HYPERSURFACE SINGULARITIES
AND
RESOLUTIONS OF JACOBI MODULES

HYPEROPPERVLAK SINGULARITEITEN
EN
RESOLUTIES VAN JACOBI MODULEN
(MET EEN SAMENVATTING IN HET NEDERLANDS)

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. O.J. DE JONG, VOLGENS BESLUIT VAN HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP MAANDAG 2 DECEMBER 1985 DES NAMIDDAGS TE 12.00 UUR

DOOR

GERARDUS RUDOLF PELLIKAAN

GEBOREN OP 30 MAART 1953 TE DJAKARTA

DRUKKERIJ ELINKWIJK BV - UTRECHT

STELLINGEN

behorende bij het proefschrift: "Hypersurface singularities and Jacobi modules" van Ruud Pellikaan

1. Het vermoeden dat S.M. Husein-Zade in 1977 in zijn artikel: "The monodromy groups of isolated hypersurface singularities", Russian Math. Surveys 32 (1977), 23-65, stelde, is reeds door J.H.C. Whitehead in 1936 opgelost.

2. Laten f en g twee analytische functies zijn met een geïsoleerde singulariteit in 0. Als de Milnor ring van f een quotient is van de Milnor ring van g dan $g \to f$, dat wil zeggen g grenst aan f.

3. Voor Noetherse topologische ruimten is de krull dimensie gelijk aan de dimensie van Menger en Urysohn.

4. Het vermoeden van D. Siersma:
$j_f = \#\{A_1 \text{ punten}\} + \#\{D_\infty \text{ punten}\}$
voor een geïsoleerde lijnsingulariteit, is juist.
Zie opmerking (7.2.3) van dit proefschrift.

5. De tekening op bladzijde 4 van het boek: "Introduction to intersection theory in algebraic geometry" van W. Fulton stelt geen "pinch point" voor, maar een zogenaamde $J_{2,\infty}$ singulariteit

6. Men dient de 1-vormen van een locale analytische ring geen Kähler differentialen te noemen.

7. Het symmetrie argument van D. Hofstadter in zijn behandeling van het "multiperson prisoners dilemma" gebruikt, is onvolledig. Zie: Scientific American, juni 198

NEDERLANDSE SAMENVATTING

- We beschouwen kiemen van analytische functies $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ die een gegeven analytische verzameling Σ, gedefinieerd door een ideaal I, in hun singuliere locus hebben. We geven de collectie van zulke functies aan met $\int I$.
- We kijken naar de groep $\mathcal{D}_I$ van alle locale isomorfismen die Σ invariant laten. We berekenen de raakruimte $\tau_I(f)$ aan de baan of $\mathcal{D}_I$ in f van de groepswerking van $\mathcal{D}_I$ op $\int I$.
- We laten zien dat eindige codimensie van $\tau_I(f)$ in $\int I$ eindige bepaaldheid impliceert en we geven een miniversele ontvouwing voor zo'n functie.
- We gaan nader in op functies met één dimensionale volledige doorsnede als singuliere locus en transversaal op $\Sigma\setminus\{0\}$ alleen A_1 singulariteiten. We laten zien dat zulke functies deformeren in functies met alleen zogenaamde A_1, A_∞ en D_∞ singulariteiten. We berekenen de aantallen van zulke singulariteiten.
- Voor een functie en een coördinaat functie x waarvoor geldt dat $x=0$ de singuliere locus alleen in 0 snijdt, bestuderen we series geïsoleerde singulariteiten $f+x^k$ voor $k\gg 0$ en brengen de invarianten van f en $f+x^k$ met elkaar in verband.
- In hoofdstuk II bewijzen we dat het quotient I/J van twee idealen I en J in een ring R met $J\subset I$, een eindige projectieve resolutie heeft, onder zekere voorwaarden. Dit hoofdstuk kan opzichzelf gelezen worden.
- Het resultaat uit hoofdstuk II wordt op enkele essentiële plaatsen in hoofdstuk I toegepast.

DANKWOORD

Ik wil op deze bladzijde mijn dank betuigen aan allen die tot mijn proefschrift hebben bijgedragen.

Mijn promotor D. Siersma voor zijn ideeën en vragen en zijn voortdurende belangstelling.

Alle deelnemers aan het singulariteiten project in het bijzonder E. Looijenga en J. Steenbrink. Ik heb altijd met groot plezier aan de twee maandelijkse bijeenkomsten deelgenomen en er veel van geleerd.

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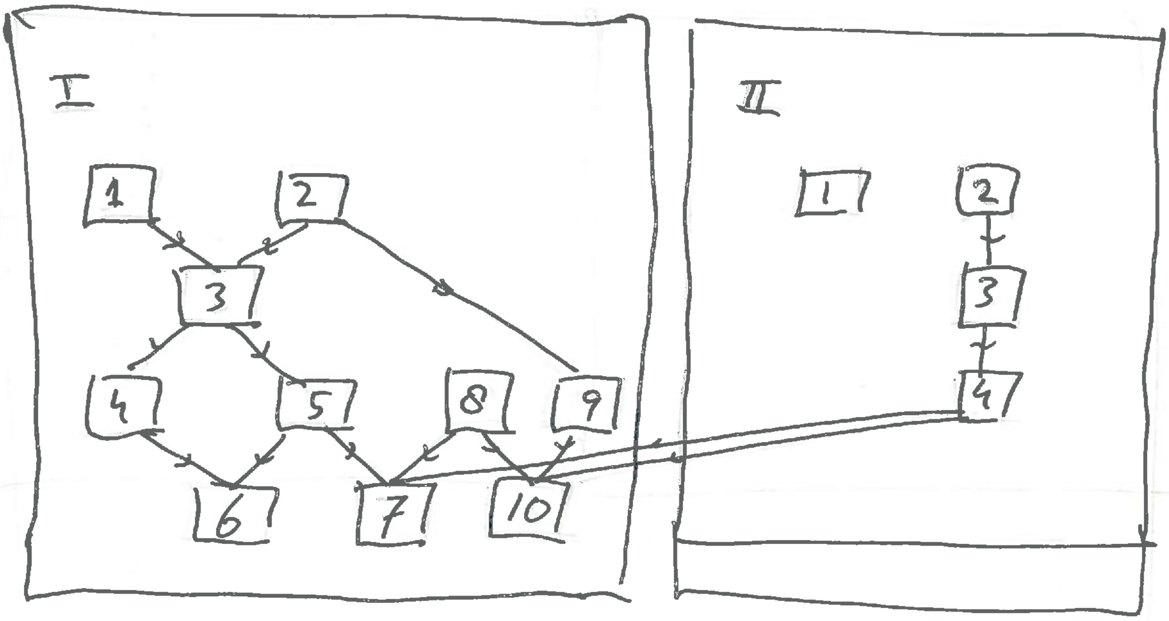
A. Koomans voor de snelle en uitstekende wijze waarop ze dit proefschrift heeft getypt.

H. van Dorp voor het ontwerp van de omslag. Hij heeft daarmee een extra dimensie toegevoegd.

Verder wil ik iedereen bedanken die mij op indirecte wijze tot steun is geweest.

CONTENTS

SCHEMATIC PLAN OF THE WORK



The two chapters have their own introduction and references. Every section starts with a short summary of the results.
The first chapter presupposes
the second one at a few essential places.
The second chapter is completely independent of the first one.

CHAPTER I NON-ISOLATED HYPERSURFACE SINGULARITIES

INTRODUCTION

We consider germs of analytic functions $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ i.e. f is analytic on a neighbourhood of 0 in $\mathbb{C}^m$ and f restricted to a smaller neighbourhood gives the same germ.
Any two of such germs f and g are called ~~right~~ right-equivalent if there exists a local analytic isomorphism $h:(\mathbb{C}^m,0)\to(\mathbb{C}^m,0)$ such that $f=g\circ h$, i.e. f is the same as g after a coördinate transformation.

(i) The ultimate goal is to classify germs of analytic function under the right-equivalence relation and give lists of normal forms and enough invariants to determine the types.

(ii) Less ambitious questions are. Is a germ of an analytic function right-equivalent with a polynomial? Or stronger: Is ~~an~~ such a germ right-equivalent with a finite part of its Taylor-expansion and can one give ~~any~~ an a priori bound? These are so called finite determinacy questions. The solution is important in order to answer (i).

(iii) Can one give universal unfoldings of f to which every unfolding of a given f belongs?

away This ~~(iv) gives rise to the concept adjacency, i.e. f is adjacent to g if f is arbitrary close to g in the universal unfolding of g. This gives an hierarchy of the singularities~~

(iv) Can one deform a function f into f_t such that for all $t\neq 0$ small enough f_t has only certain simple singularities one knows very well? What are the numbers of these simple singularities?

(v) What can one say about the topology of the fibres $f^{-1}(s)$ for $s\neq 0$?

If f has an isolated singularity, i.e. 0 is the only singularity of f on a neighbourhood of 0 in $\mathbb{C}^m$ then these questions

have nice answers: Thom and Mather answered (ii) and (iii). Arnold made a beginning of the classication. Milnor gave the first results concerning the nearby fibres $f^{-1}(s)$ beyond the curve case.
We give answers to these questions, except the first one, for functions with arbitrary singularities. If the singular locus is one dimensional and the transversal type to the singular locus is as simple as possible outside the origin, then we give the best results.

Let $\mathcal{O}$ be the local ring of germs of analytic functions $f:(\mathbb{C}^m,0)\to\mathbb{C}$ and let $\mathfrak{m}$ be its maximal ideal. We denote the ideal generated by the partial derivatives $(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_m})\mathcal{O}$ by J_f and call it the Jacobi ideal of f.
Concerning question (ii) one usually considers the group $\mathcal{D}$ of local analytic isomorphisms $h:(\mathbb{C}^m,0)\to(\mathbb{C}^m,0)$ and the orbits ~~of~~ $\mathcal{D}$ of the action of $\mathcal{D}$ on $\mathcal{O}$. The tangent space to the orbit at f is $\mathfrak{m}J_f$. For isolated singularities has $\mathfrak{m}J_f$ finite codimension in $\mathcal{O}$, which allows one to integrate a certain differential equation in order to get finite determinacy results. Since $\mathcal{O}/J_f$ is a finite dimensional $\mathbb{C}$-vector space one can define ~~$F(z,t)=f(z)+\sum t_i\xi_i$~~ $F(z,t)=f(z)+\sum_i t_i\xi_i(z)$, where $\xi_1,\dots,\xi_\mu\in\mathcal{O}$ project to a basis of $\mathcal{O}/J_f$ over $\mathbb{C}$. One can prove that this F is an universal unfolding of f. For non-isolated singularities we no longer have that $\mathfrak{m}J_f$ has finite codimension ~~and~~ We look at a smaller space of functions instead. We consider all functions which have a given analytic space Σ, defined by an ideal I in $\mathcal{O}$, in their singular locus, i.e. $(f)+J_f\subseteq I$. The set of such functions is an ideal, we call it the primitive ideal of I and denote it by $\int I$. This ideal $\int I$ contains the main objects we want to study, see §1. The corresponding right-equivalence relation is defined by the subgroup $\mathcal{D}_I$ of $\mathcal{D}$ of all local isomorphisms h which leave Σ invariant, i.e. $h(\Sigma)=\Sigma$ or what is the same $h^*(I)=I$. Then $\mathcal{D}_I$ acts

on $\int I$ and the orbits are the right-I-equivalence classes, see §2. This was done by Siersma [Si] for functions with a line as singular locus. We could generalise the concepts to an arbitrary analytic space. A new aspect is that we no longer give an ad hoc definition of the tangent space $\tau_I(f)$ to the orbit of $\mathcal{D}_I$.
In section 3 we give normal forms of certain "generic" singularities, i.e. of functions with a singular locus which is regular and of I-codimension of $\tau_I(f)$ in $\int I = 0$.
We identify the tangent space $\tau_I(f)$ in case the I-codimension $c_I(f)$ is finite and prove $\tau_I(f) = m J_f \cap \int I$.
In section 4 we obtain a finite determinacy and an unfolding theorem in case that $c_I(f)$ is finite. This is a straight forward generalization of well known methods as formalized by Damon [D].
We give two applications to functions which are right-equivalent to polynomials. We also give a counter example to a question posed by Bochnak and Kucharz.
Questions (iv) and (v) have the following answer for isolated singularities. The simples singularity is the so called Morse or A_1 singularity given by $f(z) = z_1^2 + \dots + z_m^2$ in certain local coördinates. Every isolated singularity has an approximation $f_t(z) = f(z) + tg(z)$ such that for all $t \neq 0$ small enough f_t has only Morse singularities and the number of these singularities is equal to $\mu = \dim_{\mathbb{C}} \mathcal{O}/J_f$.
This is an important step in order to give a description of the homotopy type of the nearby fibre's $f^{-1}(s)$ of $f^{-1}(0)$, the so called Milnor fibre. They are homotopy equivalent with a bouquet of μ n-spheres. The property that all these a priori different numbers μ are the same is a consequence of the fact that a deformation of $\mathcal{O}/(f)$ gives also a deformation of $\mathcal{O}/J_f$, i.e. the module $\hat{\mathcal{O}}/J_F$ is a free $\mathbb{C}\{t\}$-module of rank μ, where F is an unfolding of f with parameter t and $J_F = (\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_m})\hat{\mathcal{O}}$ and $\hat{\mathcal{O}} = \mathbb{C}\{z_1, \dots, z_m, t\}$ the convergent power series in $z_1, \dots, z_m, t$.
For non-isolated singularities $\mathcal{O}/J_f$ is no longer finite dimensional

We look at I/J_f instead. We call it a Jacobi module. and we prove that $\tilde{I}/J_F$ is a free $\mathbb{C}\{t\}$-module of rank $j_f = \dim_{\mathbb{C}}(I/J_f)$, if j_f is finite and I defines a curve in $\mathbb{C}^m$ and F is a deformation of the pair (f, Σ), see §7. This is done with the help of a result in of chapter II. The functions with j_f finite are characterised in section 5. We prove that they exist in case I defines a curve. If moreover I defines a complete intersection curve Σ then we prove in section 7 that f with j_f finite has an approximation f_t such that for all $t \neq 0$ small enough f_t has only A_1, A_∞ and D_∞ singularities, moreover

$$j_f = \#\{A_1 \text{ points}\} + \#\{D_\infty \text{ points}\}.$$

The sections 8, 9 and 10 are concerned with series of isolated singularities one obtains by considering $f + x^k$, where f is a function with a one dimensional singular locus Σ and x is a coördinate function such that $V(x) \cap \Sigma = \{0\}$. We relate the invariants of f and $f + x^k$, and we can compute the Euler characteristic of the Milnor fibre of f.

For a short summery of results we refer to the beginning of every section.
For basic definitions and facts about isolated singularities we refer to [Lo].
For results about analytic spaces we refer to [G-R] and [Na].
For definitions and facts about commutative algebra we refer to [Ma].
Most of the time $\mathcal{O}$ denotes the local ring of germs of analytic functions $f : (\mathbb{C}^m, 0) \to \mathbb{C}$, but sometimes in proofs $\mathcal{O}$ is the sheaf of analytic functions on an open set U in $\mathbb{C}^m$. ~~We~~ Then we denote by $\mathcal{O}_x$ the stalk at $x \in U$.
We denote the Krull dimension of a module M by $\operatorname{Kdim} M$.
References are at the end of this chapter.

§1 The primitive ideal

Given a germ of an analytic space $(\Sigma,0)$ in $(\mathbb{C}^m,0)$, we want to consider all germs of analytic functions $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ with $(\Sigma,0)$ in their singular locus.

Most of the time $\mathcal{O}$ will be the local ring of germs of analytic functions $f:(\mathbb{C}^m,0)\to\mathbb{C}$ and $\mathfrak{m}$ its maximal ideal. Sometimes $\mathcal{O}$ denotes the sheaf of analytic functions on an open set U of $\mathbb{C}^m$. For an analytic function f we denote by J_f its Jacobi ideal, i.e. $J_f=(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_m})\mathcal{O}$ in local coördinates. Let I be an ideal in $\mathcal{O}$ defining the analytic space Σ, then $\mathcal{O}_\Sigma=\mathcal{O}/I$.

Definition (1.1) The *primitive ideal* $\int I$ of an ideal I in $\mathcal{O}$ is defined as $\int I=\{f\in\mathcal{O}\mid (f)+J_f\subseteq I\}$.

Remark (1.2) The singular set of f is $\Sigma_f=V(J_f)$ and $\Sigma_f=V(J_f)=V((f)+J_f)$ in case $f(0)=0$, since $f^N\in J_f$ for $N\gg 0$ [B-S]. Therefore $f\in\int I$ implies $\Sigma\subseteq\Sigma_f$. The converse also holds in case I is a radical ideal and one requires $f\in\mathfrak{m}$. From the sum and product formulae for differentiation follows that $\int I$ is an ideal.

Property (1.3)
(i) Let I_1 and I_2 be ideals in $\mathcal{O}$, then $\int(I_1\cap I_2)=\int I_1\cap\int I_2$
(ii) Let I be an ideal in $\mathcal{O}$, then $I^2\subseteq\int I\subseteq I$.
These properties follow directly from the definition.

Remark (1.4) The ideal $\int I$ is known in the following setting, altough there was no special name for it. Let $\mathcal{O}$ be the local ring $\mathcal{O}_{(\mathbb{C}^m,0)}$ and I an ideal in $\mathcal{O}$ defining $(\Sigma,0)$, i.e. $\mathcal{O}_{\Sigma,0}=\mathcal{O}/I$. Let $\Omega_{(\mathbb{C}^m,0)}$ and $\Omega_{(\Sigma,0)}$ be the modules of differentials of $\mathcal{O}$ and $\mathcal{O}_{(\Sigma,0)}$ resp. over $\mathbb{C}$ (see [G-R]). Then we have the following exact sequence of $\mathcal{O}_{(\Sigma,0)}$-modules.

$$I/I^2\xrightarrow{d}\Omega_{(\mathbb{C}^m,0)}\otimes_{\mathcal{O}}\mathcal{O}_{(\Sigma,0)}\longrightarrow\Omega_{(\Sigma,0)}\longrightarrow 0$$

Let $z_1, \dots, z_m$ be local coördinates of $(\mathbb{C}^m, o)$, then $\Omega_{(\mathbb{C}^m,o)}$ is a free $\mathcal{O}$-module generated by the basis $dz_1, \dots, dz_m$. The following sequence is exact

$$0 \to I/\int I \xrightarrow{d} \Omega_{(\mathbb{C}^m,o)} \otimes_{\mathcal{O}} \mathcal{O}_{(\Sigma,o)} \to \Omega_{(\Sigma,o)} \to 0$$

, by definition of $\int I$.

__Definition__ (1.5) For a prime ideal $\mathfrak{p}$ in a noetherian ring R one defines the __n^{th} symbolic power__ $\mathfrak{p}^{(n)}$ as follows

$$\mathfrak{p}^{(n)} = R \cap (\mathfrak{p}^n R_{\mathfrak{p}}) \qquad \text{(see [Ma], §8).}$$

More generally. Let I be a radical ideal in R, we have a prime decomposition $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$, with $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ prime ideals in R such that $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$. This decomposition is unique up to the ordering of $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. We define the n^{th} symbolic power of I as follows

$$I^{(n)} = \mathfrak{p}_1^{(n)} \cap \dots \cap \mathfrak{p}_r^{(n)}.$$

__Proposition__ (1.6) Let I be a radical ideal in $\mathcal{O}$ then

$$\int I = I^{(2)} \quad \text{and more generally}$$

$$\int I^{(n)} = I^{(n+1)} = \left\{ f \in I \;\middle|\; \frac{\partial^n f}{\partial z_{i_1} \cdots \partial z_{i_n}} \in I \text{ for all } 1 \le i_1 \le \dots \le i_n \le m \right\}$$

__Proof__ The proof is a consequence of Zariski's main lemma on holomorphic functions ([Z], [B]). Let U be an open set in $\mathbb{C}^m$ and $\mathcal{O}$ the sheaf of analytic functions on U. Let the reduced analytic space Σ be defined by the radical ideal I. We may assume that I is prime by (1.3)(i) and (1.5), so Σ is irreducible. Let $\dim \Sigma = 0$ then I is maximal ideal generated by $z_1, \dots, z_m$, some local coördinates of $(\mathbb{C}^m, x)$. To prove the claims in this situation is an easy computation. Suppose $d = \dim \Sigma \geq 1$ and let V be the regular locus of Σ, then V is an open dense set in Σ, since Σ is reduced of dimension ≥ 1. For every point $x \in V$ the ideal $I\mathcal{O}_x$ is generated by $(z_1, \dots, z_{m-d})$, with $z_1, \dots, z_m$ local coördinates of $(\mathbb{C}^m, x)$ and by the same computation as in case of a maximal ideal we have $I^{(n+1)}\mathcal{O}_x = I^{n+1}\mathcal{O}_x = \int I^{(n)}\mathcal{O}_x$

Hence $I^{(n+1)}\mathcal{O}_I = \int I^{(n)}\mathcal{O}_I$. The local ring $\mathcal{O}_I$ is obtained by localizing $\mathcal{O}$ at the prime ideal I. So $I^{(n+1)}$ is a I-primary component of $\int I^{(n)}$.

Let $f \in I^{(n+1)}$, then $af \in I^{n+1}$ for some $a \in \mathcal{O}\setminus I$. So $a\frac{\partial f}{\partial z_i} \equiv \frac{\partial(af)}{\partial z_i} \equiv 0 \mod I^n$. Thus $(f) + J_f \subseteq I^{(n)}$ and we proved

$I^{(n+1)} \subseteq \int I^{(n)}$. Thus $I^{(n+1)}$ is a I-primary component of $\int I^{(n)}$ and contained in $\int I^{(n)}$. So they must be equal.

