}(\mathrm{m}) = \mathrm{m}G$

Singleton bound

Singleton bound

$$d \leq n - k + 1$$

Maximum Distance Separable (MDS)

$$d = n - k + 1$$

Inner product

The standard inner product is defined by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

Is bilinear and non-degenerate
but "positive definite"makes no sense

Two subsets A and B of $\mathbb{F}_q^n$ are perpendicular:
$A \perp B$ if and only if $\mathbf{a} \cdot \mathbf{b} = 0$ for all $\mathbf{a} \in A$ and $\mathbf{b} \in B$

Dual code

Let C be a linear code in $\mathbb{F}_q^n$
The dual code is defined by

$$C^{\perp} = \{\, \mathbf{x} \;:\; \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C \,\}$$

If C has dimension k, then $C^{\perp}$ has dimension $n - k$

Star product

The star product is defined by coordinatewise multiplication

$$\mathbf{a} * \mathbf{b} = (a_1 b_1, \dots, a_n b_n)$$

For two subsets A and B of $\mathbb{F}_q^n$

$$A * B = \langle \mathbf{a} * \mathbf{b} \mid \mathbf{a} \in A \text{ and } \mathbf{b} \in B \rangle$$

Efficient decoding algorithms

The following classes of codes:

- Generalized Reed-Solomon codes
- Cyclic codes
- Alternant codes
- Goppa codes
- Algebraic geometry codes

have efficient decoding algorithms:

- Arimoto, Peterson, Gorenstein, Zierler
- Berlekamp, Massey, Sakata
- Justesen et al., Vladut-Skrobogatov, ...
- Error-correcting pairs

Error-correcting pair

Let C be a linear code in $\mathbb{F}_q^n$

The pair (A, B) of linear subcodes of $\mathbb{F}_{q^m}^n$ is a called a t-error correcting pair (ECP) over $\mathbb{F}_{q^m}$ for C if

- E.1 $(A * B) \perp C$
- E.2 $k(A) > t$
- E.3 $d(B^\perp) > t$
- E.4 $d(A) + d(C) > n$

Generalized Reed-Solomon codes- 1

Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of mutually distinct elements of $\mathbb{F}_q$
Let $\mathbf{b} = (b_1, \ldots, b_n)$ be an n-tuple of nonzero elements of $\mathbb{F}_q$
Evaluation map:

$$\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) = (f(a_1)b_1, \ldots, f(a_n)b_n)$$

$GRS_k(\mathbf{a}, \mathbf{b}) = \{ \mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) \mid f(X) \in \mathbb{F}_q[X], \deg(f(X) < k \}$

Parameters: $[n, k, n-k+1]$ if $k \leq n$

Since a polynomial of degree $k-1$ has at most $k-1$ zeros.

Generalized Reed-Solomon codes - 2

Furthermore

$$\mathrm{ev}_{\mathbf{a},\mathbf{b}}(f(X)) * \mathrm{ev}_{\mathbf{a},\mathbf{c}}(g(X)) = \mathrm{ev}_{\mathbf{a},\mathbf{b}*\mathbf{c}}(f(X)g(X))$$

$$GRS_k(\mathbf{a}, \mathbf{b}) * GRS_l(\mathbf{a}, \mathbf{c}) = GRS_{k+l-1}(\mathbf{a}, \mathbf{b} * \mathbf{c})$$

t-ECP for $GRS_{n-2t}(\mathbf{a}, \mathbf{b})$

Let $C^\perp = GRS_{2t}(\mathbf{a}, \mathbf{1})$
Then $C = GRS_{n-2t}(\mathbf{a}, \mathbf{b})$ for some $\mathbf{b}$
has parameters: $[n, n-2t, 2t+1]$

Let $A = GRS_{t+1}(\mathbf{a}, \mathbf{1})$ and $B = GRS_t(\mathbf{a}, \mathbf{1})$
Then $(A * B) \subseteq C^\perp$

A has parameters $[n, t+1, n-t]$
B has parameters $[n, t, n-t+1]$
So $B^\perp$ has parameters $[n, n-t, t+1]$