By induction we get $I^{(n+1)} = \int\cdots\int I$, the n-fold primitive ideal of I. And $\int\cdots\int I = \{f \in I \mid \frac{\partial^n f}{\partial z_{i_1}\cdots\partial z_{i_n}} \in I \text{ for all } 1\le i_1 \le \cdots \le i_n \le m\}$ is a direct consequence of the definition of $\int$. This proves the proposition.

Example (1.7) Let $g_1,\dots,g_n$ be elements of $\mathcal{O}$. Suppose $(g_i)\mathcal{O}$ is a prime ideal in $\mathcal{O}$ for every $i=1,\dots,n$. Suppose g_i and g_j have no factor in common for $i\neq j$, i.e. $(g_i)\mathcal{O}\cap(g_j)\mathcal{O} = (g_ig_j)\mathcal{O}$. Take $I=(g)\mathcal{O}$, with $g = \prod_i g_i^{e_i}$. Then $\int I = (\prod_i g_i^{e_i+1})\mathcal{O}$. This is a corollary of proposition (1.6)

Example (1.8) Let $I = (yz, zx, xy)\mathcal{O}$ with $\mathcal{O} = \mathbb{C}\{x,y,z\}$ Then $\int I = I^2 + (xyz)\mathcal{O} \neq I^2$.

Proposition (1.9) Let I be an ideal in $\mathcal{O}$ defining a germ of a reduced complete intersection in $(\mathbb{C}^m,0)$. Then $\int I = I^2$.

proof We already noticed in (1.3)(ii) that $I^2 \subseteq \int I$. Let $f \in \int I$ then $f \in I$. The ideal I is generated by an $\mathcal{O}$-sequence $g_1,\dots,g_n$. So we can write $f = \sum_i a_i g_i$. Moreover $\frac{\partial f}{\partial z_n} \in I$, therefore $\sum_i a_i \frac{\partial g_i}{\partial z_n} \equiv \frac{\partial f}{\partial z_n} \equiv 0 \pmod I$. Thus $\bar a = (\bar a_1,\dots,\bar a_n) \in \mathcal{O}_\Sigma^n$, with $\mathcal{O}_\Sigma = \mathcal{O}/I$, is in the kernel of dg in the exact sequence

$$\mathcal{O}_\Sigma^n \xrightarrow{dg} \mathcal{O}_\Sigma^m \longrightarrow \Omega_{(\Sigma,0)} \to 0$$

But in case of a reduced complete intersection is the map dg injective, [Lo3](6.B), that is $a_i \in I$ for all i. Hence $f = \sum_i a_i g_i \in I^2$. This proves the proposition.

Example (1.10) The converse of (1.9) is not true. Take for instance

$I = (x,y)\mathcal{O} \cap (u,v)\mathcal{O}$ in $\mathcal{O} = \mathbb{C}\{x,y,u,v\}$. Then $\int I = \int(x,y) \cap \int(u,v) = (x,y)^2\mathcal{O} \cap (u,v)^2\mathcal{O} = I^2$. But I does not define a germ of a complete intersection. It is not a Cohen-Macaulay ideal.

Remark (1.11) In case I is a radical ideal in $\mathcal{O}$ defining a germ of a Gorenstein analytic space $(\Sigma,0)$ of codimension ≤ 3 one can proof $\int I = I^2$, [He].

Lemma (1.12) Let I be a radical ideal in $\mathcal{O}$ defining a germ of an analytic space $(\Sigma,0)$ in $(\mathbb{C}^m,0)$ of dimension $d \geq 1$. with an isolated singularity at 0. Then

$\int I = I^2$ if and only if $\operatorname{depth}(I/I^2) \geq 1$.

proof Suppose $\int I = I^2$, then $I/I^2 = I/\int I \xrightarrow{d} \Omega_{(\mathbb{C}^m,0)} \otimes \mathcal{O}_{(\Sigma,0)} = \mathcal{O}^m_{(\Sigma,0)}$
The analytic space $(\Sigma,0)$ is reduced of dimension $d \geq 1$. So $\operatorname{depth}(\mathcal{O}_{(\Sigma,0)}) \geq 1$ and therefore $\operatorname{depth}(I/I^2) \geq 1$, since I/I^2 is isomorphic with a submodule of $\mathcal{O}^m_{(\Sigma,0)}$, not equal to zero. The ideal $\int I$ has no embedded components, by (1.6), since I is radical. In particular is the maximal ideal $\mathfrak{m}$ not an associated prime of $\int I = I^2$. So $\operatorname{depth}(\mathcal{O}/I^2) \geq 1$. The following sequence is exact $0 \to I/I^2 \to \mathcal{O}/I^2 \to \mathcal{O}_{(\Sigma,0)} \to 0$. Thus $\operatorname{depth}(I/I^2) \geq 1$.

Conversely. Suppose $\operatorname{depth}(I/I^2) \geq 1$ and $I^2 \neq \int I$. The reduced analytic space $(\Sigma,0)$ has an isolated singularity at 0. So the finitely generated $\mathcal{O}$-module $\int I/I^2$ has support equal to $\{0\}$.
Consider the exact sequence $0 \to I/I^2 \to \mathcal{O}/I^2 \to \mathcal{O}_{(\Sigma,0)} \to 0$, then $\operatorname{depth}(\mathcal{O}/I^2) \geq 1$, since $\operatorname{depth}(I/I^2) \geq 1$ and $\operatorname{depth} \mathcal{O}_{(\Sigma,0)} \geq 1$.
From the exact sequence $0 \to \int I/I^2 \to \mathcal{O}/I^2$
we derive $\operatorname{depth}(\int I/I^2) \geq 1$, since $\operatorname{depth}(\mathcal{O}/I^2) \geq 1$.
But this contradicts $\operatorname{Kdim}(\int I/I^2) = 0$. So we conclude $I^2 = \int I$.
This proves the lemma.

Example (1.13) Let I be the ideal defining the germ of the curve $(\Sigma,0)$ in $(\mathbb{C}^5,0)$, where $(\Sigma,0)$ is parametrised by $(t^6, t^7, t^8, t^9, t^{10})$.
Pinkham proved [Pi] that $(\Sigma,0)$ is a Gorenstein curve with

obstructed deformations. Suppose $\int I = I^2$, then by (1.12) $\operatorname{depth}(I/I^2) \geq 1$. So I/I^2 is a Cohen-Macaulay, since $\dim I/I^2 = 1$. By a result of Herzog [He], $\mathcal{O}_{(\Sigma,0)}$ is strongly unobstructed. Which is a contradiction. Thus $\int I \neq I^2$, and remark (1.11) cannot be generalised to codim 4 Gorenstein varieties. For the concepts obstructed and strongly unobstructed one consults the above references.

Remark (1.14) With respect to the question whether $\int I = I^2$ we refer to an article of Hochster [Ho]

Lemma (1.15) Let I be a radical ideal in $\mathcal{O}$ defining a germ of an analytic space $(\Sigma,0)$ in $(\mathbb{C}^m,0)$. Then

$$\operatorname{Hom}_{\mathcal{O}_{(\Sigma,0)}}\left(I/I^2,\ \mathcal{O}_{(\Sigma,0)}\right) \cong \operatorname{Hom}_{\mathcal{O}_{(\Sigma,0)}}\left(I/\int I,\ \mathcal{O}_{(\Sigma,0)}\right).$$

proof Let $\dim(\Sigma,0) = 0$, then $I = \mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, so $I^2 = \mathfrak{m}^2 = \int I$ and this is a trivial case.
Suppose $\dim(\Sigma,0) \geq 1$. Let U be an open neighbourhood of 0 in $\mathbb{C}^m$ on which Σ is defined. Let V be the regular locus of Σ. Then V is an open dense subset of Σ, since Σ is reduced and of dimension ≥ 1. So I^2 and $\int I$ are equal in $\mathcal{O}_x$ for every point of $x \in V$. Hence $\int I/I^2$ is a torsion $\mathcal{O}_\Sigma$-module. Thus $\operatorname{Hom}_{\mathcal{O}_{(\Sigma,0)}}(\int I/I^2, \mathcal{O}_{(\Sigma,0)}) = (0)$. and we have the desired result.

D script.

§2 THE GROUP $\mathfrak{D}_I$ AND ITS k-JET

For a first reading one needs only the first seven pages of this section. The results at the end are of some independent interest and will be used in (4.5.a), (9.8) and to give a description of the tangent space of an orbit.

Let $\mathcal{O}$ be the local ring of germs of analytic functions $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$. We denote by m the unique maximal ideal of $\mathcal{O}$. Let $\mathfrak{D}$ be the group of germs of local analytic isomorphisms $h:(\mathbb{C}^m,0)\to(\mathbb{C}^m,0)$, which leave the origin fixed. For an ideal I in $\mathcal{O}$ we also consider a subgroup $\mathfrak{D}_I$ of $\mathfrak{D}$ defined by $\mathfrak{D}_I=\{h\in\mathfrak{D}\,|\,h^*(I)=I\}$. We will define $\mathcal{O}(q)$, $\mathfrak{D}(q)$ and $\mathfrak{D}_I(q)$ as sets of families in $\mathcal{O}$, $\mathfrak{D}$ and $\mathfrak{D}_I$ resp., depending on q parameters. We will consider the tangent space $T\mathfrak{D}_I$ of $\mathfrak{D}_I$ at the identity and an exponential map $\exp: T\mathfrak{D}_I\to\mathfrak{D}_I(1)$. We will show that the k-jet of $\mathfrak{D}_I$ is a complex linear algebraic subgroup of $J^k\mathfrak{D}$, the k-jet of $\mathfrak{D}$. Finally we will prove that $T\alpha_f(\mathfrak{D}_I)$, the tangent space of the orbit of f under $\mathfrak{D}_I$, is equal to $d\alpha_f(T\mathfrak{D}_I)$.

(2.1) We can furnish $\mathcal{O}$ with a Fréchet norm as follows. Choose some local coördinates $z_1,\dots,z_m$ of $(\mathbb{C}^m,0)$ and let $f(z)=\sum_\alpha a_\alpha z^\alpha$ be a powerseries expansion of f in $z_1,\dots,z_m$ and define

$$\|f\| = \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{|\alpha|=n}\frac{|a_\alpha|}{|a_\alpha|+1}$$

$\mathcal{O}$ becomes a Fréchet $\mathbb{C}$-vector space. The induced topology is called the weak topology on $\mathcal{O}$ or the topology of coëfficientwise convergence. This topology does not depend on the chosen coördinates and it is the weakest topology on $\mathcal{O}$ making all the projections $\mathcal{O}\to\mathcal{O}/m^k$ continuous. (The quotients $\mathcal{O}/m^k$ are finite dimensional $\mathbb{C}$-vector spaces and we furnish them with the complex topology, see [G-R]. I, 0.4 1.0.) The weak topology is a Hausdorff topology on $\mathcal{O}$. which satisfies the first axiom of countability.

(2.2) The multiplication is continuous in the weak topology. Hence $\mathcal{O}$ is a Fréchet $\mathbb{C}$-algebra. It is not possible to give $\mathcal{O}$ the structure of a Banach space (inducing the weak topology), since every open neighbourhood of 0 in $\mathcal{O}$ is unbounded (i.e. for every open $\mathcal{U}$ in $\mathcal{O}$, $0 \in \mathcal{U}$. there exist an $f \in \mathcal{U}$, $f \neq 0$ such that $\{\lambda f \mid \lambda \in \mathbb{C}\} \subseteq \mathcal{U}$). It is also not possible to furnish $\mathcal{O}$ with the structure of a Banach $\mathbb{C}$-algebra (inducing an arbitrary topology) in case $m \geq 1$. Since every Noetherian Banach $\mathbb{C}$-algebra is Artinian (see [G-R] I.5).

(2.3) The $\mathfrak{m}$-adic topology on $\mathcal{O}$ is defined by the local basis $\{\mathfrak{m}^k \mid k \in \mathbb{N}\}$ of open neighbourhoods of 0. One has the final topology also called the topology of analytic convergence ([G-R], I.3). We will not use this topology

(2.4) On every finitely generated $\mathcal{O}$-module M one constructs in the same way as for $\mathcal{O}$ the weak resp. $\mathfrak{m}$-adic topology. We will use the following fact. Every $\mathcal{O}$-submodule N of a finitely generated $\mathcal{O}$-module M is closed in the weak (and also in the $\mathfrak{m}$-adic) topology ([G-R], I.4).

(2.5) Altough it is possible to furnish the group $\mathcal{D}$ with the structure of an infinite dimensional differentiable Fréchet manifold, such that $\mathcal{D}$ becomes a "Fréchet Liegroup" we will not do this. First of all because there are different ways to do it. [0 ,1], [0,2] and in the second place we won't need it. What we need is a topology on $\mathcal{D}$ and a tangentspace of $\mathcal{D}$ at the identity.

(2.6) For a subset V of a Fréchet $\mathbb{C}$-vectorspace we define the tangent space T_vV of V at $v \in V$ as the set of tangent vectors of differentiable curves $\gamma : (\mathbb{C}, 0) \to (V, v)$. Remark that for an arbitrary subset V the tangentspace T_vV needs not to be a $\mathbb{C}$-vectorspace.

(2.7) The topology of $\mathcal{D}$ is defined (after choosing local coördinates $z_1, \ldots, z_m$) by considering $\mathcal{D}$ as a subspace of $\mathcal{O}^m$ and taking the induced topology. The multiplication map

$$\mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$$

defined by $(h_1, h_2) \mapsto h_2 \circ h$, is continuous. The action of $\mathcal{D}$ on $\mathcal{O}$, defined by $(h, f) \mapsto f \circ h$, is continuous.

Definition (2.8) Let $\Theta_{(\mathbb{C}^m,0)} := Der_{\mathbb{C}}(\mathcal{O}_{(\mathbb{C}^m,0)})$, the $\mathbb{C}$-derivations of $\mathcal{O}_{(\mathbb{C}^m,0)}$

Remark (2.9) We shall denote $\Theta_{(\mathbb{C}^m,0)}$ by Θ. The elements of Θ are also called germs of vector fields in $(\mathbb{C}^m,0)$. and $m\Theta$ are the germs of vector fields in $(\mathbb{C}^m,0)$, which are zero at the origin.

Let $\gamma : (\mathbb{C},0) \longrightarrow (\mathcal{D}, id)$ be a germ of a differentiable curve in $\mathcal{D}$. Let $f \in \mathcal{O}$ then

$$\left(\frac{d}{dt}(f \circ \gamma)\right)\Big|_{t=0} = \sum_i \left(\frac{d\gamma_i}{dt}\right)\Big|_{t=0} \frac{\partial f}{\partial z_i}$$

We identify the tangent vector of the curve γ at 0, with the derivation $\sum_i \left(\frac{d\gamma_i}{dt}\right)\Big|_{t=0} \frac{\partial}{\partial z_i}$

Since elements of $\mathcal{D}$ leave the origin fixed, is the tangent vector of γ at 0 an element of $m\Theta$.

Definition (2.10) Let $\mathcal{D}(q)$ be the set of germs of analytic maps $H : (\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow (\mathbb{C}^m, 0)$ such that $h_t : (\mathbb{C}^m,0) \to (\mathbb{C}^m,0)$ is an element of $\mathcal{D}$ for all $t \in \mathbb{C}^q$ small enough, where $h_t(z) = H(z,t)$.

(2.11) Let $\xi \in \Theta$, then there exist a unique germ of an analytic map $H : (\mathbb{C}^m \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}^m, 0)$ such that

$$\begin{cases} \dfrac{\partial H}{\partial t} = \xi \circ H \\ h_0 = id \end{cases}$$

, by the existence and uniqueness theorem of analytic differential equations. In case $\xi \in m\Theta$ is $h_t(0) = 0$ for all t small enough. So $h_t : (\mathbb{C}^m,0) \longrightarrow (\mathbb{C}^m,0)$ is a germ of a local analytic isomorphism, which leaves the origin fixed, for t small enough. Hence we have a map

$$\exp : m\Theta \longrightarrow \mathcal{D}(1)$$

which we call the *exponential map*. It is a classical result that $\{h_t\}$ defines a local one parameter subgroup of $\mathcal{D}$, i.e. $h_s \circ h_t = h_{s+t}$, for s and t small enough. The curve $\gamma : (\mathbb{C}, 0) \to (\mathcal{D}, id)$ defined by $\gamma(t) = h_t$ is a differentiable curve in $\mathcal{D}$, starting at id and with tangent vector $\xi \in m\Theta$.

Conclusion (2.12) We can identify $T\mathcal{D}$, the tangent space of $\mathcal{D}$ at id, with $m\Theta$ and there exists an exponential map

$$\exp : T\mathcal{D} \longrightarrow \mathcal{D}(1)$$

, which is a right inverse of the map $d : \mathcal{D}(1) \to \mathcal{D}$ defined by $d(H) = \left(\frac{\partial h_t}{\partial t}\right)_{t=0}$.

Remark (2.13) Let $\xi \in T\mathcal{D}$ and $H = \exp(\xi)$. Then the local analytic map h_t is defined on a neighbourhood U_t of 0 in $\mathbb{C}^m$. One can show that h_t is defined for every $t \in \mathbb{C}$. Thus there exist a time 1 exponential map

$$\exp : T\mathcal{D} \longrightarrow \mathcal{D}$$

, we also denote by exp. The derivative of this map is the identity, but it does not define a local isomorphism (in the weak topology). The inverse function theorem does not hold for $\mathcal{D}$, as one sees in the following example.

Example (2.14) A diffeomorphism h with an isolated periodic point at x (i.e. for some $n \in \mathbb{N}$ $h^n(x) = x$ and there is a neighbourhood U of x such that for all $y \in U$, $y \neq x$, $h^m(y) = y$ implies $m = 0$), which is in a one parameter flow h_t (i.e. $h_s \circ h_t = h_{s+t}$ and $h_0 = id$ and $h_1 = h$) must have a fixed point at x, see [Pa]. It is possible to give an example of a local analytic isomorphism $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, which is arbitrary close to the identity (in the weak topology), with the property that there exist a converging sequence $\{x_n\}$ in $\mathbb{C}^2$ to 0, such that x_n is an isolated periodic point of h, with period q_n and $q_n \to \infty$ as $n \to \infty$, see [Mo].
Thus there are elements h in $\mathcal{D}$, arbitrary close to the identity in the weak topology, which are not in the image

of the exponential map. This is one of the difficulties in the theory of infinite dimensional "Lie groups", which are not Banach Lie groups, see [D-K].

Definition (2.15) Let $\mathcal{O}(q)$ be the local ring of germs of analytic functions with q parameters, i.e. germs of analytic functions $F: (\mathbb{C}^m \times \mathbb{C}^q, 0) \to \mathbb{C}$
Let $\Theta(q)$ be the set of germs of vector fields in $(\mathbb{C}^m, 0)$ with q-parameters, i.e. in local coördinates $z_1, \dots, z_m$ of $(\mathbb{C}^m, 0)$ is $\Theta(q) = \{ \sum_{i=1}^m A_i \frac{\partial}{\partial z_i} \mid A_1, \dots, A_m \in \mathcal{O}(q) \}$.

Definition (2.16) Let $\mathcal{D}_e$ be the pseudo group of germs of local analytic isomorphisms $h: (\mathbb{C}^m, 0) \to \mathbb{C}^m$, which have an analytic representative $h: U \to V$, with U and V open sets in $\mathbb{C}^m$ and $0 \in U$, such that $h: U \to V$ is analytic and has an analytic inverse $h^{-1}: V \to U$.
We also consider $\mathcal{D}_e(q)$, the set of all germs of analytic maps $H: (\mathbb{C}^m \times \mathbb{C}^q, 0) \to \mathbb{C}^m$ such that $h_t \in \mathcal{D}_e$ for all $t \in \mathbb{C}^q$ small enough.

Remark (2.17) One can define exponential maps
$\exp: T\mathcal{D}(q) \to \mathcal{D}(q+1)$ and $\exp: T\mathcal{D}_e(q) \to \mathcal{D}_e(q+1)$
and show that $T\mathcal{D}(q) = m\,\Theta(q)$ and $T\mathcal{D}_e(q) = \Theta(q)$, in the same way as (2.10) and (2.11).

Definition (2.18) Let I be an ideal in $\mathcal{O}$. Define

$$\mathcal{D}_I = \{ h \in \mathcal{D} \mid h^*(I) = I \}$$

$$\mathcal{D}_I(q) = \{ H \in \mathcal{D}(q) \mid h_t^*(I) = I \text{ for all } t \in \mathbb{C}^q \text{ small enough} \}.$$

Remark (2.19) $\mathcal{D}_I$ is a subgroup of $\mathcal{D}$. Define a composition of $\mathcal{D}(q)$ as follows $H_1, H_2 \in \mathcal{D}(q)$, then $(H_1 \circ H_2)(z,t) = H_2(H_1(z,t),t)$. Then $\mathcal{D}(q)$ becomes a group and $\mathcal{D}_I(q)$ a subgroup of $\mathcal{D}(q)$.

Remark (2.20) $\mathcal{D}_I = \{ h \in \mathcal{D} \mid h^*(I) \subseteq I \}$, since $h^*(I) \subseteq I$ implies $I \subseteq (h^{-1})^*(I)$. So we have a sequence of ideals

$$I \subseteq (h^{-1})^*(I) \subseteq (h^{-2})^*(I) \subseteq \dots\dots$$

, which must be stationary, since $\mathcal{O}$ is a noetherian ring. Hence $(h^{-n+1})^*(I) = (h^{-n})^*(I)$ for some n. Thus $h^*(I) = I$.
In the same way is $\mathcal{D}_I(q) = \{ H \in \mathcal{D}(q) \mid h_t^*(I) \subseteq I$ for all $t \in \mathbb{C}^q$ small enough

Remark (2.21) Let I be an ideal in $\mathcal{O}$, then $\mathcal{D}_I \subseteq \mathcal{D}_{\int I}$. This follows directly from the definition and the chain rule. Thus we have an action of $\mathcal{D}_I$ on $\int I$ and $T\mathcal{D}_I \subseteq T\mathcal{D}_{\int I}$. Suppose moreover that I is radical, then $\mathcal{D}_I = \mathcal{D}_{\int I}$. Since $I = \operatorname{rad}(\int I)$ and for $h \in \mathcal{D}_{\int I}$ we have

$$h^*(I) = h^*(\operatorname{rad}(\int I)) = \operatorname{rad}(h^*(\int I)) = \operatorname{rad}(\int I) = I.$$

So $h \in \mathcal{D}_I$.

Definition (2.22) Let U be an open neighbourhood of 0 in $\mathbb{C}^m$. Let $\mathcal{O}_U$ be the sheaf of analytic functions on U. Let I be an ideal in $\mathcal{O} = \mathcal{O}_{(U,0)}$ and $\mathcal{J}$ an ideal sheaf in $\mathcal{O}_U$ with stalk $\mathcal{J}_0 = I$. Define $\mathcal{D}_{I,e}$ as the set of all $h \in \mathcal{D}_e$, which have a representative $h: V \to W$, with V and W open sets in U and $0 \in V$, such that $h^*(\mathcal{J}(W)) = \mathcal{J}(V)$.
Let $\mathcal{D}_{I,e}(q) = \{ H \in \mathcal{D}_e(q) \mid h_t \in \mathcal{D}_{I,e}$ for all $t \in \mathbb{C}^q$ small enough$\}$.

Remark (2.23) $\mathcal{D}_e(q)$ is a pseudo group, with a product analogous to (2.19), and $\mathcal{D}_{I,e}(q)$ is a sub pseudogroup of $\mathcal{D}_e(q)$.

Lemma (2.24) Let I be an ideal in $\mathcal{O}$. Then

$$T\mathcal{D}_I(q) = \{ \xi \in T\mathcal{D}(q) \mid \xi(I) \subseteq I\,\mathcal{O}(q) \}$$
$$T\mathcal{D}_{I,e}(q) = \{ \xi \in T\mathcal{D}_e(q) \mid \xi(I) \subseteq I\,\mathcal{O}(q) \}$$

and the restrictions of the exponential maps from $T\mathcal{D}(q)$ and $T\mathcal{D}_e(q)$ to $T\mathcal{D}_I(q)$ and $T\mathcal{D}_{I,e}(q)$ resp., define maps

$$\exp : T\mathcal{D}_I(q) \longrightarrow \mathcal{D}_I(q+1)$$
$$\exp : T\mathcal{D}_{I,e}(q) \longrightarrow \mathcal{D}_{I,e}(q+1).$$

Proof We shall give a proof in case of $\mathcal{D}_I(q)$, with $q=0$. The other cases are proved in the same way.
Let $\gamma: (\mathbb{C},0) \to (\mathcal{D}_I, id)$ be a germ of a differentiable curve in $\mathcal{D}_I$ and ξ its tangent vector at 0, i.e.

$$\xi = \sum_i \left(\frac{d\gamma_i}{dt}\right)_{t=0} \frac{\partial}{\partial z_i} \quad \text{in local coördinates.}$$

Let $g \in I$, then for all t small enough is $\gamma_t^*(g) = g \circ \gamma_t$ an element of I, by definition of $\mathcal{D}_I$. Hence

$$\left(\frac{d}{dt}(g \circ \gamma_t)\right)\Big|_{t=0} = \lim_{t \to 0} \left(\frac{(g \circ \gamma_t) - g}{t}\right) \in I$$

, since I is closed in the weak topology (2.4). Thus

$$\xi(g) = \sum_i \left(\frac{d\gamma_i}{dt}\right)\Big|_{t=0} \frac{\partial g}{\partial z_i} = \left(\frac{d}{dt}(g \circ \gamma_t)\right)\Big|_{t=0} \in I$$

and we have proved $\xi(I) \subseteq I$. So $T\mathcal{D}_I \subseteq \{\xi \in T\mathcal{D} \mid \xi(I) \subseteq I\}$.
Now, let $H = \exp(\xi)$, with $\xi \in T\mathcal{D}$ and suppose $\xi(I) \subseteq I$.
Then

$$\begin{cases} \frac{\partial H}{\partial t} = \xi \circ H \\ h_0 = id \end{cases}$$

Let $g \in I$ and define by induction $g_0 = g$ and $g_{n+1} = \xi(g_n)$.
Then $g_n \in I$ for all $n \in \mathbb{N}$, since we assumed $\xi(I) \subseteq I$! Moreover

$$\frac{\partial}{\partial t}(g_n \circ H) = \xi(g_n) \circ H = g_{n+1} \circ H.$$

Thus $\left(\frac{\partial^n}{\partial t^n}(g \circ H)\right)\Big|_{t=0} = g_n$ and $g \circ H = \sum_{n=0}^{\infty} \frac{g_n}{n!} t^n$

so $h_t^*(g) \in I$ for all t small enough, since I is closed in $\mathcal{O}$!
Hence for all t small enough $h_t^*(I) \subseteq I$ and $h_t^{-1} = h_{-t}$.
Thus $h_t \in \mathcal{D}_I$ by (2.20). Furthermore $\left(\frac{\partial h}{\partial t}\right)\Big|_{t=0} = \xi$.
Conclusion. The restriction of exp to $T\mathcal{D}_I$

$$\exp : T\mathcal{D}_I \to \mathcal{D}_I(1)$$

is well defined and $T\mathcal{D}_I = \{\xi \in T\mathcal{D} \mid \xi(I) \subseteq I\}$
This proves the lemma.

<u>Remark</u> (2.25) One can define an exponential map

$$\exp : T\mathcal{D}(1) \longrightarrow \mathcal{D}(1)$$

as follows. Let $E \in T\mathcal{D}(1) = \mathfrak{m}.\,\Theta(1)$. and H the unique solution of the differential equation

$$\begin{cases} \frac{\partial H}{\partial t}(z,t) = E(H(z,t),t) \\ h_0 = id \end{cases}$$

The restriction of this exponential map to $T\mathcal{D}_I(1)$ gives a well defined map

$$\exp : T\mathcal{D}_I(1) \longrightarrow \mathcal{D}_I(1).$$

as one proofs the same way as (2.24).

<u>Remark</u> (2.26) The tangent space $T\mathcal{D}_{I,e} = \{\xi \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}) \mid \xi(I) \subseteq I\}$ also appears in the work of K. Saito on free divisors, see [Sa, 2].
Let $I = (g)\mathcal{O}$ define a germ of a reduced hypersurface $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$. Then $\xi \in T\mathcal{D}_{I,e}$ is called a germ of a vector field in $(\mathbb{C}^m, 0)$ tangent at Σ. This vector field ξ is also called logarithmic along Σ, since

$$T\mathcal{D}_{I,e} = \mathrm{Hom}_{\mathcal{O}}(\Omega^1(\log \Sigma), \mathcal{O}), \text{ with}$$

$$\Omega^1(\log \Sigma) = \left\{ \begin{array}{l} \text{germs of meromorphic 1-forms } \omega \text{ on} \\ (\mathbb{C}^m, 0), \text{ such that } g\omega \text{ and } g d\omega \text{ are} \\ \text{holomorphic at } 0 \end{array} \right\}$$

<u>Remark</u> (2.27) Define the <u>k-jet of $\mathcal{O}$</u>, by $J^k\mathcal{O} = \mathcal{O}/m^{k+1}$ and let $J^k : \mathcal{O} \to J^k\mathcal{O}$ be the projection map. We call $J^k(f)$ the k-jet of the function f in $\mathcal{O}$.
Let $\mathcal{D}^{(k)} = \{ h \in \mathcal{D} \mid h \equiv id \bmod m^{k+1} \}$. The <u>$k$-jet of $\mathcal{D}$</u> is defined by $J^k\mathcal{D} = \mathcal{D}/\mathcal{D}^{(k)}$
Let I be an ideal in $\mathcal{O}$. Define $\mathcal{D}_I^{(k)} = \mathcal{D}_I \cap \mathcal{D}^{(k)}$ and let $J^k\mathcal{D}_I = \mathcal{D}_I / \mathcal{D}_I^{(k)}$, the k-jet of $\mathcal{D}_I$.

<u>Remark</u>. (2.28) The $\mathcal{D}^{(k)}$ is a normal subgroup of $\mathcal{D}$ and $J^k\mathcal{D}$ is a complex linear algebraic group acting algebraically on the finite dimensional $\mathbb{C}$-vector space $J^k\mathcal{O}$.
The $\mathcal{D}_I^{(k)}$ is a normal subgroup of $\mathcal{D}_I$ and $J^k\mathcal{D}_I$ can be viewed as a subgroup of $J^k\mathcal{D}$, since

$$J^k\mathcal{D}_I = \mathcal{D}_I / \mathcal{D}_I \cap \mathcal{D}^{(k)} \cong \mathcal{D}_I \cdot \mathcal{D}^{(k)} / \mathcal{D}^{(k)} \subseteq J^k\mathcal{D}$$

All the projection maps $J^k : \mathcal{O} \to J^k\mathcal{O}$, $J^k : \mathcal{D} \to J^k\mathcal{D}$ and $J^k : \mathcal{D}_I \to J^k\mathcal{D}_I$ are continuous with respect to the weak topology. Let for $l \geq k$, $J^k_l : J^l\mathcal{O} \to J^k\mathcal{O}$, $J^k_l : J^l\mathcal{D} \to J^k\mathcal{D}$ and $J^k_l : J^l\mathcal{D}_I \to J^k\mathcal{D}_I$ be the projection maps.
Then $(J^k\mathcal{O}, J^k_l)$, $(J^k\mathcal{D}, J^k_l)$ and $(J^k\mathcal{D}_I, J^k_l)$ are inverse systems of topological vector spaces resp. topological groups.

<u>Definition</u> (2.29) By taking the inverse limit, we define

$$\hat{\mathcal{O}} = \varprojlim J^k\mathcal{O}, \text{ the formal function on } (\mathbb{C}^m, 0)$$

$$\hat{\mathcal{D}} = \varprojlim J^k\mathcal{D}, \text{ the formal isomorphisms from } (\mathbb{C}^m, 0) \text{ to itself}$$

$$\hat{\mathcal{D}}_I = \varprojlim J^k\mathcal{D}_I$$

Remark (2.30) Choose local coördinates $z_1, \dots, z_m$ of $(\mathbb{C}^m; 0)$. Then $\hat{\mathcal{O}} \cong \mathbb{C}[[z_1, \dots, z_m]]$ the local ring of formal power series in $z_1, \dots, z_m$. We denote by $\hat{m}$ the ideal $m \cdot \hat{\mathcal{O}}$. Furthermore $\hat{\mathcal{D}} \cong \{\hat{h} = (\hat{h}_1, \dots, \hat{h}_m) \in \hat{\mathcal{O}}^m \mid \hat{h}(0) = 0$ and $d\hat{h}(0)$ is an invertible matrix$\}$. The group $\hat{\mathcal{D}}$ acts on $\hat{\mathcal{O}}$. In the same way as for $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{D}_I$ we can define the k-jet of $\hat{\mathcal{O}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}_I$ resp. We have canonical isomorphisms

$$J^k \hat{\mathcal{O}} \cong J^k \mathcal{O} \quad , \quad J^k \hat{\mathcal{D}} \cong J^k \mathcal{D} \text{ and } J^k(\hat{\mathcal{D}}_I) \cong J^k(\mathcal{D}_I).$$

From now on we shall identify these two k-jets.

Remark (2.31) Let I be an ideal in $\mathcal{O}$ and define $\hat{I} = I\hat{\mathcal{O}}$. Let

$$\hat{\mathcal{D}}_{\hat{I}} = \{ \hat{h} \in \hat{\mathcal{D}} \mid \hat{h}^*(\hat{I}) = \hat{I} \}$$

We can consider $\hat{\mathcal{D}}_I$ as a subgroup of $\hat{\mathcal{D}}$ and it is by definition equal to $\{ \hat{h} \in \hat{\mathcal{D}} \mid$ for all k there exist an $h_k \in \mathcal{D}$ such that $J^k(\hat{h}) = J^k(h_k)$ and $h_k^*(I) = I\}$. Thus $\hat{\mathcal{D}}_I$ is a priori a subgroup of $\hat{\mathcal{D}}_{\hat{I}}$. In fact they are equal as we shall see. One can find the idea of the proof in ([To, 1], III. 5)

Lemma (2.32) $\hat{\mathcal{D}}_I = \hat{\mathcal{D}}_{\hat{I}}$

proof We already noted that $\hat{\mathcal{D}}_I \subseteq \hat{\mathcal{D}}_{\hat{I}}$. We shall prove their equality by means of Artin approximation. Let $g_1, \dots, g_n$ be generators of the ideal I in $\mathcal{O}$. Let $z_1, \dots, z_m$ be local coördinates of $(\mathbb{C}^m; 0)$. Consider the following analytic functions

$$F_i(u, v, z) = g_i(u) - \sum_j v_{ij}\, g_j(z) \quad \text{in} \quad \mathbb{C}\{z, u, v\}$$

where $z = (z_1, \dots, z_m)$, $u = (u_1, \dots, u_m)$ and $v = (v_{ij}, 1 \le i, j \le n)$. Now, suppose $\hat{h} \in \hat{\mathcal{D}}_{\hat{I}}$, then $\hat{h}(\hat{I}) = \hat{I}$. Hence we can find $(\hat{a}_{ij})$ in $\hat{\mathcal{O}}$ such that

$$g_i \circ \hat{h} = \sum_j \hat{a}_{ij}\, g_j$$

Thus $(\hat{h}, \hat{a})$ is a solution of $F_1(u, v) = \cdots = F_n(u, v) = 0$ ...... (*)

By Artin approximation [A, 2], we can find $h_k \in \mathcal{O}^m$ and $a_k \in \mathcal{O}^{m^2}$ such that $J^k(h_k) = J^k(\hat{h})$ and $J^k(a_k) = J^k(\hat{a})$. and (h_k, a_k) is a solution of (*), so $h_k^*(I) \subseteq I$. The matrix $dh_k(0)$ is invertible, since $d\hat{h}(0)$ is invertible (for $k \ge 1$). So h_k is an element of $\mathcal{D}_I$, by (2.20), such that $J^k(h_k) = J^k(\hat{h})$. Thus $\hat{h} \in \hat{\mathcal{D}}_I$. This proves the lemma.

Remark (2.33) In the same way one defines $\hat{\mathcal{D}}_I(q)$ and $\hat{\mathcal{D}}_{\hat{I}}(q)$ and proves their equality.

Proposition (2.34) The k-jet $J^k(\mathcal{D}_I)$ is a complex linear algebraic subgroup of $J^k(\mathcal{D})$.

proof The idea of this proof one can find in ([Mü T, satz 3]).
Let $D^k = J^k\mathcal{D}$ and $E^k = (J^k\mathcal{D})_{J^kI}$, i.e.
$E^k = \{J^k h \in D^k \mid (J^k h)(J^k I) = J^k I \text{ in } J^k\mathcal{O}\}$. Let $J^k_\ell : D^\ell \to D^k$, $\ell \geq k$, be the projection map from D^ℓ to D^k. Then $J^k = J^k_\ell \circ J^\ell$.
Let $E^k_\ell = J^k_\ell(E^\ell)$ for $\ell \geq k$. The D^k is a complex linear algebraic group and $\{E^k_\ell\}_{\ell \geq k}$ is a decreasing sequence of subgroups, which are closed in the Zariski topology, hence it must be a stationary sequence. So for every k there exist an $\ell(k) \geq k$ such that for all $m \geq \ell(k)$ one has $E^k_m = E^k_{\ell(k)}$. By an induction argument one proves $E^k_{\ell(k)} = J^k(\hat{\mathcal{D}}_{\hat{I}})$.
Lemma (2.32) implies $J^k(\hat{\mathcal{D}}_{\hat{I}}) = J^k(\hat{\mathcal{D}}_I) = J^k(\mathcal{D}_I)$. So $J^k(\mathcal{D}_I) = E^k_{\ell(k)}$ is a linear algebraic subgroup of $J^k\mathcal{D}$.
This proves the proposition.

Proposition (2.35) The following is an exact sequence of complex (infinite dimensional) Liealgebra's.

$$0 \to T(\mathcal{D}_I^{(k)}) \longrightarrow T\mathcal{D}_I \longrightarrow T(J^k\mathcal{D}_I) \longrightarrow 0$$

proof Since $\mathcal{D}_I^{(k)}$ is a subgroup of $\mathcal{D}_I$ we conclude $0 \to T(\mathcal{D}_I^{(k)}) \to T\mathcal{D}_I$ is exact. For every ℓ and k, $\ell \geq k$ is the map $J^k_\ell : J^\ell\mathcal{D}_I \to J^k\mathcal{D}_I$ a submersion. Hence every germ of a differentiable curve $\gamma : (\mathbb{C},0) \to J^k\mathcal{D}_I$ can be extended to a curve $\gamma_\ell : (\mathbb{C},0) \to J^\ell\mathcal{D}_I$, such that $J^k_\ell \circ \gamma_\ell = \gamma$, for every $\ell \geq k$. So there exist a differentiable curve $\hat{\gamma} : (\mathbb{C},0) \to \hat{\mathcal{D}}_I$ such that $J^k \circ \hat{\gamma} = \gamma$. Thus $T\hat{\mathcal{D}}_I \longrightarrow T(J^k\mathcal{D}_I) \to 0$ is a surjection.

Let $\xi_k \in T(J^k\mathcal{D}_I)$ and $\hat{\xi} \in T(\hat{\mathcal{D}}_{\hat{I}}) = T(\hat{\mathcal{D}}_I)$ such that $J^k(\hat{\xi}) = \xi_k$. Let $\hat{H} = \exp(\hat{\xi}) \in \hat{\mathcal{D}}_{\hat{I}}(1)$. Then $\hat{\mathcal{D}}_{\hat{I}}(1) = \hat{\mathcal{D}}_I(1)$ by (2.33). Hence there exist an $H_k \in \mathcal{D}_I(1)$ such that $J^k(H_k) = J^k(\hat{H})$. Take $\xi = \left(\frac{\partial H_k}{\partial t}\right)\Big|_{t=0}$. Then

$$J^k(\xi) = J^k\left(\left(\frac{\partial H_k}{\partial t}\right)\Big|_{t=0}\right) = J^k\left(\left(\frac{\partial \hat{H}}{\partial t}\right)\Big|_{t=0}\right) = J^k(\hat{\xi}) = \xi_k$$

Hence $T\mathcal{D}_I \to T(J^k\mathcal{D}_I)$ is surjective.

We claime $T(\mathcal{D}_I^{(k)}) = T\mathcal{D}_I \cap T\mathcal{D}^{(k)}$ ..... (*)

The inclusion $\subseteq$ is obvious, since $\mathcal{D}_I^{(k)} \subseteq \mathcal{D}_I \cap \mathcal{D}^{(k)}$. Let ξ be an element of $T\mathcal{D}_I \cap T\mathcal{D}^{(k)}$ and take $H = \exp(\xi)$. Then $H \in \mathcal{D}_I(1) \cap \mathcal{D}^{(k)}(1) = \mathcal{D}_I^{(k)}(1)$ and $\xi = \left(\frac{\partial H}{\partial t}\right)\Big|_{t=0} \in T(\mathcal{D}_I^{(k)})$.

this proves the claim

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T(\mathcal{D}_I^{(k)}) & \longrightarrow & T\mathcal{D}_I & \longrightarrow & T(J^k\mathcal{D}_I) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T\mathcal{D}^{(k)} & \longrightarrow & T\mathcal{D} & \longrightarrow & TJ^k\mathcal{D} & \to & 0
\end{array}$$

The columns and bottom row are exact and from (*) we conclude that the upper row is exact in the middle. That was the remaining part to proof.

<u>Definition</u> (2.36) Let I be an ideal in $\mathcal{O}$. Let $F \in \mathcal{O}(q)$. Let the map $\alpha_F : \mathcal{D}_I(q) \to \mathcal{O}(q)$ be defined by
$\alpha_F(H) = F \circ \tilde{H}$, with $\tilde{H}(z,t) = (H(z,t),t)$

Let $\hat{I} = I.\hat{\mathcal{O}}$. In the same way one defines the maps $\hat{\alpha}_{\hat{F}} : \hat{\mathcal{D}}_{\hat{I}}(q) \to \hat{\mathcal{O}}(q)$ for $\hat{F} \in \hat{\mathcal{O}}(q)$. and $\alpha_{J^kF} : J^k\mathcal{D}_I(q) \to J^k\mathcal{O}(q)$

The derivatives of α_F, $\hat{\alpha}_{\hat{F}}$ and α_{J^kF} are denoted by $d\alpha_F$, $d\hat{\alpha}_{\hat{F}}$ and $d\alpha_{J^kF}$.

<u>Proposition</u> (2.37) Let I be an ideal in $\mathcal{O}$. Let $\hat{F} \in \hat{\mathcal{O}}(q)$. Let $\gamma : (\mathbb{C},0) \to (\hat{\alpha}_{\hat{F}}(\hat{\mathcal{D}}_{\hat{I}}(q)), \hat{F})$ be a germ of a differentiable curve. Then

(i) There exist a germ $\hat{\gamma} : (\mathbb{C},0) \to (\hat{\mathcal{D}}_{\hat{I}}(q), id)$ of a differentiable curve such that $\hat{\alpha}_{\hat{F}} \circ \hat{\gamma} = \gamma$

(ii) $d\hat{\alpha}_{\hat{F}}(T\hat{\mathcal{D}}_{\hat{I}}(q)) = T\hat{\alpha}_{\hat{F}}(\hat{\mathcal{D}}_{\hat{I}}(q))$

Proof We shall give the proof for $q=0$. The proof for arbitrary q goes the same. Let $\hat f \in \hat{\mathcal{O}}$.
(i) Let $\gamma_k : (\mathbb{C},0) \longrightarrow (\alpha_{J^k\hat f}(J^k \mathcal{D}_I), J^k f)$ be the germ of the curve defined by $\gamma_k = J^k \circ \gamma$. The $J^k \mathcal{D}_I$ is a linear algebraic group, by (2.34), acting algebraically on $J^k\mathcal{O}$ and on $J^{k-1}\mathcal{D}_I$. We need the following theorem.