Hence (A, B) is a t-error-correcting pair for C

Kernel of a received word

Let A and B be linear subspaces of $\mathbb{F}_{q^m}^n$
and $\mathbf{r} \in \mathbb{F}_q^n$ a received word
Define the kernel

$$K(\mathbf{r}) = \{ \mathbf{a} \in A \mid (\mathbf{a} * \mathbf{b}) \cdot \mathbf{r} = 0 \text{ for all } \mathbf{b} \in B \}$$

Lemma
Let C be an $\mathbb{F}_q$-linear code of length n
Let $\mathbf{r}$ be a received word with error vector $\mathbf{e}$
So $\mathbf{r} = \mathbf{c} + \mathbf{e}$ for some $\mathbf{c} \in C$
If $(A * B) \subseteq C^{\perp}$, then

$$K(\mathbf{r}) = K(\mathbf{e})$$

Kernel for a GRS code

Let $A = GRS_{t+1}(\mathbf{a}, \mathbf{1})$ and $B = GRS_t(\mathbf{a}, \mathbf{1})$ and $C = \langle A * B \rangle^{\perp}$

Let
$\mathbf{a}_i = \mathrm{ev}_{\mathbf{a},1}(X^{i-1})$ for $i = 1, \ldots, t+1$
$\mathbf{b}_j = \mathrm{ev}_{\mathbf{a},1}(X^{j})$ for $j = 1, \ldots, t$
$\mathbf{h}_l = \mathrm{ev}_{\mathbf{a},1}(X^{l})$ for $l = 1, \ldots, 2t$

Then
$\mathbf{a}_1, \ldots, \mathbf{a}_{t+1}$ is a basis of A
$\mathbf{b}_1, \ldots, \mathbf{b}_t$ is a basis of B
$\mathbf{h}_1, \ldots, \mathbf{h}_{2t}$ is a basis of $C^{\perp}$

Furthermore

$$\mathbf{a}_i * \mathbf{b}_j = \mathrm{ev}_{\mathbf{a},1}(X^{i+j-1}) = \mathbf{h}_{i+j-1}$$

Matrix of syndromes for a GRS code

Let r be a received word and
$(s_1, \ldots, s_{2t}) = \mathrm{r}H^T$ its syndrome
Then

$$(\mathbf{b}_j * \mathbf{a}_i) \cdot \mathbf{r} = s_{i+j-1}.$$

To compute the kernel $K(\mathbf{r})$ we have to compute
the null space of the matrix of syndromes

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_t & s_{t+1} \\ s_2 & s_3 & \cdots & s_{t+1} & s_{t+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_t & s_{t+1} & \cdots & s_{2t-1} & s_{2t} \end{pmatrix}$$

Let (A, B) be a t-ECP for C
Let J be a subset of $\{1, \ldots, n\}$
Define the subspace of A of error-locating vectors:

$$A(J) = \{ \mathbf{a} \in A \mid a_j = 0 \text{ for all } j \in J \}$$

Lemma
Let $(A * B) \perp C$
Let $\mathbf{e}$ be an error vector of the received word $\mathbf{r}$
If $I = \mathrm{supp}(\mathbf{e}) = \{ i \mid e_i \neq 0 \}$, then

$$A(I) \subseteq K(\mathbf{r})$$

Lemma

Let $(A * B) \perp C$

Let $\mathbf{e}$ be an error vector of the received word $\mathbf{r}$

Assume $d(B^{\perp}) > \mathrm{wt}(\mathbf{e}) = t$

If $I = \mathrm{supp}(\mathbf{e}) = \{ i \mid e_i \neq 0 \}$, then

$$A(I) = K(\mathbf{r})$$

If $\mathbf{a}$ is a nonzero element of $K(\mathbf{r})$

J zero positions of $\mathbf{a}$

Then

$$I \subseteq J$$

Basic algorithm

Let (A, B) be a t-ECP for C with $d(C) \geq 2t + 1$
Suppose that $\mathbf{c} \in C$ is the code word sent and $\mathbf{r} = \mathbf{c} + \mathbf{e}$ is the received word for some error vector $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) \leq t$

The basic algorithm for the code C:

- Compute the kernel $K(\mathbf{r})$

  This kernel is nonzero since $k(A) > t$
- Take a nonzero element $\mathbf{a}$ of $K(\mathbf{r})$