Theorem (2.38) Let G be a complex Lie group, acting analytically on an analytic manifold M. Then
(i) All orbits Gx are non singular for $x \in M$
(ii) There exist a germ (S,x) of a non singular analytic subspace of (M,x), transversal to Gx at x, and there exist a germ $(H,1)$ of a non singular analytic subspace of $(G,1)$ such that $(H \times S, (1,x))$ is locally analytic isomorphic with a neighbourhood of x in M, and the isomorphism is induced by the action $G \times M \longrightarrow M$.

Proof [V] (2.9.2) and (2.9.7).

Now we continue the proof of (2.37). By induction one can find a sequence $\{\beta_k\}$ of germs of curves $\beta_k : (\mathbb{C},0) \to (J^k\mathcal{D}_I, id)$ such that $\alpha_{J^kf} \circ \beta_k = \gamma_k$ and $J^{k-1}_k \circ \beta_k = \beta_{k-1}$, by applying theorem (2.38) with $G = J^k\mathcal{D}_I$ and $V = (J^{k-1}\mathcal{D}_I) \times J^k\mathcal{O}$.
Let $\hat\beta = \lim_{\leftarrow} \beta_k$. Then $\hat\beta : (\mathbb{C},0) \longrightarrow \hat{\mathcal{D}}_I$ is a germ of a differentiable curve in $\hat{\mathcal{D}}_I = \hat{\mathcal{D}}_{\hat I}$ such that $\hat\alpha_{\hat f} \circ \hat\beta = \gamma$.
(ii) The inclusion $\subseteq$ is obvious. The converse inclusion is a consequence of (i).
This proves proposition (2.37)

Proposition (2.39) Let I be an ideal in $\mathcal{O}$. Let $f \in \mathcal{O}(q)$. Then

$$d\alpha_f(T\mathcal{D}_I(q)) = T\alpha_f(\mathcal{D}_I(q)) \quad \text{and}$$

Proof The inclusion $\subseteq$ is obvious. Let $f \in \mathcal{O}$ and let ξ be a tangent vector of $\alpha_f(\mathcal{D}_I)$ at f. Then

$$\xi \in T\alpha_f(\mathcal{D}_I) \subseteq T\hat\alpha_f(\hat{\mathcal{D}}_{\hat I}) = d\hat\alpha_f(T\hat{\mathcal{D}}_{\hat I}), \text{ by } (2.37)$$

Let $z_1, \ldots, z_m$ be local coordinates of $(\mathbb{C}^m, 0)$. Then

$\Theta \cong \mathcal{O}^m$, with basis $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}$. Let $g_1, \ldots, g_n$ be generators of I. Then the following sequences are exact

$$0 \to T\mathcal{D}_I \to \mathfrak{m}\,\mathcal{O}^m \xrightarrow{dg} (\mathcal{O}/I)^n$$

$$0 \to T\mathcal{D}_{\hat{I}} \to \hat{\mathfrak{m}}\,\hat{\mathcal{O}}^m \xrightarrow{dg} (\hat{\mathcal{O}}/\hat{I})^n$$

So, the completion $(T\mathcal{D}_I)^\wedge$ of the $\mathcal{O}$-module $T\mathcal{D}_I$ is equal to $T\hat{\mathcal{D}}_{\hat{I}}$, since $\hat{\mathcal{O}}$ is a flat $\mathcal{O}$-module, see [Se, 2].
The following sequences are exact.

$$T\mathcal{D}_I \xrightarrow{d\alpha_f} \mathcal{O} \longrightarrow \mathcal{O}/_{d\alpha_f(T\mathcal{D}_I)} \to 0$$

$$T\hat{\mathcal{D}}_{\hat{I}} \xrightarrow{d\hat{\alpha}_f} \hat{\mathcal{O}} \longrightarrow \hat{\mathcal{O}}/_{d\hat{\alpha}_f(T\hat{\mathcal{D}}_{\hat{I}})} \to 0$$

So, $(d\alpha_f(T\mathcal{D}_I))^\wedge = d\hat{\alpha}_f(T\hat{\mathcal{D}}_{\hat{I}})$, since $(T\mathcal{D}_I)^\wedge = T\hat{\mathcal{D}}_{\hat{I}}$ and $\hat{\mathcal{O}}$ is a flat $\mathcal{O}$-module.
Thus $\xi \in \mathcal{O}$ and $\xi \in (d\alpha_f(T\mathcal{D}_I))^\wedge$ in $\hat{\mathcal{O}}$, Hence $\xi \in d\alpha_f(T\mathcal{D}_I)$, since $\hat{\mathcal{O}}$ is a faithfully flat $\mathcal{O}$-module, see [Se, 2]. So $T\alpha_f(\mathcal{D}_I) = d\alpha_f(T\mathcal{D}_I)$. The proof for arbitrary q is the same.

<u>Remark</u> (2.40) The pseudo group $\mathcal{D}_{I,e}$ does not act on $\mathcal{O}$, since for $h \in \mathcal{D}_{I,e}$ one does not have $h(0)=0$ in general. But for every $f \in \mathcal{O}$ there exist an open ball B_ε of radius $\varepsilon > 0$ and centre 0 in $\mathbb{C}^m$ on which f is defined. Let $V = \{ h \in \mathcal{D}_{I,e} \mid |h(0)| < \varepsilon \}$, then V is an open neighbourhood of id in $\mathcal{D}_{I,e}$ in the weak topology and $f \circ h \in \mathcal{O}$ for all $h \in V$. Hence the map $\alpha_f : V \to \mathcal{O}$, with $\alpha_f(h) = f \circ h$, is well defined. Now let the germ $(\alpha_f(\mathcal{D}_{I,e}), f)$ be defined by $(\alpha_f(V), f)$. Then one can prove $T\alpha_f(\mathcal{D}_{I,e}) = d\alpha_f(T\mathcal{D}_{I,e})$ in the same way as (2.39). A similar result holds for a parametrized version.

§3 FUNCTIONS OF FINITE I-CODIMENSION

In this paragraph we will consider functions f with f and the Jacobi ideal of f contained in a given ideal I, i.e. $f \in \int I$. We take the right-equivalence relation defined by the action of the group $\mathcal{D}_I$ of local analytic isomorphisms which preserve I. In this context one can speak about: the orbit of f under the group $\mathcal{D}_I$, the tangentspace of the orbit of f, and I-codimension of f. It is possible to have finite I-codimension, eventhough f has a non-isolated singularity. We will proof a finite I-determinacy theorem and a I-versality theorem in §4.

We consider the generic case first, i.e. functions f with a regular analytic space (Σ, o) as singular set and with (extended) I-codimension equal to zero. and give normal forms of these functions.

We restrict ourselves to the complex analytic context. One could however derive the same results in the real analytic or C^∞ category. We also restrict to the case of R(= right)-equivalence and donot consider K (= contact)- nor the A(= right-left)-equivallence for convenience.

Most of the time $\underline{\mathcal{O}}$ denotes the local ring of germs of analytic functions $f : (\mathbb{C}^m, o) \to \mathbb{C}$ with maximal ideal $\underline{m}$. But sometimes in proofs we denote by $\underline{\mathcal{O}}$ the sheaf of analytic functions on an open set U of $\mathbb{C}^m$ and by $\underline{\mathcal{O}}_p$ the stalk of $\underline{\mathcal{O}}$ at $p \in U$.

$\underline{\mathcal{D}}$ is the group of germs of local analytic isomorphisms $h : (\mathbb{C}^m, o) \to (\mathbb{C}^m, o)$. Let I be an ideal in $\underline{\mathcal{O}}$. Then $\mathcal{D}_I = \{ h \in \underline{\mathcal{D}} \mid h^*(I) = I \}$. For $f \in \int I$ and $h \in \mathcal{D}_I$ we have $h^*(f) \in \int I$ as a direct consequence of the chain rule and the definitions of $\mathcal{D}_I$ and $\int I$. Thus $\mathcal{D}_I$ acts on $\int I$. For $f \in \int I$ we denote by α_f the orbit map (see 2.36)

$$\alpha_f : \mathcal{D}_I \longrightarrow \int I \text{ , defined by } \alpha_f(h) = h^*(f) = f \circ h .$$

The derivative
$$d\alpha_f : T\mathcal{D}_I \longrightarrow \int I$$
is defined by the action of the vector field $\eta \in T\mathcal{D}_I$ on f, i.e. $d\alpha_f(\eta) = \eta(f)$. In local coördinates $z_1, \dots, z_m$, we can express η as $\eta = \sum_i \eta_i \frac{\partial}{\partial z_i}$ and we have $d\alpha_f(\eta) = \sum_i \eta_i \frac{\partial f}{\partial z_i}$. The map $d\alpha_f$ is well defined, since $\eta \in T\mathcal{D}_I \subseteq T\mathcal{D}_{\int I}$ by (2.21), thus $\eta(\int I) \subseteq \int I$.
Analogously, for every $f \in \int I$, there exist a neighbourhood V of id in $\mathcal{D}_{I,e}$, which acts on f. We have a germ of an orbit map $\alpha_f : (\mathcal{D}_{I,e}, id) \longrightarrow (\int I, f)$ and a derivative $d\alpha_f : T\mathcal{D}_{I,e} \to \int I$, by (2.40).
The tangent space $T\alpha_f(T\mathcal{D}_I)$ of the orbit $\alpha_f(\mathcal{D}_I)$ at f is equal to $d\alpha_f(T\mathcal{D}_I)$, by (2.39) and in the same way $T\alpha_f(\mathcal{D}_{I,e}) = d\alpha_f(T\mathcal{D}_{I,e})$ by (2.40).

<u>Notation</u> (3.1) We shall write $\tau_I(f)$ instead of $T\alpha_f(\mathcal{D}_I)$ and $\tau_{I,e}(f)$ instead of $T\alpha_f(\mathcal{D}_{I,e})$.

<u>Remark</u> (3.2) We have the following two inclusions for $f \in \int I$
$$\tau_I(f) \subseteq (\mathfrak{m}\cdot J_f) \cap \int I \qquad \text{and} \qquad \tau_{I,e}(f) \subseteq J_f \cap \int I$$
We shall give in (3.15) a condition which implies equality of both inclusions.

<u>Definition</u> (3.3)
$$c(f) := c_I(f) := \underline{I\text{-codimension of } f} = \dim_{\mathbb{C}}\left(\int I / \tau_I(f) \right)$$
$$c_e(f) := c_{I,e}(f) := \underline{\text{extended } I\text{-codimension of } f} = \dim_{\mathbb{C}}\left(\int I / \tau_{I,e}(f) \right)$$

<u>Remark</u> (3.4) One always has: $c_I(f) < \infty$ if and only if $c_{I,e}(f) < \infty$. Since $\tau_I(f) \subseteq \tau_{I,e}(f)$ and the quotient M of the two is a finitely generated $\mathcal{O}$-module. One checks directly that $\mathfrak{m}\cdot\tau_{I,e}(f) \subseteq \tau_I(f)$. Thus $\mathfrak{m}\cdot M = (0)$, hence $\dim_{\mathbb{C}} M < \infty$. This proves the claim.

<u>Remark</u> (3.5) One might think that:

<u>Splitting lemma</u> (3.6) Let $f:(\mathbb{C}^m,0) \to (\mathbb{C},0)$ be a germ of an analytic function such that $d^2f(0)$ is a $(m\times m)$-matrix of rank r, then one can find local coördinates $z_1,\dots,z_q, W_1,\dots W_r$ of $(\mathbb{C}^m,0)$, with $q+r=m$, such that

$$f(z,w) = g(z) + w_1^2 + \dots + w_r^2$$

and $g \in (z_1,\dots,z_q)^3\, \mathbb{C}\{z\}$.

<u>proof</u> [G-M]

<u>Definition</u> (3.7) The function g of lemma (3.6) is called the <u>residual function</u> of f. The R-equivalence class of g is determined by the R-equivalence class of f, [G-M]. Two germs of analytic functions $f_1:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ and $f_2:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ are called <u>stably equivalent</u> when their residual functions are R-equivalent.

<u>Lemma</u> (3.8) Let I be an ideal in $\mathcal{O} = \mathbb{C}\{z_1,\dots,z_q\}$ and $g \in \int I$. Take $\tilde{I} = I.\tilde{\mathcal{O}} + (w_1,\dots,w_r)\tilde{\mathcal{O}}$ with $\tilde{\mathcal{O}} = \mathbb{C}\{z_1,\dots,z_q,w_1,\dots,w_r\}$. Let $f(z,w) = g(z) + w_1^2+\dots+w_r^2$. Then $f\in\int\tilde{I}$ and

$$c_{\tilde{I}}(f) = c_I(g) \quad\text{and}\quad c_{\tilde{I},e}(f) = c_{I,e}(g)$$

<u>proof</u> Let $(w) := (w_1,\dots,w_r)\tilde{\mathcal{O}}$ and $\left(\frac{\partial f}{\partial z}\right) := \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_q}\right)\tilde{\mathcal{O}}$. Then $\int\tilde{I} = (\int I).\tilde{\mathcal{O}} + I.(w).\tilde{\mathcal{O}} + (w)^2$, thus $f\in\int\tilde{I}$. Moreover $\tau_{\tilde{I}}(f) = \tau_I(g)\tilde{\mathcal{O}} + \left(\frac{\partial f}{\partial z}\right).(w) + \tilde{I}(w) = \tau_I(g).\tilde{\mathcal{O}} + I.(w).\tilde{\mathcal{O}} + (w)^2$, since $\left(\frac{\partial f}{\partial z}\right) \subseteq \tilde{I} = I\cdot\tilde{\mathcal{O}} + (w)$. Hence there is a well defined $\mathbb{C}$-linear map

$$\int I/\tau_I(g) \longrightarrow \int\tilde{I}/\tau_{\tilde{I}}(f)$$

This map is surjective, since $(\int I).(w)\tilde{\mathcal{O}} \subseteq I.(w)\tilde{\mathcal{O}} \subseteq \tau_{\tilde{I}}(f)$. It is injective too, since $\mathcal{O}\cap\tau_{\tilde{I}}(f) \subseteq \tau_I(g)$. Hence $c_{\tilde{I}}(f) = c_I(g)$ and in the same way $c_{\tilde{I},e}(f) = c_{I,e}(g)$.

<u>Definition</u> (3.9) A germ of an analytic function $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ is called of <u>type $A(d)$</u> in case one can find local coördinates $z_1,\dots,z_d, w_1,\dots,w_r$ of $(\mathbb{C}^m,0)$ such that

$$f(z,w) = w_1^2+\dots+w_r^2$$

Remark (3.10) A function f of type $A(d)$ has 0 as its residual function. The singular locus of f is regular, of dimension d and given by the ideal $I=(w_1,\dots,w_k)$. Sometimes we shall call them functions with transversal A_1 (or transversal Morse) singularities. A function of type $A(0)$ is also called a Morse function or an ordinary double point. In case $d=1$ Siersma [Si] denotes $A(1)$ by A_∞. One verifies that $f\in\int I$ and $c_{I,e}(f)=c_I(f)=0$.

Definition (3.11) Let $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ be a germ of an analytic function, such that its residual function is of the form

$$g(x,y,z)=\sum_{1\le i\le j\le p} x_{ij}\,y_iy_j$$

in certain local coordinates x_{ij} $(1\le i\le j\le p)$, $y_1,\dots,y_p$, $z_1,\dots,z_q$. Then f is called of type $D(d,p)$, with $d=\frac12 p(p+1)+q$.

Remark (3.12) The singular locus of a function f of type $D(d,p)$ is regular and of dimension $d=\frac12 p(p+1)+q$. A function of type $D(1,1)$ is called D_∞ by Siersma [Si]. One sees that $D(d,0)=A(d)$. A singularity at 0 in $\mathbb{C}^3$ defined by a
Take $I=(y_1,\dots,y_p)\mathcal{O}$ then $g\in I^2=\int I$ and

$$I^2=\left(\frac{\partial g}{\partial x}\right)\subseteq \mathcal{I}_{I,e}(g)=I.\left(\frac{\partial g}{\partial y}\right)+\left(\frac{\partial g}{\partial x}\right)\subseteq I^2.$$ Thus $c_{I,e}(g)=0$

10 And $\mathcal{I}_I(g)=I.\left(\frac{\partial g}{\partial y}\right)+m\left(\frac{\partial g}{\partial x}\right)=m.I^2$. Thus $\{y_iy_j \mid 1\le i\le j\le p\}$ is a basis of $\int I/\mathcal{I}_I(g)$ over $\mathbb{C}$. Hence $c_I(g)=\frac12 p(p+1)$.

Proposition (3.13) Let I be an ideal in $\mathcal{O}$ defining a germ of a regular analytic space of dimension d in $(\mathbb{C}^m,0)$ and suppose $f\in\int I$. Then
(i) $c_I(f)=0$ if and only if f is of type $A(d)$
(ii) $c_{I,e}(f)=0$ if and only if f is of type $D(d,p)$ for some p.

proof One way of the implications is already proved in (3.12). It is enough to prove: $c_{I,e}(f)=0$ implies f is of type $D(d,p)$ for some p. Since $c_I(f)=0$ gives $c_{I,e}(f)=0$, hence f is of

type $D(d,p)$, so $c_I(f) = \frac{1}{2}p(p+1) = 0$. Thus $p=0$ and f is of type $D(d,0) = A(d)$.
By the splitting lemma (3.6) and lemma (3.8), we may assume that $d^2(f) = (0)$.
The ideal I defines a germ of a regular analytic space. We can choose local coördinates $u_1,\dots,u_t, y_1,\dots,y_p$ of $(\mathbb{C}^m,0)$ such that $I = (y_1,\dots,y_p)\mathcal{O}$. Let $(\frac{\partial f}{\partial u}) = (\frac{\partial f}{\partial u_1},\dots,\frac{\partial f}{\partial u_t})\mathcal{O}$ and $(y) = (y_1,\dots,y_p)\mathcal{O}$. Then $\tau_{I,e}(f) = (\frac{\partial f}{\partial u}) + (y)(\frac{\partial f}{\partial y})$.
We assumed $f \in \int I = I^2 = (y)^2$. Since $d^2 f(0) = 0$ we have $f \in \mathfrak{m}$
Hence we can write $f = \sum_{i\le j} (\sum_k a_{ijk} u_k + \sum_\ell b_{ij\ell} y_\ell) y_i y_j$
Differentiating gives:

$$\frac{\partial f}{\partial u_k} \equiv \sum_{i\le j} \alpha_{ijk} y_i y_j \pmod{\mathfrak{m} I^2}, \quad \text{with } \alpha_{ijk} = a_{ijk}(0).$$

$$\frac{\partial f}{\partial y_\ell} \equiv 0 \pmod{\mathfrak{m} I}$$

Thus $\tau_{I,e}(f) \equiv (\frac{\partial f}{\partial u}) \pmod{\mathfrak{m} I^2}$. We assumed $c_{I,e}(f) = 0$, so $\tau_{I,e}(f) = \int I = I^2$. Therefore $I^2 \equiv (\frac{\partial f}{\partial u}) \pmod{\mathfrak{m} I^2}$. The set $\{y_i y_j \mid 1 \le i \le j \le p\}$ is a basis of $I^2/\mathfrak{m} I^2$ over $\mathbb{C}$.
So we can write

$$y_i y_j \equiv \sum_k \gamma_{k,ij} \frac{\partial f}{\partial u_k} \pmod{\mathfrak{m} I^2}$$

such that the $\mathbb{C}$-linear mappings α and γ with matrices $(\alpha_{ij,k})$ and $(\gamma_{k,ij})$ resp., satisfy $\alpha\gamma = id$.

$$\begin{tikzcd} \mathbb{C}^{\frac{1}{2}p(p+1)} \arrow[r, "\gamma"] \arrow[rr, bend right, "id"'] & \mathbb{C}^t \arrow[r, "\alpha"] & \mathbb{C}^{\frac{1}{2}p(p+1)} \end{tikzcd}$$

Thus α is surjective and ${}^t\alpha : \mathbb{C}^{\frac{1}{2}p(p+1)} \longrightarrow \mathbb{C}^t$ is injective.
Let $u_{ij} := \sum_k \alpha_{ij,k} u_k$ for $1 \le i \le j \le p$. Then $\{u_{ij} \mid 1 \le i \le j \le p\}$ is part of a regular system of parameters in $\mathbb{C}\{u_1,\dots,u_t\}$, i.e. there are coördinate functions $z_1,\dots,z_r$ such that
u_{ij} $(1 \le i \le j \le p)$, $z_1,\dots,z_r$ are local coördinates in $(\mathbb{C}^t,0)$
Let $x_{ij} := \sum_k a_{ijk} u_k + \sum_\ell b_{ij\ell} y_\ell$. Then $x_{ij} \equiv u_{ij} \pmod{\mathfrak{m}^2 + (y)}$.
Hence x_{ij} $(1 \le i \le j \le p)$, $y_1,\dots,y_p$, $z_1,\dots,z_r$ are local coördinates in $(\mathbb{C}^m,0)$ and

$$f(x,y,z) = \sum_{1\le i\le j\le p} x_{ij} y_i y_j$$

Hence f is of type $D(d,p)$. This proves the proposition.

~~**Example** (3.14) Let I be an ideal in $\mathcal{O} = \mathbb{C}\{z_1, \dots, z_m\}$ generated by $g_1, \dots, g_n$. Take $\tilde{\mathcal{O}} = \mathbb{C}\{x_{ij}, 1 \le i \le j \le n, z_1, \dots, z_m\}$ and~~

$$\cancel{f(x,z) = \sum_{i \le j} x_{ij}\, g_i(z)\, g_j(z)}$$

~~Let $\tilde{I} = I \cdot \tilde{\mathcal{O}}$ then $f \in \int \tilde{I}$ and $c_{\tilde{I},e}(f) = 0$, thus $c_I(f) < \infty$ (by (3.4)).~~

Example (3.14) We will give an example with a non-radical ideal. Let $I = (x^{m-1}, y^{n-1})\mathcal{O}$ with $\mathcal{O} = \mathbb{C}\{x, y, z\}$. Then $\int I = (x^m, x^{m-1}y^{n-1}, y^n)\mathcal{O}$. Take $f(x,y,z) = x^m + y^n z$ then $f \in \int I$ and $c_I(f) = 1$ and $c_{I,e}(f) = 0$.

Proposition (3.15) Let I be a radical ideal in $\mathcal{O}$ defining an analytic space $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$ of dimension $d \ge 1$. Suppose $f \in \int I$ such that $c_I(f) < \infty$. Then

$$\tau_I(f) = (\mathfrak{m} \cdot J_f) \cap \int I$$
$$\tau_{I,e}(f) = J_f \cap \int I$$

proof Take a neighbourhood U of 0 in $\mathbb{C}^m$ such that $f: U \to \mathbb{C}$ and Σ are well defined on U. Let Σ be defined by the analytic functions $g_1, \dots, g_n$ on U. Let $\mathcal{O}$ be the sheaf of holomorphic functions on U, with stalk $\mathcal{O}_a$ for $a \in U$. The ideal sheaf $I = (g_1, \dots, g_n)\mathcal{O}$ defines Σ on U. and $\int I$ and $\tau_{I,e}(f)$ are ideal sheafs too on U. Let $\mathcal{M} = \int I / \tau_{I,e}(f)$. Then $\mathcal{M}$ is a sheaf of coherent $\mathcal{O}$-modules on U, with stalks $\mathcal{M}_a$ for $a \in U$ and such that $\dim_{\mathbb{C}} \mathcal{M}_a = c_{I_a,e}(f_a) = I_a$-codimension of the germ $f_a : (U, a) \to \mathbb{C}$. Possibly after shrinking U, we may assume that $\mathcal{M}_a = (0)$ for all $a \in U \setminus \{0\}$, since $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}$-modules with a finite dimensional $\mathbb{C}$-vectorspace as stalk at 0. Let V be the regular locus of Σ. Then V is an open dense set in Σ, since Σ is reduced and of dimension $d \ge 1$. At all points $a \in V \setminus \{0\}$ we have $c_{I_a,e}(f_a) = 0$ and (Σ, a) is a germ of a regular analytic space. The Jacobi ideal J_f is contained in I, since $f \in \int I$.

Thus $\frac{\partial f}{\partial z_i} = \sum_k \varphi_{ik} g_k$ and $\frac{\partial^2 f}{\partial z_i \partial z_j} \equiv \sum_k \varphi_{ik} \frac{\partial g_k}{\partial z_j} \pmod{I}$

Let $\varphi : \mathcal{O}_\Sigma^n \longrightarrow \mathcal{O}_\Sigma^n$ be the map with matrix (φ_{ik})

The following diagram is commutative

$$\begin{tikzcd} \mathcal{O}^m \arrow[r, "df"] \arrow[d, "dg"'] \arrow[dr, "\overline{d^2f}"] & \mathcal{O} \\ \mathcal{O}_\Sigma^n \arrow[r, "\varphi"] & \mathcal{O}_\Sigma^n \end{tikzcd}$$

One always has $\ker(\overline{d^2 f}) \subseteq \ker(\overline{dg})$.

We can apply proposition (3.13) and a local computation at $a \in V \setminus \{0\}$ shows that $\ker(\overline{d^2 f_a}) = \ker(\overline{dg_a})$ for all $a \in V \setminus \{0\}$. Let $\xi \in \ker(\overline{d^2 f})$, then the composition of mappings: $\Sigma \xrightarrow{\xi} U \xrightarrow{dg} \mathbb{C}^n$, is zero on the open dense subset $V \setminus \{0\}$ of Σ. The analytic space Σ is reduced, hence $dg(\xi)$ is zero on Σ and we have proved $\ker(\overline{d^2 f}) = \ker(\overline{dg})$.

We already have shown in (3.2) that $\tau_{I,e}(f) \subseteq J_f \cap \int I$

Now suppose $\psi \in J_f \cap \int I$, then $\psi = \sum_i \xi_i \frac{\partial f}{\partial z_i}$ with $\frac{\partial f}{\partial z_i} \in I$.

Thus $\frac{\partial \psi}{\partial z_j} \equiv \sum_i \xi_i \frac{\partial^2 f}{\partial z_j \partial z_i} \equiv 0 \pmod{I}$, for all $j = 1, \dots, m$,

since $\psi \in \int I$. Hence $\xi = (\xi_1, \dots, \xi_m) \in \ker(\overline{d^2 f}) = \ker(\overline{dg})$ and therefore $\xi \in T(\Theta_{I,e})$. Thus $\psi = \xi(f) \in \tau_{I,e}(f)$.

We have proved $\tau_{I,e}(f) = J_f \cap \int I$. In case $\psi \in \mathfrak{m} J_f \cap \int I$ we may assume $\psi = \sum_i \xi_i \frac{\partial f}{\partial z_i}$ with $\xi_i \in \mathfrak{m}$, so $\xi \in T(\Theta_I)$ and $\psi = \xi(f) \in \tau_I(f)$. This proves the proposition.

<u>Remark</u> (3.16) There is in general no coherent sheaf $\mathcal{N}$ of $\mathcal{O}$-modules on U such that for all $a \in U$ $c_{I_a}(f_a) = \dim_{\mathbb{C}} \mathcal{N}_a$: this in contrast with $c_{I,e}(f)$.

<u>Example</u> (3.17) We will give an example which shows that the assumption $c_I(f) < \infty$ in proposition (3.16) is necessary. Take $I = (xy)\mathcal{O}$ with $\mathcal{O} = \mathbb{C}\{x,y\}$ and let $f = (xy)^3$. Then $\int I = (xy)^2 \mathcal{O}$ and $\tau_I(f) = \tau_{I,e}(f) = (xy)^3 \mathcal{O}$.

But $J_f \cap \int I = (x^3y^2, x^2y^3)\mathcal{O}$ and $(\mathfrak{m} J_f) \cap \int I = (x^4y^2, x^3y^3, x^2y^4)\mathcal{O}$.

We finish with the case that the ideal I defines a line and $f \in \int I$ such that $c_I(f) < \infty$. These functions define so called <u>isolated line singularities</u> [Si]

<u>Proposition</u> (3.18) Let I be an ideal in $\mathcal{O} = \mathcal{O}_{(\mathbb{C}^{n+1},0)}$ defining a germ of a regular analytic space of dimension 1. Suppose $f \in \int I$ and $c_I(f) < \infty$. Then one can find local coördinates $x, y_1, \ldots, y_n$ in $(\mathbb{C}^{n+1},0)$ such that $I = (y_1, \ldots, y_n)\mathcal{O}$ and

$$f \equiv \sum_{i=1}^{n} x^{\tau_i} y_i^2 \ (\mathrm{mod}\ (y)^3) \quad , \text{ with } \tau_i \in \mathbb{N}$$

and

$$c_I(f) = \begin{cases} c_{I,e}(f) & \text{in case } f \text{ is of type } A_\infty \\ c_{I,e}(f) + 1 & \text{otherwise} \end{cases}$$

<u>Remark</u> (3.19) Let $\delta = \sum_i \tau_i$, Then $\delta = \#\{D_\infty \text{ points of a generic approximation of } f\}$, see [Si] and (7.18).

<u>Proof</u> (3.18) We can find local coördinates $x, u_1, \ldots, u_n$ of $(\mathbb{C}^{n+1},0)$ such that $I = (u_1, \ldots, u_n)\mathcal{O}$, since I defines a regular analytic space of dimension 1. We can write

$$f = \sum_{i,j} h_{ij} u_i u_j \quad , \text{ since } f \in \int I = I^2 .$$

Thus $f \equiv Q_f \ (\mathrm{mod}\ I^3)$, with Q_f a homogeneous quadratic form in $u_1, \ldots, u_n$, with coëfficients in $\mathbb{C}\{x\}$. Since $\frac{1}{2} \in \mathbb{C}\{x\}$ and $\pm\sqrt{v} \in \mathbb{C}\{x\}$ for every unit $v \in \mathbb{C}\{x\}$, we can find local coördinates $y_1, \ldots, y_n$ such that

$$f(x,y) \equiv \sum_{i=1}^{n} x^{\tau_i} y_i^2 \ (\mathrm{mod}\ I^3)$$

This is an easy exercise in quadratic forms over $\mathbb{C}\{x\}$ and proves the first claim.

$$\tau_I(f) = m \cdot \left(\frac{\partial f}{\partial x}\right) + (y)\left(\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \tau_{I,e}(f) = \left(\frac{\partial f}{\partial x}\right) + (y)\left(\frac{\partial f}{\partial y}\right)$$

Thus $c_{I,e}(f) \leq c_I(f) \leq c_{I,e}(f) + 1$. and $c_{I,e}(f) = c_I(f)$ if and only if $\frac{\partial f}{\partial x} \in \tau_I(f)$. Suppose f is of type A_∞ then $c_I(f) = c_{I,e}(f) = 0$, by (3.10). Now suppose $\frac{\partial f}{\partial x} \in \tau_I(f)$. Then $\frac{\partial f}{\partial x} \in (y)\left(\frac{\partial f}{\partial y}\right)$

$$\frac{\partial f}{\partial x} \equiv \sum_i \tau_i x^{\tau_i - 1} y_i^2 \ (\mathrm{mod}\ (y)^3)$$

$$(y)\left(\frac{\partial f}{\partial y}\right) \equiv (y)\left(x^{\tau_1} y_1, \ldots, x^{\tau_n} y_n\right) (\mathrm{mod}\ (y)^3)$$

Hence $\tau_i = 0$ for all $i = 1, \ldots, n$ and f is of type A_∞.

opm. In deze paragraaf komt geen ρ voor het zijn allemaal $\mathcal{D}$

§4 FINITE DETERMINACY AND VERSAL UNFOLDINGS OF FUNCTIONS WITH A FIXED SINGULAR LOCUS

We shall define what an unfolding of a function with a fixed singular locus is. We will show that the group $\mathcal{D}_I$ is a geometric subgroup of $\mathcal{D}$ in the sense of Damon [D]. As a consequence we have a finite determinacy and a versal unfolding theorem with respect to $\mathcal{D}_I$ for all functions of finite I-codimension.

Consider all germs of analytic functions $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ which have a given analytic space $(\Sigma,0)$, defined by an ideal I, in their singular locus. We determined this set in §1 and showed it to be an ideal and called it the primitive ideal $\int I$ of I. In §2 we introduced the group $\mathcal{D}_I$ of germs of local analytic isomorphisms $h:(\mathbb{C}^m,0)\to(\mathbb{C}^m,0)$, which leave Σ invariant. This group acts on $\int I$. In §3 we considered the case that the tangent space to the orbit $\tau_f(\mathcal{D}_I)$ of an $f\in\int I$ has finite codimension in $\int I$.

Definition (4.1) Let I be an ideal in $\mathcal{O}$ and $f\in\int I$. Then f is called <u>k-determined in $\int I$</u> (or (k,I)-determined) if for every $g\in\int I$ with $J^k(f)=J^k(g)$ we have $g=f\circ h$ for some $h\in\mathcal{D}_I$

Remark (4.2) This definition is slightly different from the one given by Damon [D] in a more general context. His definition is as follows. The function $f\in\int I$ is called k-determined in $\int I$ if for every $g\in\int I$ with $g-f\in m^k\cdot\int I$ one has $g=f\circ h$ for some $h\in\mathcal{D}_I$.
These two definitions don't differ too much, as one sees in the following lemma.

Lemma (4.3) Let (R,m) be a local noetherian ring with maximal ideal m. Let J be an ideal in R, then there exists an $r\in\mathbb{N}$ such that for every $k\in\mathbb{N}$ $\quad m^{k+r}\cap J\subseteq m^k\cdot J$

Proof This is a special case of Artin-Rees lemma, see [Ma] 11.c.

Definition (4.4) Let $r(J)$ denote the minimal number r for which the inclusion of lemma (4.3) holds.

Theorem (4.5) Let $r = r(\int I)$ and $f \in \int I$

a) suppose f is k-determined in $\int I$, then
$$\mathfrak{m}^{k+1} \cap \int I \subseteq \tau_I(f)$$
b) suppose $\mathfrak{m}^{k+1} \cdot \int I \subseteq \mathfrak{m}\, \tau_I(f) + \mathfrak{m}^{k+2} \cdot \int I$,
then f is $(k+r)$-determined in $\int I$.

Proof The idea of the proof is well known after Moser and Thom-Levine. Part b) also follows after we have proved that $\mathcal{D}_I$ is a geometric subgroup of $\mathcal{D}$.

a) The function f is k-determined in $\int I$. Hence
$$f + (\mathfrak{m}^{k+1} \cap \int I) \subseteq \alpha_f(\mathcal{D}_I)$$
Taking the tangent space at f and identifying $T_f\mathcal{O}$ with $\mathcal{O}$ we get $\quad \mathfrak{m}^{k+1} \cap \int I \subseteq T\alpha_f(\mathcal{D}_I) = \tau_I(f)$,
since $\mathfrak{m}^{k+1} \cap \int I$ is a $\mathbb{C}$-vector space.

b) Suppose $\quad \mathfrak{m}^{k+1} \cdot \int I \subseteq \mathfrak{m} \cdot \tau_I(f) + \mathfrak{m}^{k+2} \cdot \int I$

Let $g \in \int I$ and $g - f \in \mathfrak{m}^{k+r+1}$. Take $F(z,t) = f(z) + t(g(z) - f(z))$ and let $f_t(z) = F(z,t)$. Then F is an element of $\mathcal{O}(1)$. The ring $\mathcal{O}$ is contained in $\mathcal{O}(1)$. The maximal ideal of $\mathcal{O}$ is $\mathfrak{m}$ and we shall denote the ideal $\mathfrak{m}.\mathcal{O}(1)$ by $\mathfrak{m}$, by abuse of notation. We denote the maximal ideal of $\mathcal{O}(1)$ by $\mathfrak{m}^*$. ~~Define~~

Define $\tau_I^*(F) = \{ E(F) \mid E \in T\mathcal{D}_I(1) \}$, then we claim $\tau_I(f) \subseteq \tau_I(F) + \mathfrak{m}^{k+1} \cdot \int I$

Since, let $\xi(f) \in \tau_I(f)$, then $\xi(f) = \xi(F) - t\xi(g-f)$. Further, $g - f \in \mathfrak{m}^{k+r+1} \cap \int I$, thus $\xi(g-f) \in \mathfrak{m}^{k+r+1} \cap \int I$, by (2.21). So $\xi(g-f) \in \mathfrak{m}^{k+1} \cdot \int I$. This proves the claim. Thus
$$\mathfrak{m}^{k+1} \cdot \int I \subseteq \mathfrak{m}\, \tau_I(f) + \mathfrak{m}^{k+2} \cdot \int I \subseteq \mathfrak{m}^* \tau_I^*(F) + \mathfrak{m}^* \cdot \mathfrak{m}^{k+1} \cdot \int I$$
By Nakayama's lemma is $\mathfrak{m}^{k+1} \cdot \int I \subseteq \mathfrak{m}^* \tau_I^*(F)$. Moreover
$g - f \in \mathfrak{m}^{k+r+1} \cap \int I \subseteq \mathfrak{m}^{k+1} \cdot \int I$. Thus $f - g = E(F)$, for some $E \in T\mathcal{D}_I(1)$.

Let $H = \exp(E)$, then $H \in \mathcal{D}_I(1)$ by (2.25). Hence
$$\frac{d}{dt}(F \circ H) = \sum_i \left(\frac{\partial F}{\partial z_i} \circ H \right) \cdot \frac{\partial H_i}{\partial t} + \left(\frac{\partial F}{\partial t} \circ H \right) = \sum_i \left(E_i \frac{\partial F}{\partial z_i} + g - f \right) \circ H = 0$$

Thus $f = f_0 = f_t \circ h_t$ and $h_t \in \mathcal{D}_I$ for all t sufficiently small. This can be proved for every $t_0 \in \mathbb{C}$, i.e. every $t_0 \in \mathbb{C}$ has a neighbourhood U_{t_0} such that f_t is in the orbit $\alpha_{f_{t_0}}(\mathcal{D}_I)$ for every $t \in U_{t_0}$. Since 0 and 1 are connected by a compact curve in $\mathbb{C}$, we conclude that $g = f_1$ is in the orbit of f. Thus f is $(k+1)$-determined in $\int I$. This proves the proposition.

Remark (4.6) We will show that $\mathcal{D}_I$ is a geometric subgroup of $\mathcal{D}$ in the sense of Damon [D]. We shall not define for a general subgroup $\mathcal{G}$ of $\mathcal{D}$, what it means to be a geometric subgroup, but we shall refer to [D] for the general definition and check the four conditions:

I) naturality under pull-back
II) algebraic structure of the tangentspace.
III) exponential map
IV) filtration property

in the particular case of $\mathcal{D}_I$ in $\mathcal{D}$.
Remark that the sentence: "$\mathcal{D}_I$ is a geometric subgroup of $\mathcal{D}$", is an abuse of language, since not only $\mathcal{D}_I$ and $\mathcal{D}$ are involved in the definition, but $\int I$ and the unfoldings $\int I(q)$, $\mathcal{D}_I(q)$ and $\mathcal{D}(q)$ too, as we shall see.

Definition (4.7) An unfolding (with q parameters) of a germ of an analytic function $f: (\mathbb{C}^m, 0) \to \mathbb{C}$ is a germ of an analytic function $F: (\mathbb{C}^m \times \mathbb{C}^q, 0) \to \mathbb{C}$, i.e. $F \in \mathcal{O}(q)$ see (2.15), such that $F(z,0) = f(z)$. Let $f_t : (\mathbb{C}^m, 0) \to \mathbb{C}$ be defined by $f_t(z) = F(z,t)$.
Let I be an ideal in $\mathcal{O}$ and $f \in \int I$. An element $F \in \mathcal{O}(q)$ is called an unfolding (with q parameters) of f which preserves I (or an I-unfolding) iff $f_0 = f$ and $f_t \in \int I$ for all $t \in \mathbb{C}^q$ small enough, i.e. $F \in \int I \cdot \mathcal{O}(q)$. We denote $\int I \cdot \mathcal{O}(q)$ by $\int I(q)$.

Definition (4.8) An unfolding (with q parameters) of a germ of an analytic isomorphism $h: (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ is a germ of an analytic map $H: (\mathbb{C}^m \times \mathbb{C}^q, 0) \to (\mathbb{C}^m, 0)$, such that $H(z,0) = h(z)$

and $h_t \in \mathcal{D}$ for all t small enough.
An element $H \in \mathcal{D}_e(q)$, see (2.16), is called an unfolding (with q parameters) of $h \in \mathcal{D}_e$ if $H(z,0) = h(z)$.
Let I be an ideal in $\mathcal{O}$. An unfolding (with q parameters) of $h \in \mathcal{D}_I$ which preserves I is an element $H \in \mathcal{D}(q)$ such that $H(z,0) = h(z)$ and $h_t \in \mathcal{D}_I$ for all $t \in \mathbb{C}^q$ small enough, see (2.10). In the same way one defines an unfolding of $h \in \mathcal{D}_{I,e}$, see (2.23).

Remark (4.9) There are some differences between the definitions in [D] and ours. In the first place, Damon considers $F : (\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow \mathbb{C} \times \mathbb{C}^q$ with $F(z,t) = (F_1(z,t), t)$ as unfoldings, instead of F_1. This difference is not important, since we consider the case of right equivalence only. In the second place, Damon makes a difference between "unfoldings with q parameters" and "q parameter families". Our definition of unfolding would be a q parameter family in the sense of Damon. An unfolding of an $h \in \mathcal{D}$ would be a germ of an analytic map $H : (\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow (\mathbb{C}^m \times \mathbb{C}^q, 0)$ with $H(z,t) = (H_1(z,t), t)$ and $H_1(z,0) = h(z)$, without requiring $H_1(0,t) = 0$ for all t small enough. Damon does not consider the pseudo groups $\mathcal{D}_e(q)$ and $\mathcal{D}_{I,e}(q)$, but defines so called extended tangent spaces $(T\mathcal{D}(q))_e$ and $(T\mathcal{D}_I(q))_e$, which are the ordinary tangent spaces of $\mathcal{D}_e(q)$ and $\mathcal{D}_{I,e}(q)$ at the identity.

(I) Naturality under pull-back (4.10)
Let $F \in \int I(q)$ be an I-unfolding of $f \in \int I$. Let $\lambda : (\mathbb{C}^r, 0) \longrightarrow (\mathbb{C}^q, 0)$ be a germ of an analytic map. The pull-back of F via λ, $\lambda^* F$ is defined by $(\lambda^* F)(z,t) = F(z, \lambda(t))$. Then $\lambda^* F \in \int I(r)$, and $\lambda^* F$ is an unfolding of f, since $\lambda(0) = 0$.
Let $H \in \mathcal{D}_I(q)$ be an unfolding of $h \in \mathcal{D}_I$, which preserves I. The pull-back of H via λ, $\lambda^* H$ is defined by $(\lambda^* H)(z,t) = H(z, \lambda(t))$. Then $\lambda^* H \in \mathcal{D}_I(r)$ is an unfolding of h, since $\lambda(0) = 0$.
In the same way is $\lambda^* H \in \mathcal{D}_{I,e}(r)$ for an $H \in \mathcal{D}_{I,e}(q)$.
Hence the condition of naturality under pull-back is satisfied.