  $K(\mathbf{r}) = K(\mathbf{e})$ since $(A * B) \perp C$
- Determine the set J of zero positions of $\mathbf{a}$

  $\mathrm{supp}(\mathbf{e}) \subseteq J$ since $d(B^\perp) > t$
- Compute the error values by erasure decoding

  $|J| < d(C)$ since $n - d(A) < d(C)$

t-ECP corrects t errors efficiently

Theorem

Let C be an $\mathbb{F}_q$-linear code of length n
Let (A, B) be a t-error-correcting pair over $\mathbb{F}_{q^m}$ for C

Then the basic algorithm corrects t errors
for the code C with complexity $\mathcal{O}((mn)^3)$

Code based PKC systems - 1

McEliece:
Let $\mathcal{C}$ be a class of codes that have
efficient decoding algorithms correcting t errors with $t \leq (d-1)/2$

Secret key: (S, G, P)

– S an invertible $k \times k$ matrix
– G a $k \times n$ generator matrix of a code C in $\mathcal{C}$.
– P an $n \times n$ permutation matrix

Public key: $G' = SGP$

Code based PKC systems - 2

McEliece:

Encryption with public key $G' = SGP$ and message $\mathbf{m}$ in $\mathbb{F}_q^k$:

$$\mathbf{y} = \mathbf{m}G' + \mathbf{e}$$

with random chosen $\mathbf{e}$ in $\mathbb{F}_q^n$ of weight t

Decryption with secret key (S, G, P):

$$\mathbf{y}P^{-1} = (\mathbf{m}G' + \mathbf{e})P^{-1} = \mathbf{m}SG + \mathbf{e}P^{-1}$$

SG and G are generator matrices of the same code C
$\mathbf{e}P^{-1}$ has weight t
Decoder gives $\mathbf{c} = \mathbf{m}SG$ as closest codeword

Code based PKC systems - 3

Minimum distance decoding is NP-hard
(Berlekamp-McEliece-Van Tilborg)

It is assumed that:

1. $P \neq NP$
2. Decoding up to half the minimum distance is hard
3. One cannot distinguish nor retrieve the original code by disguising it by S and P

Attacks on code based PKC systems - 1

Generic attack – decoding algorithms:

– McEliece 1978

.....

– Brickell, Lee 1988

– Leon 1988

– van Tilburg 1988

– Stern 1989

– Canteaut, Chabaud, Sendrier 1998

– Finiasz-Sendrier 2009

– Bernstein-Lange-Peters 2008-2011

– Becker-Joux-May-Meurer Eurocrypt 2012

Attacks on code based PKC systems - 2

Structural attacks:

– GRS codes (Sidelnikov-Shestakov)
– subcodes of GRS codes (Wieschebrink, Márquez-Martínez-P)
– Alternant codes: open
– Goppa codes: open
– Algebraic geometry codes: (Faure-Minder, genus $g \leq 2$)
– VSAG codes: (Márquez-Martínez-P-Ruano, arbitrary g)
– Polynomial attack on AG codes: (Couvreur-Márquez-P, using ECP's)

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Codes with t-ECP

$\mathscr{P}(n, t, q)$ is the collection of pairs (A, B) that satisfy

$$\begin{array}{ll} E.2 & k(A) > t \\ E.3 & d(B^\perp) > t \\ E.5 & d(A^\perp) > 1 \\ E.6 & d(A) + 2t > n \end{array}$$

Let

$$C = \mathbb{F}_q^n \cap (A * B)^\perp$$

Then $d(C)$ is at least $2t + 1$
and (A, B) is a t-ECP for C

ECP one-way function

$\mathscr{F}(n, t, q)$ is the collection of $\mathbb{F}_q$-linear codes of length n and minimum distance $d \geq 2t + 1$

Consider the following map

$$\begin{array}{rccc} \varphi_{(n,t,q)}: & \mathscr{P}(n,t,q) & \longrightarrow & \mathscr{F}(n,t,q) \\ & (A,B) & \longmapsto & C \end{array}$$

Question:
Is this a one-way function?

Conclusion

- Many known classes of codes
- that have decoding algorithm correcting t-errors
- have a t-ECP
- and are not suitable for a code based PKC

Question for future research
Is the ECP map a one-way function?

Thank you for your attention!