(II) Algebraic structure of the tangent space (4.11)

(i) We showed that $T\mathcal{D}_I(q) = \{ E \in T\mathcal{D}(q) \mid E(I) \subseteq I.\mathcal{O}(q) \}$ and $T\mathcal{D}_{I,e}(q) = \{ E \in T\mathcal{D}_e(q) \mid E(I) \subseteq I.\mathcal{O}(q) \}$, see (2.24). Moreover $T\mathcal{D}(q) \cong \mathfrak{m}.\mathcal{O}^m(q)$ and $T\mathcal{D}_e(q) \cong \mathcal{O}^m(q)$ as $\mathcal{O}(q)$-modules, see (2.12), (2.15) and (2.17). Hence $T\mathcal{D}_{I,e}(q)$ and $T\mathcal{D}_I(q)$ are finitely generated $\mathcal{O}(q)$-modules.

(ii) Let $F \in \int I(q)$, we defined an orbit map $\alpha_F : \mathcal{D}_I(q) \to \int I(q)$ see (2.36), and a germ of an orbit map $\alpha_F : (\mathcal{D}_{I,e}(q), id) \to (\int I(q), F)$, see (2.40). The corresponding derivatives $d\alpha_F : T\mathcal{D}_I(q) \to \int I(q)$ and $d\alpha_F : T\mathcal{D}_{I,e}(q) \to \int I(q)$ are $\mathcal{O}(q)$-linear maps.

(iii) One easily sees that

$$\int I(q) \Big/ (t).\int I(q) \cong \int I \qquad \text{and} \qquad T\mathcal{D}_{I,e}(q) \Big/ (t).T\mathcal{D}_{I,e}(q) \cong T\mathcal{D}_{I,e}$$

as $\mathcal{O}(q)$-modules. Where $(t) = (t_1, \ldots, t_q)\,\mathcal{O}(q)$, with $t_1, \ldots, t_q$ some local coordinates of the parameter space $(\mathbb{C}^q, 0)$.

(iv) We already noted in (3.4) that $\mathfrak{m}.T\mathcal{D}_{I,e} \subseteq T\mathcal{D}_I$

Hence the conditions on the algebraic structure of the tangent space are satisfied.

__Definition__ (4.12) Let $F \in \int I(q)$ and $G \in \int I(r)$ be two I-unfoldings of $f \in \int I$. A __morphism of I-unfoldings__ $\Phi : F \to G$ is a pair (H, λ) with $H \in \mathcal{D}_{I,e}(q)$ an I-unfolding of the identity in $\mathcal{D}_{I,e}$ and $\lambda : (\mathbb{C}^q, 0) \to (\mathbb{C}^r, 0)$ a germ of an analytic map such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^m \times \mathbb{C}^q & \xrightarrow{F} & \mathbb{C} \\ {\scriptstyle (H,\lambda)} \downarrow & & \| \\ \mathbb{C}^m \times \mathbb{C}^r & \xrightarrow{G} & \mathbb{C} \end{array}$$

, with $(H, \lambda)(z, t) = (H(z,t), \lambda(t))$. my example?

__Definition__ (4.13) An I-unfolding $F \in \int I(q)$ of $f \in \int I$ is called __versal__ if for any I-unfolding $G \in \int I(r)$ of f, there exist a morphism of I-unfoldings $\Phi : G \to F$. Two I-unfoldings F and G in $\int I(q)$ of $f \in \int I$ are called __equivalent__ if there exist two morphisms of I-unfoldings $\Phi : F \to G$ and $\Psi : G \to F$ such that $\Phi\Psi = id$ and $\Psi\Phi = id$.

Definition (4.14) The set of such equivalencies (with q parameters, $q \geq 1$) we denote by $\mathcal{E}_{I,e}(q)$.

Remark (4.15) The set $\mathcal{E}_{I,e}(q)$ of equivalencies form a pseudo group and its tangent space at id, $T(\mathcal{E}_{I,e}(q))$ is equal to $T\mathcal{D}_{I,e}(q) \oplus (t)T(\mathcal{O}_{(\mathbb{C}^q,0)})$, where $(t) = (t_1, \dots, t_q)\mathcal{O}_{(\mathbb{C}^q,0)}$ and $t_1, \dots, t_q$ some local coördinates of $(\mathbb{C}^q, 0)$.

(III) Exponential map (4.16)
There exists an exponential map

$$\exp : T(\mathcal{E}_{I,e}(q)) \longrightarrow \mathcal{E}_{I,e}(q+1)$$

, which is defined in the same way as for $T\mathcal{D}_{I,e}(q)$, by integrating vector fields.

(IV) Filtration property (4.17)
Let $H \in \mathcal{D}_I(q)$ then $H^*(m^k \cdot \int I(q)) = m^k \cdot \int I(q)$. Thus $\mathcal{D}_I(q)$ preserves the filtration $\{m^k \cdot \int I(q)\}_{k \in \mathbb{N}}$ on $\int I(q)$. and induces an action on the quotient $\int I(q) / m^k \cdot \int I(q)$ for all $k \geq 0$.

Hence all the conditions (I), ..., (IV) are satisfied and we have proved the following proposition.

Proposition (4.18) The group $\mathcal{D}_I$ is a geometric subgroup of $\mathcal{D}$.

Definition (4.19) An I-unfolding $F \in \int I(q)$ of $f \in \int I$ extends the I-unfolding $G \in \int I(k)$ of f, if there exists a germ of an analytic map $i : (\mathbb{C}^k, 0) \to (\mathbb{C}^q, 0)$, which is an inclusion, such that $G = i^* F$.

Definition (4.20) Let $F \in \int I(q)$ and $G \in \int I(k)$. Then F is called a trivial I-unfolding of G if F is equivalent to $G \times id_{\mathbb{C}^{q-k}}$, by an equivalence which restricts to the identity in $\mathcal{E}_{I,e}(k)$.

<u>Notation</u> (4.21) Let $F \in \int I(q)$ and $t_1, \ldots, t_q$ local coördinates of the parameter space $(\mathbb{C}^q, 0)$. We denote $\left(\frac{\partial F}{\partial t_i}\right)\Big|_{t=0}$ by $\partial_i F$ for $i = 1, \ldots, q$.

<u>Unfolding theorem</u> (4.22). Let $F \in \int I(q)$ be an I-unfolding of $f \in \int I(q)$. Then the following statements are equivalent.

(i) $\tau_{I,e}(f) + (\partial_1 F, \ldots, \partial_q F)\,\mathcal{O} = \int I$

(ii) F is a versal I-unfolding of f

(iii) Every I-unfolding G of f, which extends F, is a trivial I-unfolding of F.

<u>Definition</u> (4.23) Let $F \in \int I(q)$ be an I-unfolding of $f \in \int I$. Then F is called a <u>miniversal I-unfolding</u> of f if F is a versal I-unfolding and $q = c_{I,e}(f)$.

<u>Corollary</u> (4.24)

$c_{I,e}(f) < \infty$

(i) A function $f \in \int I$ has a versal I-unfolding if and only if

(ii) Any two versal I-unfoldings $F, G \in \int I(q)$ of $f \in \int I$ are equivalent

(iii) Any versal I-unfolding F of $f \in \int I$ is equivalent to a trivial unfolding of a miniversal I-unfolding of f.

<u>Proof</u> (4.22) and (4.24) are consequences of [D](9.3) and (9.9), since $\mathbb{D}_I$ is a geometric subgroup of $\mathbb{D}$ by (4.15).

<u>Example</u> (4.25) (i) Let $f(x,y,z) = xyz + z^r$, with $r \in \mathbb{N}$ and $r \geq 2$. Let $I = (xy, z)\,\mathcal{O}$, with $\mathcal{O} = \mathbb{C}\{x,y,z\}$. Define $F(x,y,z,t) = f(x,y,z) + t_1 z^2 + \cdots + t_{r-2} z^{r-1}$. Then F is a miniversal I-unfolding of $f \in \int I$.

(ii) Let $f(x,y) = x^2y^2(x^2 - y^2)$ and $I = (xy)\,\mathcal{O}$, with $\mathcal{O} = \mathbb{C}\{x,y\}$. Let $F(x,y,t) = x^2y^2(x^2 - y^2 + t_1 + t_2 x + t_3 y + t_4 xy)$. Then F is a miniversal I-unfolding of f.

§5 FUNCTIONS WITH TRANSVERSAL A_1 SINGULARITIES

Let $f : (\mathbb{C}^m, 0) \to \mathbb{C}$ be a germ of an analytic function and I an ideal in $\mathcal{O}$ defining a germ of an analytic space $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$. We shall construct a map $h_f : \mathrm{Hom}(I/I^2, \mathcal{O}) \to I/\int I$. Let $\delta_f = \dim_{\mathbb{C}}(\mathrm{Coker}\, h_f)$ and $\tau(\Sigma, 0)$ be the Tyurina number of $(\Sigma, 0)$. Suppose $f \in I^2$ and $(\Sigma, 0)$ has an isolated singularity and $j_f = \dim_{\mathbb{C}}(I/J_f) < \infty$. Then f has transversally only A_1 singularities on $\Sigma \setminus \{0\}$ and

$$j_f = c_{I,e}(f) + \tau(\Sigma, 0) + \delta_f$$

We shall prove the existence of functions $f \in I^2$ such that $j_f < \infty$ in case I defines a reduced curve.

<u>Definition</u> (5.1) Let I be an ideal in $\mathcal{O}$. Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function such that $J_f \subseteq I$. Then we call I/J_f a <u>Jacobi module</u>. Suppose moreover $I = \mathrm{rad}(J_f)$, then we define $j_f = \dim_{\mathbb{C}}(I/J_f)$.

<u>Lemma</u> (5.2) Let I be a radical ideal defining a germ of an analytic space $(\Sigma, 0)$ of dimension at least 1. Suppose $f \in \int I$. Then $c_{I,e}(f) \leq j_f$.

<u>Proof</u> We may assume $j_f < \infty$. First we prove $c_{I,e}(f) < \infty$. There exists an $N \in \mathbb{N}$ such that $m^N I \subseteq J_f$, since $j_f < \infty$. Let $g_1, \dots, g_n$ be generators of I. We can write $\frac{\partial f}{\partial z_i} = \sum_k \varphi_{ik} g_k$, since $f \in \int I$. Let $\varphi : \mathcal{O}_\Sigma^n \to \mathcal{O}_\Sigma^m$ be the $\mathcal{O}_\Sigma$-linear map given by the matrix (φ_{ik}). From the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}^m \arrow[r, "\overline{d^2 f}"] \arrow[dr, "dg"'] & \mathcal{O}_\Sigma^m \\
& \mathcal{O}_\Sigma^n \arrow[u, "\varphi"']
\end{tikzcd}$$

, we derive $\ker(dg) \subseteq \ker(\overline{d^2 f})$. But $m^N I \subseteq J_f$, hence $m^N \ker(\overline{d^2 f}) \subseteq \ker(dg)$. Take $a \in m^N$ and $\psi \in \int I$, then

$a\psi \in J_f$. Thus $a\psi = \sum_i \xi_i \frac{\partial f}{\partial z_i}$ and $\sum_i \xi_i \frac{\partial^2 f}{\partial z_i \partial z_j} \equiv \frac{\partial a\psi}{\partial z_j} \equiv 0 (\mathrm{mod}$ $)$, since $a\psi \in \int I$. Let $\xi = (\xi_1, \dots, \xi_m)$, then $\xi \in \ker(d^2 f)$.
Take $b \in m^N$, then $b\xi \in \ker(dg)$. Thus $b\xi \in T\Theta_{I,e}$ and $ab\psi = (b\xi)(f) \in \tau_{I,e}(f)$. So we have proved $m^{2N} \int I \subseteq \tau_{I,e}(f)$ hence $C_{I,e}(f) < \infty$.
By (3.16), we have $\tau_{I,e}(f) = J_f \cap \int I$. So $\int I / \tau_{I,e}(f) \cong J_f + \int I / J_f$.
Hence $C_{I,e}(f) \leq j_f$. This proves the lemma.

<u>Remark</u> (5.3) Suppose $d \geq \frac{1}{2}p(p+1)$ and $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is of type $D(d,p)$, see (3.13). Then

$$j_f = \begin{cases} 0 & \text{if } p = 0 \\ p & \text{if } p \geq 1 \text{ and } d = \frac{1}{2}p(p+1) \\ \infty & \text{if } p \geq 1 \text{ and } d > \frac{1}{2}p(p+1) \end{cases}$$

<u>Remark</u> (5.4) One may ask: which functions $f \in \int I$ have $j_f = 0$ Or what amounts to the same: which functions have a radical Jacobi ideal? There are at least two ways to produce such functions:
(i) Let $f(x,y) = g(x) + h(y)$.
If $\mathrm{rad}(J_g) = J_g$ and $\mathrm{rad}(J_h) = J_h$ then $\mathrm{rad}(J_f) = J_f$.
(ii) Let $f(x,y) = g(x) \cdot h(y)$ and g and h be squarefree.
If $\mathrm{rad}(J_g) = J_g$ and $\mathrm{rad}(J_h) = J_h$ then $\mathrm{rad}(J_f) = J_f$.

<u>Proposition</u> (5.5) Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function. Let I be the radical ideal of J_f defining an analytic space $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$ of dimension $d \geq 1$, with an isolated singularity. Suppose f has only $A(d)$ singularities on a punctured neighbourhood of 0 in Σ. Then $j_f < \infty$.

<u>Proof</u> Let $f : U \to \mathbb{C}$ be a representative of the given germ. We may assume that $(U \cap \Sigma) \setminus \{0\}$ has no singularities. Let $\mathcal{O}$ be the sheaf of analytic functions on U. Let $\mathcal{I}$ be the ideal sheaf of Σ on U with stalk $\mathcal{I}_0 = I$ at 0. Let $\mathcal{J}_f$ be the Jacobi ideal sheaf of f in $\mathcal{O}$. Let $f_x : (U,x) \to \mathbb{C}$ be the germ of f at x. Now $\mathcal{I}_x = \mathcal{O}_x = \mathcal{J}_{f_x}$ at all $x \in U \setminus \Sigma$, since

$\mathrm{rad}(J_f) = I$. Further $j_{f_x} = 0$ for all $x \in (U \cap \Sigma) \setminus \{0\}$, since f_x has an $A(d)$ singularity at x, after possibly shrinking U. Thus $\mathrm{Supp}(I/J_f) \subseteq \{0\}$ and therefore $j_f = \dim_{\mathbb{C}}(I/J_f) < \infty$.

(5.6) Let I be an ideal in $\mathcal{O}$ defining a germ of an analytic space $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$. Suppose $g_1, \dots, g_n$ are generators of I. Let $\mathcal{O}^p \xrightarrow{\psi} \mathcal{O}^n \xrightarrow{g} I \to 0$ be a presentation of I as an $\mathcal{O}$-module. Then $\mathcal{O}_\Sigma^p \xrightarrow{\psi} \mathcal{O}_\Sigma^n \xrightarrow{g} I/I^2 \to 0$ is a presentation of I/I^2 as an $\mathcal{O}_\Sigma$-module. Hence the following sequence is exact: $0 \to \mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma) \xrightarrow{{}^t g} \mathcal{O}_\Sigma^n \xrightarrow{{}^t\psi} \mathcal{O}_\Sigma^p$

The $\mathcal{O}_\Sigma$-linear map $dg : \mathcal{O}_\Sigma^m \to \mathcal{O}_\Sigma^n$, with matrix $\left(\frac{\partial g_i}{\partial z_j}\right)$, factorizes, i.e. the following diagram is commutative

$$\begin{tikzcd}
\mathcal{O}_\Sigma^m \arrow[r, "dg"] \arrow[dr, "dg"'] & \mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma) \arrow[d, "{}^t g"] \\
& \mathcal{O}_\Sigma^n
\end{tikzcd}$$

We denote the map $dg : \mathcal{O}_\Sigma^m \to \mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma)$ by dg too, by abuse of notation. See [Ar, 3].

<u>Definition</u> (5.7) Let $T^1_{(\Sigma,0)} = \mathrm{Coker}\left(dg : \mathcal{O}_\Sigma^m \to \mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma)\right)$. Then $T^1_{(\Sigma,0)}$ are called the <u>first order deformations</u> of the germ $(\Sigma, 0)$. The <u>Tyurina number</u> is by definition $\tau(\Sigma,0) = \dim_{\mathbb{C}} T^1_{(\Sigma,0)}$

<u>Remark</u> (5.8) The $\mathcal{O}_\Sigma$-module $T^1_{(\Sigma,0)}$ does not depend on the chosen generators of I. If $(\Sigma, 0)$ is a reduced analytic space then $T^1_{(\Sigma,0)} \cong \mathrm{Ext}^1_{\mathcal{O}_\Sigma}(\Omega^1_{(\Sigma,0)}, \mathcal{O}_\Sigma)$, see [Ar, 3]. The Tyurina number is finite if $(\Sigma, 0)$ has an isolated singularity.
In the same way one defines a sheaf $\mathcal{T}^1_\Sigma$ on Σ of $\mathcal{O}_\Sigma$-modules if Σ is an analytic space. The support of $\mathcal{T}^1_\Sigma$ is contained in the singular locus of Σ.

wlog $\left(\begin{array}{l}\operatorname{rad}(J_f) = I. \text{ Further, } j_{f_x}=0 \text{ for all } x\in (U\cap\Sigma)\setminus\{0\}, \text{ since} \\ f_x \text{ has an } A(d) \text{ singularity at } x, \text{ after possibly shrinking } U. \\ \text{Thus } U\cap \operatorname{Supp}(I/J_f)\subseteq\{0\} \text{ and therefore } j_f=\dim_{\mathbb{C}}(I/J_f)<\infty.\end{array}\right.$

<u>Construction</u> (5.9) Let I be a radical ideal in $\mathcal{O}$, defining a germ of an analytic space $(\Sigma,0)$ in $(\mathbb{C}^m,0)$. Suppose $f\in I^2$. Let I be generated by $g_1,\dots,g_n$. Then $f=\sum_{i,j} h_{ij}g_ig_j$ where $h_{ij}=h_{ji}$. Let $\mathcal{O}^p \xrightarrow{\psi} \mathcal{O}^n \xrightarrow{g} I\to 0$ be a presentation of the ideal I as an $\mathcal{O}$-module. Then $\mathcal{O}_\Sigma^p \xrightarrow{\psi} \mathcal{O}_\Sigma^n \xrightarrow{g} I/I^2\to 0$ is a presentation of I/I^2 as an $\mathcal{O}$-module. Hence

$$0\to \operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma)\xrightarrow{{}^tg}\mathcal{O}_\Sigma^n\xrightarrow{{}^t\psi}\mathcal{O}_\Sigma^p$$

is an exact sequence. Define the $\mathcal{O}_\Sigma$-linear map $h:\mathcal{O}_\Sigma^n\to\mathcal{O}_\Sigma^n$ by the matrix (h_{ij}). Define the map

$$h_f:\operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma)\longrightarrow I/JI$$

by the composition of the maps

$$\operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma)\xrightarrow{{}^tg}\mathcal{O}_\Sigma^n\xrightarrow{h}\mathcal{O}_\Sigma^n\xrightarrow{g} I/I^2\longrightarrow I/JI$$

<u>Lemma</u> (5.10) The map h_f depends only on f and I

<u>Proof</u> We have to prove that the map h_f does not depend on the chosen generators $g_1,\dots,g_n$ of I and does not depend on the chosen symmetric matrix (h_{ij}) such that $f=\sum_{i,j}h_{ij}g_ig_j$.

Suppose $I=(g_1,\dots,g_n)\mathcal{O}=(g_1,\dots,g_n,g_{n+1})\mathcal{O}$ and $f=\sum_{i,j\le n}h_{ij}g_ig_j$. Then $f=\sum_{i,j\le n+1}\tilde h_{ij}g_ig_j$, where $\tilde h_{ij}=h_{ij}$ in case $i,j\le n$ and $\tilde h_{ij}=0$ otherwise. Hence, if $g_1,\dots,g_n$ and $\bar g_1,\dots,\bar g_{\bar n}$ are two sets of generators of I then we may assume that $n=\bar n$. So, suppose $I=(g_1,\dots,g_n)\mathcal{O}=(\bar g_1,\dots,\bar g_n)\mathcal{O}$ and $f=\sum_{i,j}h_{ij}g_ig_j=\sum_{i,j}\bar h_{ij}\bar g_i\bar g_j$. An exercise in linear algebra shows that there exists an invertible $(n\times n)$-matrix (u_{ij}) such that $\bar g_i=\sum_j u_{ij}g_j$, see [Gi] p. 146. Then

$$f=\sum_{i,j}\bar h_{ij}\bar g_i\bar g_j=\sum_{i,j,k,l}{}^tu_{ki}\bar h_{ij}u_{jl}g_kg_l=\sum_{k,l}h_{kl}g_kg_l$$

Hence, the following diagram is commutative

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma) \arrow[rrr, "\bar h_f"] \arrow[dr, equal] \arrow[ddd, "{}^t\bar g"'] & & & I/\int I \\
 & \mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma) \arrow[r, "h_f"] \arrow[d, "{}^t g"] & I/\int I \arrow[ur, equal] & \\
 & \mathcal{O}_\Sigma^n \arrow[r, "h"] & \mathcal{O}_\Sigma^n \arrow[u, "g"'] \arrow[dr, "u"] & \\
\mathcal{O}_\Sigma \arrow[ur, "{}^t u"] \arrow[rrr, "\bar h"] & & & \mathcal{O}_\Sigma^n \arrow[uuu, "\bar g"']
\end{tikzcd}$$

So $\bar h$ and $u h\, {}^t u$ induce the same map $\bar h_f$. Therefore, we only need to consider the case $f = \sum_{ij} h_{ij} g_i g_j = \sum_{ij} \bar h_{ij} g_i g_j$ where (h_{ij}) and $(\bar h_{ij})$ are symmetric matrices.

Take $v_{ij} = \bar h_{ij} - h_{ij}$. Then (v_{ij}) is a symmetric matrix and $\sum_{ij} v_{ij} g_i g_j = 0$. So $\sum_{ij} v_{ij} g_i \frac{\partial g_j}{\partial z_k} \equiv 0 \pmod{I^2}$.

Let ξ be an element of $\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma)$, considered as an element of $\mathcal{O}_\Sigma^n$ by means of the map $\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma) \xrightarrow{{}^t g} \mathcal{O}_\Sigma^n$.

Then $\xi = (\xi_1,\dots,\xi_n)$ and

$$\frac{\partial}{\partial z_k}\Big(\sum_{ij} v_{ij}\xi_i g_j\Big) \equiv \sum_{ij} v_{ij}\xi_i \frac{\partial g_j}{\partial z_k} \equiv \xi\Big(\sum_{ij} v_{ij} g_i \frac{\partial g_j}{\partial z_k}\Big) \equiv 0 \pmod I$$

for all $k = 1,\dots,m$. Thus $\sum_{ij} v_{ij}\xi_i g_j \in \int I$.

So $\bar h_f(\xi) \equiv \sum_{ij} \bar h_{ij}\xi_i g_j \equiv \sum_{ij}(h_{ij}+v_{ij})\xi_i g_j \equiv \sum_{ij} h_{ij}\xi_i g_i \equiv h_f(\xi) \pmod{\int I}$

Therefore: the map h_f depends only on f and I.

<u>Remark</u> (5.11) The map h_f factorises as follows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2,\mathcal{O}_\Sigma) \arrow[r, "\tilde h_f"] \arrow[dr, "h_f"'] & I/I^2 \arrow[d] \\
 & I/\int I
\end{tikzcd}$$

This is a direct consequence of the construction of h_f. It could be that $\tilde h_f$ depends on the choice of some generators of I or on the symmetric matrix (h_{ij}). But we have no example.

<u>Lemma</u>(5.12) Let I be a radical ideal in $\mathcal{O}$ defining a germ of an analytic space $(\Sigma,0)$ in $(\mathbb{C}^m,0)$ of dimension $d \geq 1$.

Suppose Σ is locally a complete intersection at every point of a punctured neighbourhood of 0 in Σ. Suppose $f \in I^2$ and $j_f < \infty$. Then $\ker(h_f) = 0$ and $\dim_{\mathbb{C}} \operatorname{Coker}(h_f) < \infty$.

Proof Let U be an open neighbourhood of 0 in $\mathbb{C}^m$ such that Σ and f are defined on U and Σ is locally a complete intersection at all points in $(U \cap \Sigma) \setminus \{0\}$. Let $\mathcal{O}$ be the sheaf of analytic functions on U. Let I be the ideal sheaf of Σ in $\mathcal{O}$. Let I be generated by $g_1, \ldots, g_n$, then $f = \sum_{k,l} h_{kl} g_k g_l$ where $h_{kl} = h_{lk}$, since $f \in I^2$. Hence $\frac{\partial f}{\partial z_i} = \sum_k \varphi_{ik} g_k$, where $\varphi_{ik} \equiv \sum_l 2 h_{kl} \frac{\partial g_k}{\partial z_i} \pmod{I}$. Let $\varphi : \mathcal{O}_\Sigma^m \longrightarrow I/\int I$ be the map defined by $\varphi(\xi) \equiv \sum_{i,k} \xi_i \varphi_{ik} g_k \pmod{\int I}$ for $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_\Sigma^m$. The $\mathcal{O}_\Sigma$-linear map $dg : \mathcal{O}_\Sigma^m \to \operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma)$ is induced by the map $dg : \mathcal{O}_\Sigma^m \to \mathcal{O}_\Sigma^n$ with matrix $\left(\frac{\partial g_k}{\partial z_i}\right)$, see (5.6).

The following diagram is exact and commutative

$$\begin{tikzcd}
\mathcal{O}_\Sigma^m \arrow[r, "\varphi"] \arrow[d, "dg"'] & I/\int I \arrow[r] \arrow[d, equal] & I/\int I + J_f \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma) \arrow[r, "h_f"] & I/\int I \arrow[r] & \operatorname{Coker}(h_f) \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

The sheaf of $\mathcal{O}$-modules I/J_f has finite length, since $j_f < \infty$. Hence $l(\operatorname{Coker}(h_f)) \leq l(I/\int I + J_f) \leq j_f < \infty$. So $\operatorname{supp}(\operatorname{Coker}(h_f)) \subseteq \{0\}$. Let f_x be the germ of f at x and I_x the stalk of I at x. Let $x \in (U \cap \Sigma) \setminus \{0\}$, then by assumption I_x is generated by an $\mathcal{O}_x$-sequence $\bar{g}_1, \ldots, \bar{g}_{\bar{n}}$. So $f_x = \sum_{k,l} \bar{h}_{kl} \bar{g}_k \bar{g}_l$. Moreover $\int I_x = I_x^2$ and $I_x/I_x^2 \cong \mathcal{O}_{(\Sigma,x)}^{\bar{n}}$ and $\operatorname{Hom}(I_x/I_x^2, \mathcal{O}_{(\Sigma,x)}^{\bar{n}}) \cong \mathcal{O}_{(\Sigma,x)}^{\bar{n}}$. Let $\bar{h} : \mathcal{O}_{(\Sigma,x)}^{\bar{n}} \to \mathcal{O}_{(\Sigma,x)}^{\bar{n}}$ be the map with matrix $(\bar{h}_{kl})$. By the uniqueness of the map h_f (5.7), we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma) \otimes \mathcal{O}_{(\Sigma,x)} \arrow[r, "\cong"] \arrow[d, "(h_f)"'] & \operatorname{Hom}_{\mathcal{O}_{(\Sigma,x)}}(I_x/I_x^2, \mathcal{O}_{(\Sigma,x)}) \arrow[r, "\cong"] \arrow[d, "h_{f_x}"] & \mathcal{O}_{(\Sigma,x)}^{\bar{n}} \arrow[d, "\bar{h}"] \\
I/\int I \otimes \mathcal{O}_{(\Sigma,x)} \arrow[r, "\cong"] & I_x/I_x^2 \arrow[r, "\cong"] & \mathcal{O}_{(\Sigma,x)}^{\bar{n}}
\end{tikzcd}$$

Now $\mathrm{Coker}(h_f)_x = 0$ for all $x \in (U \cap \Sigma) \setminus \{0\}$, since $\mathrm{Supp}(\mathrm{Coker}(h_f)) \subseteq \{0\}$. Thus $(h_f)_x$ is surjective and $\bar{h}$ too. So $\bar{h}$ is injective, since it is given by an $(n \times n)$-matrix. Hence $\mathrm{Supp}(\ker(h_f)) \subseteq \{0\}$. The analytic space Σ is reduced and of dimension $d \geq 1$, so $\mathrm{depth}(\mathcal{O}_\Sigma) \geq 1$. Thus $\ker(h_f)$ is a sheaf of $\mathcal{O}$-modules of finite length and a subsheaf of $\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma)$, which is a sub-sheaf of $\mathcal{O}_\Sigma^n$ of depth ≥ 1. This is a contradiction unless $\ker(h_f) = 0$. This proves the lemma.

Definition (5.13) Let I be a radical ideal in $\mathcal{O}$. Let $f \in I^2$. Define $\delta_f := \dim_{\mathbb{C}} \mathrm{Coker}(h_f)$.

Theorem (5.14) Let I be an ideal in $\mathcal{O}$ defining a germ of a reduced analytic space $(\Sigma, 0)$ in $(\mathbb{C}^m, 0)$ of dimension $d \geq 1$. Suppose Σ is locally a complete intersection at all points in a punctured neighbourhood of 0 in Σ. Suppose $f \in I^2$ and $j_f < \infty$. Then $(\Sigma, 0)$ has an isolated singularity and f has transversally to Σ only A_1 singularities on a punctured neighbourhood of 0 in Σ. Moreover

$$j_f = c_{I_\Sigma}(f) + \tau(\Sigma, 0) + \delta_f$$

Remark (5.15) (i) The assumption $f \in I^2$, instead of $f \in \int I$, is essential. Take $I = (yz, xz, xy)\mathcal{O}$, where $\mathcal{O} = \mathbb{C}\{x, y, z\}$. Let $f(x,y,z) = xyz$. Then $J_f = I$, so $f \in \int I$ and $j_f = 0$. But $\tau(\Sigma, 0) = 3$. The Σ is the union of the coördinate lines in $\mathbb{C}^3$, hence it is locally a complete intersection outside the origin.

(ii) We shall identify δ_f with the number of D_∞ points in a generic unfolding of f, in case $(\Sigma, 0)$ is a complete intersection, see (7.10).

Proof of (5.14). Let U be a neighbourhood of 0 in $\mathbb{C}^m$ such that Σ and f are defined on U and Σ is a complete intersection at all points in $(U \cap \Sigma) \setminus \{0\}$. Let $\mathcal{O}$ be the sheaf of analytic functions on U. Let I be the ideal sheaf of Σ in $\mathcal{O}$. Let J_f be the Jacobi ideal sheaf of f. The stalk of I at 0

is I and the stalk $\mathcal{J}_0$ is equal to J_f.
Consider the first diagram in the proof of (5.12).

$$\begin{tikzcd}
\mathcal{O}_{\Sigma}^{m} \arrow[r, "\varphi"] \arrow[d, "d\varphi"'] & I/\mathcal{S}I \arrow[r] \arrow[d, equal] & I/\mathcal{S}I + J_f \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathcal{O}_{\Sigma}}(I/I^2, \mathcal{O}_{\Sigma}) \arrow[r, "h_f"] & I/\mathcal{S}I \arrow[r] & \mathrm{Coker}(h_f) \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

Then $\ker(h_f) = 0$, since $j_f < \infty$ by (5.12). Further $\mathrm{Coker}(d\varphi) = T_{\Sigma}^1$, see (5.8). Diagram chasing gives an exact sequence

$$0 \to T_{\Sigma}^1 \longrightarrow I/\mathcal{S}I + J_f \longrightarrow \mathrm{Coker}(h_f) \to 0$$

Looking at the stalks at 0 we get $\tau(\Sigma, 0) + \delta_f = \dim_{\mathbb{C}}(I/\mathcal{S}I + J_f)$
Hence $\dim_{\mathbb{C}} T^1_{(\Sigma,0)}$ is finite, so $\dim_{\mathbb{C}} T^1_{(\Sigma,x)} = 0$ for all $x \in (U \cap \Sigma) \setminus \{0\}$ after possibly shrinking U. The germ (Σ, x) is locally a complete intersection at all $x \in (U \cap \Sigma) \setminus \{0\}$ and $T^1_{(\Sigma,x)} = 0$, hence (Σ, x) is non singular. Therefore $(\Sigma, 0)$ has an isolated singularity. The following sequence is exact

$$0 \to \mathcal{S}I + J_f / J_f \longrightarrow I/J_f \longrightarrow I/\mathcal{S}I + J_f \to 0$$

and $c_{I,e}(f) = \dim_{\mathbb{C}}(\mathcal{S}I + J_f / J_f)$, see (5.2). Thus $j_f = c_{I,e}(f) + \tau(\Sigma, 0) + \delta_f$
Let f_x be the germ $f : (U, x) \longrightarrow \mathbb{C}$. At all points $x \in U \setminus \{0\}$ is $c_{I,e}(f_x) \leq j(f_x)$, by (5.2) and $j(f_x) = 0$. Now (Σ, x) is non singular and $c_{I_x,e}(f_x) = 0$. So f_x is of type $D(d,0) = A(d)$, by (3.) and (5.3). Hence f has transversally to Σ only A_1 singularities on $(U \cap \Sigma) \setminus \{0\}$.
This proves the theorem.

In the sequel we shall consider under which circumstances we can find $f \in \mathcal{S}I$ such that $j_f < \infty$.

Proposition (5.16) Let I be an ideal in $\underline{\mathcal{O}}$ defining a germ of a reduced equidimensional analytic space $(\underline{\Sigma}, 0)$ in $(\mathbb{C}^m, 0)$ of dimension $d \geq 1$. Then there exists a germ of an analytic function $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}, 0)$ such that the singular locus of f is $\underline{\Sigma}$ and f has transversal A_1 singularities on an open dense subset of $\underline{\Sigma}$.

Corrollary (5.17) Let I be an ideal in $\underline{\mathcal{O}}$ defining a germ of a reduced curve $(\underline{\Sigma}, 0)$ in $(\mathbb{C}^m, 0)$. Then there exists a germ of an analytic function $f : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}, 0)$ with $\underline{\Sigma}$ as singular locus and transversal A_1 singularities on a punctured neighbourhood of 0 in $\underline{\Sigma}$. Further $f \in \underline{\int I}$ and $j_f < \infty$.

Proof of (5.17). This is a direct consequence of (5.14) and (5.16).

In order to prove proposition (5.16) we need Bertini's theorem and a lemma, which is an exercise in prime avoidance. Bertini's theorem is well known and the lemma is probably known.

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Lemma (5.18) Let R be a Noetherian regular ring and I a radical ideal in R such that all associated primes of I have the same height n. Then there exist $g_1,\dots,g_n \in I$ such that for all $\mathfrak{p} \in \mathrm{Ass}(R/I)$ we have $(g_1,\dots,g_n)R_{\mathfrak{p}} = I_{\mathfrak{p}}$.

Proof The ideal I is radical in a Noetherian ring R, hence I has a prime decomposition $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$, with $\mathfrak{p}_i$ a prime ideal and $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for all $i \neq j$. All prime ideals $\mathfrak{p}_i$ have the same height n. Hence by induction we can construct a sequence of elements $g_{i1},\dots,g_{in}$ in $\mathfrak{p}_i$ such that $g_{i1} \in \mathfrak{p}_i \setminus (\bigcup_{j\neq i} \mathfrak{p}_j \cup \mathfrak{p}_i^2)$ and

$$g_{ik+1} \in \mathfrak{p}_i \setminus \left(\bigcup_{j\neq i} \mathfrak{p}_j \cup \mathfrak{p}_i^2 \cup (R \cap (g_{i1},\dots,g_{ik})R_{\mathfrak{p}_i})\right)$$

for all $i=1,\dots,r$ and $1 \le k < n$, see [Ma] 1.B.
Let k_i be the residue field of the local ring $R_{\mathfrak{p}_i}$. Then by construction $g_{i1},\dots,g_{in}$ are linear independent over k_i. The localization $R_{\mathfrak{p}_i}$ of the regular ring R is regular again. Thus $R_{\mathfrak{p}_i}$ is a regular local ring of Krull dimension $=$ height $\mathfrak{p}_i = n$. So $(g_{i1},\dots,g_{in})R_{\mathfrak{p}_i} = \mathfrak{p}_i R_{\mathfrak{p}_i} = I_{\mathfrak{p}_i}$.
Take $g_j = \prod_{i=1}^{r} g_{ij}$, then $(g_1,\dots,g_n)R_{\mathfrak{p}_i} = (g_{i1},\dots,g_{in})R_{\mathfrak{p}_i} = I_{\mathfrak{p}_i}$, since $g_{ij} \notin \mathfrak{p}_k$ for all j and $i \neq k$. This proves the lemma.

Bertini's theorem (5.19) Let $f, \varphi_1,\dots,\varphi_p$ be germs of analytic functions from $(\mathbb{C}^m,0)$ to $(\mathbb{C},0)$ and define $f_\lambda = f + \sum \lambda_i \varphi_i$ for $\lambda = (\lambda_1,\dots,\lambda_p) \in \mathbb{C}^p$. Then the singular locus of f_λ is contained in $V(\varphi_1,\dots,\varphi_p)$ for all $\lambda \in U$, where U is a dense subset of $\mathbb{C}^p$.

Proof Let V be an open neighbourhood of 0 in $\mathbb{C}^m$ and let $f: V \to \mathbb{C}$ and $\varphi_i: V \to \mathbb{C}$, for $i=1,\dots,p$, be representatives of the given germs. Consider the map $F: V \times \mathbb{C}^p \to \mathbb{C}$ defined by $F(z,\lambda) = f(z) + \sum_i \lambda_i \varphi_i(z)$. The critical locus of F is $V\left(\frac{\partial f}{\partial z_j} + \sum_i \lambda_i \frac{\partial \varphi_i}{\partial z_j}, \varphi_1,\dots,\varphi_p\right)$ and is contained in $B \times \mathbb{C}^p$, $B := V(\varphi_1,\dots,\varphi_p) \subseteq \mathbb{C}^m$. Let M be defined by $(F^{-1}(0) \setminus B \times \mathbb{C}^p)$. Then M is a regular analytic space. The set of critical values D_π of the projection map $\pi: M \to \mathbb{C}^p$ is a set of measure zero, by Sard's theorem [St]. Define $U = \mathbb{C}^p \setminus D_\pi$

The analytic subspace $\pi^{-1}(\lambda)\cap M$ of M is regular for all $\lambda\in U$. Moreover $\pi^{-1}(\lambda)\cap M = (f_\lambda^{-1}(0)\setminus B)\times\{\lambda\}$. Thus the singular locus of $f_\lambda^{-1}(0)$ is contained in B for all $\lambda\in U$ and U is a dense subset of $\mathbb{C}^p$. This proves the theorem.

<u>Proof</u> of (5.16). Let U be an open neighbourhood of 0 in $\mathbb{C}^m$ such that f and Σ are defined on U. Let $\mathcal{O}$ be the sheaf of analytic functions on . Let $\mathcal{I}$ be an ideal sheaf in $\mathcal{O}$ with stalk $\mathcal{I}_0 = I$ defining the reduced equidimensional analytic space Σ in U of dimension ≥ 1. Let $\Sigma = \Sigma_1\cup\dots\cup\Sigma_r$ be a decomposition of Σ in irreducible components. Let $\mathfrak{p}_i$ be the ideal sheaf in $\mathcal{O}$ such that $V(\mathfrak{p}_i) = \Sigma_i$. Then there exist $g_1,\dots,g_n\in \mathcal{I}$ such that $(g_1,\dots,g_n)\mathcal{O}_{\mathfrak{p}_i} = \mathcal{I}_{\mathfrak{p}_i}$, by (5.10). Take $f = g_1^2+\dots+g_n^2$ and let $\varphi_1,\dots,\varphi_p$ be generators of $\mathcal{I}^3$. Apply Bertini's theorem (5.19) to $f_\lambda = f + \sum_i \lambda_i\varphi_i$. So we may assume that the singular locus of f_λ is contained in $V(\varphi_1,\dots,\varphi_p) = V(\mathcal{I}^3) = \Sigma$ for some $\lambda\in\mathbb{C}^p$. But $f_\lambda\in \mathcal{I}^2\subseteq \int\mathcal{I}$, thus the singular locus of f_λ is equal to Σ. The functions $g_1,\dots,g_n$ are part of local coördinates at every point of some open dense subset V_i of Σ_i, since $(g_1,\dots,g_n)\mathcal{O}_{\mathfrak{p}_i} = \mathcal{I}_{\mathfrak{p}_i} = \mathfrak{p}_i\mathcal{O}_{\mathfrak{p}_i}$. Let $x\in V_i$, then there exist $g_{n+1},\dots,g_m$ such that $g_1,\dots,g_m$ are local coördinates of $(\mathbb{C}^m, x)$. Further $\mathcal{I}\mathcal{O}_x = (g_1,\dots,g_n)\mathcal{O}_x$ and $f_\lambda(g) = g_1^2+\dots+g_n^2+\sum_i\lambda_i\varphi_i$, where $\sum_i\lambda_i\varphi_i\in \mathcal{I}^3\mathcal{O}_x$. Thus f_λ is transversal A_1 at x, by the splitting lemma (3.16). Take $V = \bigcup_i V_i$, then V is an open dense subset of Σ and f_λ is transversal A_1 at all points of V. This proves proposition (5.16)

<u>Example</u> (5.20) Let $f\in\int I$, then $j_f<\infty$ implies $c_{I,e}(f)<\infty$, by (5.2). The converse does not hold. Take $\mathcal{O} = \mathbb{C}\{x,y,u,v\}$ and $I = (x,y)\mathcal{O}\cap(u,v)\mathcal{O}$. We already noted that $\int I = I^2$ in (1.10). Thus $f = ax + b(xv)^2 + c(yu)^2 + d(yv)^2 + px^2uv + qy^2uv + rxyu^2 + sxyv^2 + txyuv$, where $a,b,\dots,t\in\mathcal{O}$. Let Σ_1 be the plane $V(x,y)$. Now

$$f = (au^2+bv^2+puv)x^2 + (cu^2+dv^2+quv)y^2 + (ru^2+sv^2+tuv)xy$$

So we can write $f = h_{11}x^2 + h_{22}y^2 + (h_{12}+h_{21})xy$. Let $\Delta_1 = \det(h_{ij})$ then $\Delta_1 = (au^2+bv^2+puv)(cu^2+dv^2+quv) - \frac{1}{4}(ru^2+sv^2+tuv)$.

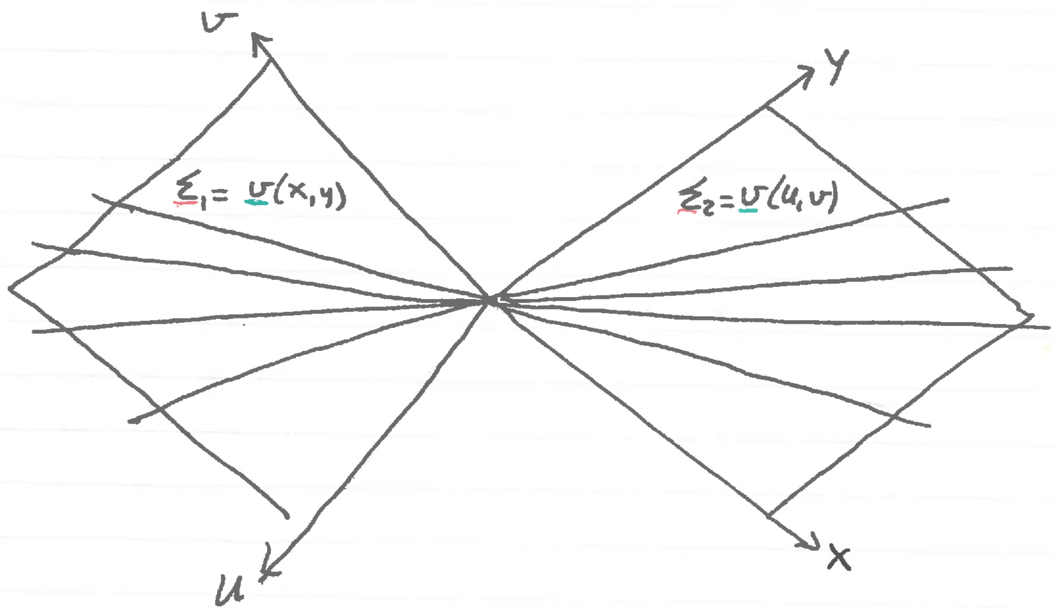
Let f_p be the germ of f at p, then $j_{f_p}\neq 0$ at all points

C_1 C_2 = furst

$c_{I,e}$ = klein

furst.

$p \in C_1 = V(x,y,\Delta_1)$. The dimension of C_1 is at least 1, since Δ_1 is not a unit in $\mathcal{O}$. Thus $j_f = \infty$.
Eventhough $j_f = \infty$ for every $f \in \int I$, there exists an $f \in \int I$ such that klein $c_{I,e}(f) < \infty$.
Take $f = \frac{1}{2}[(xu)^2 + (xv)^2 + (yu)^2 - (yv)^2\}$. Let $\Sigma_1 = V(x,y)$ and $\Sigma_2 = V(u,v)$, then Σ_1 and Σ_2 are two planes such that $\Sigma_1 \cap \Sigma_2 = \{0\}$ and $\Sigma = \Sigma_1 \cup \Sigma_2$ is the singular locus of f.
Let C_1 be the curve $V(x,y,(u^2+v^2)(u^2-v^2))$ in Σ_1, then C_1 consists of four lines. Let $C_2 = V(u,v,(x^2+y^2)(x^2-y^2))$, then C_2 consists of four lines in Σ_2. The function f is transversal A_1 at all points of $\Sigma \setminus (C_1 \cup C_2)$ and has type $D(2,1)$ at all points of $(C_1 \cup C_2) \setminus \{0\}$. Thus the sheaf $\int I / \tau_{I,e}(f)$ is concentrated at $\{0\}$ and therefore $c_{I,e}(f) < \infty$.



§5 Quasihomogeneous and polynomial functions

We shall state a result of K. Saito concerning quasi homogeneous functions and give two applications of the finite determinacy theorem of §4.
We shall give an example of a function defining an algebraic hypersurface but which is not equivalent with a polynomial

Definition (5.1) A germ of an analytic function $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ is called quasi homogeneous (with respect to the local coördinates $z_0,\dots,z_n$) with weights $w_0,\dots,w_n \in \mathbb{N}$ and of degree $d \in \mathbb{N}$ if $f(\lambda^{w_0} z_0,\dots,\lambda^{w_n} z_n) = \lambda^d f(z)$

Theorem (5.2) Let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function then:
$f \in \underline{m} J_f$ if and only if f is quasi homogeneous with respect to some local coördinates

Proof, see [Sa, 1].

Remark (5.3) In case f has an isolated singularity or more generally if f is of isolated singularity type, i.e. $\{x \in \mathbb{C}^{n+1} \mid (f^{-1}(0),x) \cong (f^{-1}(0),0)\} = \{0\}$, one has a sharper version since then $f \in J_f$ if and only if $f \in \underline{m} J_f$, see [Sa, 1] [G-H].

Example (5.4) Let $g(x,y) = x^5 + x^2y^2 + y^5$ then $g \notin J_g$.
Take $f(x,y,z) = (1+z)\,g(x,y)$ then $f \in J_f$ but $f \notin \underline{m} J_f$.
Let $h(x,y,z) = z\,g(x,y)$ then the locus of points where h is not quasi homogeneous (in certain local coördinates) is $\{(0,0,z) \in \mathbb{C}^3 \mid z \neq 0\}$. Hence the "non quasi homogeneous locus" of an analytic function is in general not a closed set.

Definition (5.5) $\Sigma_{NQH}(f) := \{ x \in \mathbb{C}^{n+1} \mid f_x \notin J_{f,x} \}$

§6 QUASI-HOMOGENEOUS AND POLYNOMIAL FUNCTIONS

We shall state a result of K. Saito concerning weakly quasi-homogeneous functions and give two applications of the finite determinacy theorem of §4. We shall give an example of a function defining an algebraic hypersurface but which is not equivalent with a polynomial.

Definition (6.1) A formal function $f \in \mathbb{C}[[z_0, \dots, z_n]]$ is called weakly quasi-homogeneous with weights $w_0, \dots, w_n \in \mathbb{Z}$ and of degree $d \in \mathbb{Z}, d \geq 1$ if $f = \sum_{\underline{\alpha}} a_{\underline{\alpha}} z^{\underline{\alpha}}$ and $a_{\underline{\alpha}} = 0$ for all $\underline{\alpha}$ such that $\alpha_0 w_0 + \dots + \alpha_n w_n \neq d$. If f is a convergent powerseries we call f quasi-homogeneous with respect to $z_0, \dots, z_n$.

Theorem (6.2) Let $f \in \mathbb{C}[[z_0, \dots, z_n]]$ then $f \in \hat{\mathfrak{m}} J_f$ if and only if f is weakly quasi-homogeneous after a formal coördinate transformation

Proof see [Sa, 1] (3.3).

Remark (6.3) In case f is a germ of an analytic function with an isolated singularity one has a stronger version: $f \in J_f$ if and only if f is quasi-homogeneous with positive weights after an analytic coördinate transformation see [Sa, 1] (4.1).

Example (6.4) Let $g(x,y) = x^5 + x^2y^2 + y^5$ then $g \notin J_g$.
Take $f(x,y,z) = (1+z)\, g(x,y)$ then $f \in J_f$ but $f \notin \mathfrak{m} J_f$
Let $h(x,y,z) = z g(x,y)$ then the locus of points where h is not quasi-homogeneous (in certain local coördinates)

Remark (6.6) We denoted by f_x the germ $f:(\mathbb{C}^{n+1},x) \to \mathbb{C}$ and by $J_{f,x}$ the corresponding Jacobi ideal in the local ring $\mathcal{O}_x$ of germs of analytic functions $f:(\mathbb{C}^{n+1},x) \to \mathbb{C}$. $\Sigma_{NQH}(f)$ is an analytic set in $\mathbb{C}^{n+1}$, since it is the support of the coherent sheaf $(f)+J_f/J_f$. It contains the points where f is not quasi homogeneous, but it may be bigger as (6.4) shows.
Finally one has: $\Sigma_{NQH}(f) \subseteq \{0\}$ if and only if $\underline{m}^N(f) \subseteq J_f$ for some $N \in \mathbb{N}$

The following proposition is due to Bochnak [Bo]. We shall prove it as a consequence of the finite determinacy theorem (4.5)

Proposition (6.7) Let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function. Suppose $f^{-1}(0)$ is algebraic and $\Sigma_{NQH}(f) \subseteq \{0\}$ then f is right equivalent with a polynomial.

proof We have $f^{-1}(0)$ is algebraic, i.e. $f^{-1}(0) = P^{-1}(0)$ with P a polynomial. Hence $f = u.P$ with u a unit in the local ring $\underline{\mathcal{O}}$. Take $I = J_P + (P)\underline{\mathcal{O}}$ then $f \in \int I$

Lemma (6.8) In the above situation we have:
if $\underline{m}^N.(f) \subseteq J_f$ then $\underline{m}^N. \int I \subseteq \tau_{I,e}(f)$

proof (6.8) Suppose $\underline{m}^N(f) \subseteq J_f$. Let $a \in \underline{m}^N$ and $\varphi \in \int I$. Then $a.\varphi \in \underline{m}^N((f)+J_f) \subseteq J_f$, Hence $a\varphi = \sum_i \xi_i \frac{\partial f}{\partial z_i}$ and

$$\frac{\partial(a\varphi)}{\partial z_j} \equiv \sum_i \xi_i \frac{\partial^2 f}{\partial z_i \partial z_j} \pmod{J_f}.$$

Since $\varphi \in \int I$, we have $a\varphi \in \int I$, so $\frac{\partial(a\varphi)}{\partial z_j} \in I$, thus $\sum_i \xi_i \frac{\partial^2 f}{\partial z_i \partial z_j} \in I$. Let ξ be the vectorfield $\sum_i \xi_i \frac{\partial}{\partial z_i}$, then $\xi(I) \subseteq I$, so $\xi \in T(\underline{\mathcal{D}}_{I,e})$ and $a\varphi = \xi(f) \in \tau_{I,e}(f)$
This proves lemma (6.8).

We continue with the proof of (6.7). By assumption we have $\Sigma_{NQH}(f) \subseteq \{0\}$, thus $\underline{m}^N(f) \subseteq J_f$ for some $N \in \mathbb{N}$ and so

$m^N \cdot SI \subseteq T_{I,e}(f)$. Hence f has finite I-codimension and proposition (4.) gives: f is finitely I-determined. Let u_k be the Taylorexpansion of u up to order k, then $u_k \cdot P$ is a polynomial and an element of SI. $J_k(f) = J_k(u_k \cdot P)$, since $f = u \cdot P$. Thus for all $k \gg 0$ we have f is holomorphically equivalent with $u_k \cdot P$. More precisely there exist a k_0 such that for all $k \geq k_0$ there excist a $h_k \in \mathcal{D}_I$ with $f \circ h_k = u_k \cdot P$. This proves (5.7).

Proposition (5.7) gives rise to the following question posed by Bochnak and Kucharz [B-K].

<u>Question</u> (5.9) Suppose $P \in \mathbb{C}[z_0, \ldots, z_n]$ and u is an analytic unit, i.e. $u \in \mathbb{C}\{z_0, \ldots, z_n\}^*$. Is there always a germ of a local analytic isomorfism h such that $h^*(uP)$ is a polynomial ?

6.4 The following example gives a negative answer.

<u>Example</u> (5.10) Let $P(x,y,z) = xyz(x^4 + y^4 + x^3yz + x^2y^3z)$ and $u(x,y,z) = \exp(z)$ (or any other transcendental function of z). Then $f = uP$ is not equivalent with a polynomial. In order to prove this claim we need a fact of Nash functions. $N_0(\mathbb{C}^n) := \{ f \in \mathcal{O}_{(\mathbb{C}^n,0)} \mid$ there excist a $P(z,w) \in \mathbb{C}[z_1, \ldots, z_n, w]$ such that $P(z, f(z)) = 0 \}$
These functions are called germs of (complex) <u>Nashfunctions</u>. $N_0(\mathbb{C}^n)$ is the henselization of the local ring $\mathbb{C}[z_1, \ldots, z_n]_{(z)}$, see [N]. We have the following fact: let $f \in N_0(\mathbb{C}^n)$ then f is irreducible in $N_0(\mathbb{C}^n)$ if and only if f is irreducible in $\mathbb{C}[[z_1, \ldots, z_n]]$, the formal power series in n variables, [N]. We also need the following fact. Let $f(x_1, \ldots, x_n, z) = \sum f_\alpha(z) x^\alpha$ and $f \in N_0(\mathbb{C}^n \times \mathbb{C})$ then $f_\alpha \in N_0(\mathbb{C})$ for all $\alpha \in \mathbb{N}$.
Now let's prove the claim of the example and suppose $f = uP$ is equivalent with a polynomial Q, i.e. there excists a germ of a local analytic isomorfism $h: (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $f \circ h = Q$.

Since f factorises in $f = f_1 f_2 f_3 f_4$ with $f_1 = x$, $f_2 = y$, $f_3 = z$ we can factorize Q in $\hat{\mathcal{O}} = \mathbb{C}[[x,y,z]]$ as $Q = Q_1 Q_2 Q_3 Q_4$ with $(Q_i)\hat{\mathcal{O}} = (f_i \circ h)\hat{\mathcal{O}}$. But, since Q is a polynomial and a priori a Nash function, we may assume that the Q_i's are Nash function's, by the remark above. So after the local Nash isomorphism $(x,y,z) \mapsto (Q_1, Q_2, Q_3)$, we may assume that f is right equivalent with a Nash function $\tilde{Q}$. i.e. $\tilde{Q} = f \circ \tilde{h}$ with $\tilde{h}$ a germ of a local analytic isomorphism of the form $h(x,y,z) = (ax, by, cz)$ with a,b,c elements of $\mathbb{C}\{x,y,z\}^*$. Take $\psi(z) = \exp(c \cdot z)$.

$$\begin{aligned}\tilde{Q}(x,y,z) &= \psi abc\, xyz\,(a^4x^4 + b^4y^4 + a^3bc\,x^3yz + a^2b^3c\,x^2y^3z) \\ &= \psi a^5bcz\,x^5y + \psi ab^5cz\,xy^5 + \psi a^4b^2c^2z^2\,x^4y^2 + \psi a^3b^4c^2z^2\,x^3y^4\end{aligned}$$

diagram
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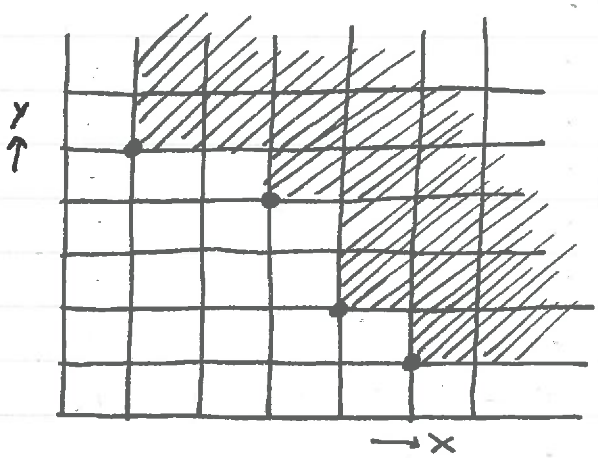
For any function $\eta \in \mathbb{C}\{x,y,z\}$ we define $\eta_0 \in \mathbb{C}\{z\}$ by $\eta_0(z) = \eta(0,0,z)$. By the particular choice of the monomials of P (see diagram, which is the Newton diagram of $\tilde{Q}$ with respect to x and y and z as a parameter), we have that
$\psi_0 a_0^5 b_0 c_0 z$, $\psi_0 a_0 b_0^5 c_0 z$, $\psi_0 a_0^4 b_0^2 c_0^2 z^2$, $\psi_0 a_0^3 b_0^4 c_0^2 z^2$ are Nash functions, i.e. elements of $N_0(\mathbb{C})$.
Since $N_0(\mathbb{C}) \subseteq \overline{\mathbb{C}(z)}$, the algebraic closure of the field of rational functions in z, we conclude:



$$\left(\frac{a_0}{b_0}\right)^4 = \frac{\psi_0 a_0^5 b_0 c_0 z}{\psi_0 a_0 b_0^5 c_0 z} \in \overline{\mathbb{C}(z)}, \text{ so}$$

$$\frac{a_0}{b_0} \in \overline{\mathbb{C}(z)}.$$

And $\dfrac{a_0}{b_0^2} = \dfrac{\psi_0 a_0^4 b_0^2 c_0^2 z^2}{\psi_0 a_0^3 b_0^4 c_0^2 z^2} \in \overline{\mathbb{C}(z)}$. Hence $a_0, b_0 \in \overline{\mathbb{C}(z)}$.

So $\psi_0 c_0$ and $\psi_0 c_0^2$ are elements of $\overline{\mathbb{C}(z)}$. Thus $\psi_0, c_0 \in \overline{\mathbb{C}(z)}$. c_0 is a unit of $\overline{\mathbb{C}(z)}$. Hence $\overline{\mathbb{C}(v)}$ is isomorphic with $\overline{\mathbb{C}(z)}$ by the isomorphism $\sigma: u \longmapsto c_0 \cdot z$. Now $\psi_0 = \exp(c_0 z) \in \overline{\mathbb{C}(z)}$. Thus $\exp(v) \in \overline{\mathbb{C}(v)}$, which is a contradiction.
Conclusion: f is not equivalent with a polynomial.

<u>Remark</u> (6.11) Example (6.10) also shows that for this function f there exists no ideal I' such that $f \in \int I'$ and $c_{I'}(f) < \infty$. Remember that $I = (f) + J_f$ and $f = u.P$, with $u = \exp(z)$. Suppose otherwise, that there exists an ideal I' such that $f \in \int I'$ and $c_{I'}(f) < \infty$. Then, for all $g \in \int I'$ and $k \gg 0$ with $J^k(f) = J^k(g)$ we have f and g are I'-equivalent, by (4.5). Now $(P) + J_P = (f) + J_f \subseteq I'$, thus $P \in \int I'$. Moreover $J^k(u_k P) = J^k(f)$, with $u_k = \sum_{n \leq k} \frac{1}{n!} z^n$. We would have f is I'-equivalent with $u_k P$ a polynomial, which contradicts (6.10)

<u>Proposition</u> (6.12) Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a one dimensional singular set Σ, such that f has transversally to Σ, outside 0, only A_1 singularities. Then f is right equivalent with a polynomial.

Before we give a proof we need a theorem.

<u>Theorem</u> (6.13) Let $(V, 0)$ be a germ of a reduced analytic space in $(\mathbb{C}^m, 0)$ such that $(V, 0)$ has an isolated singularity and all components of $(V, 0)$ are equidimensional. Then there are local coördinates $z_1, \ldots, z_m$ in $(\mathbb{C}^m, 0)$ such that $(V, 0)$ is defined by an ideal generated by polynomials in $z_1, \ldots, z_m$.

<u>Proof</u> of (6.13), see [Ar, 2], [To, 2].

<u>Proof</u> of (6.12). Let J_f be the Jacobi ideal of f in the local ring $\mathcal{O}$. Define $I = \operatorname{rad}(J_f)$, then I defines a germ of a one dimensional (reduced) curve. There are local coördinates $z_1, \ldots, z_m$ of $(\mathbb{C}^m, 0)$, such that I is generated by polynomials in $z_1, \ldots, z_m$, by (6.13). We have $f \in \int I$, by (1.2). Transversally to Σ, outside 0, f has only A_1 singularities, hence $\dim_{\mathbb{C}} (I/J_f) < \infty$, by (5.5). Thus $c_I(f) < \infty$, by (5.2). From (4.5) we derive: f is finitely I-determined.
The ideal I is generated by some polynomials $g_1, \ldots, g_n$. So $f = \sum_i a_i g_i$. Moreover $f \in \int I$, thus $\sum_i a_i \frac{\partial g_i}{\partial z_j} \equiv 0 \pmod I$.
Let $d = \max\{\deg(g_i) \mid i = 1, \ldots, n\}$, then $\sum_i J^k(a_i) \frac{\partial g_i}{\partial z_j} \equiv 0 \pmod I$

, for all $k \geq d$. Hence $\sum_i J^k(a_i) g_i \in JI$ and $J^k(f) = J^k(\sum_i J^k(a_i) g_i)$. Thus f is I-equivalent with $\sum_i J^k(a_i) g_i$ for $k \gg 0$, by the finite I-determinacy theorem (4.5). Remark that $\sum_i J^k(a_i) g_i$ is a polynomial. Hence we have proved proposition (6.12).

We end with a question.

<u>Question</u> (6.14) Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a one dimensional singular set Σ, such that f has transversally to Σ, outside 0, only simple singularities. Is it true that f is right equivalent with a polynomial?

§7 THE RESIDUAL DISCRIMINANT AND THE NUMBER OF A_1 AND D_∞ POINTS OF A DERFOMATION

We give a definition of a deformation of a pair (f, Σ) where Σ is the singular locus of the function f, which generalizes the notion of an I-unfolding of §4. We show that the number of j_f behaves well under deformations in case Σ is a curve. We show that certain nice deformation of (f,Σ) exist in case that Σ is a complete intersection curve and $j_f < \infty$ and prove a formula for j_f in terms of A_1 points and D_∞ points of such a deformation. We give a proof of a conjecture of Siersma.

Definition (7.1) Let R be a Noetherian commutative local ring with a unit. Let M be a finitely generated R-module of krull dimension n. Then M is called CM or a Cohen-Macaulay R-module if there exists an M-regular sequence $f_1, \ldots, f_n$ in the maximal ideal of R, i.e. f_i is not a zero-divisor on $M/(f_1,\ldots,f_{i-1})M$ for $i = 1,\ldots,n$.

Proposition (7.2) Let X be an analytic space and $\mathcal{M}$ a coherent sheaf of $\mathcal{O}_X$-modules. Let x be a point of X. Let m_s be the maximal ideal of $\mathcal{O}_{(\mathbb{C}^n,s)}$. Suppose $\mathcal{M}_x$ is a Cohen-Macaulay $\mathcal{O}_{(X,x)}$-module of krull dimension n. Let $f : (X,x) \to (\mathbb{C}^n,0)$ be a germ of an analytic map such that $\dim_{\mathbb{C}} (\mathcal{M}_x / f^*(m_0)\mathcal{M}_x) < \infty$. Then there exist open neighbourhoods U of x in X and V of 0 in $\mathbb{C}^n$ such that:

$$\dim_{\mathbb{C}} \left(\mathcal{M}_x / f^*(m_0)\mathcal{M}_x \right) = \sum_{y \in f^{-1}(s) \cap U} \dim_{\mathbb{C}} \left(\mathcal{M}_y / f^*(m_s)\mathcal{M}_y \right)$$

for all $s \in V$.

Proof. We denote $\dim_{\mathbb{C}} M$ by $\ell(M)$ and $\mathcal{O}_{(\mathbb{C}^n,0)}$ by $\mathcal{O}_0$. Now $\mathcal{M}_x$ is a finitely generated $\mathcal{O}_0$-module, via the map $f^* : \mathcal{O}_0 \to \mathcal{O}_{(X,x)}$, since $\ell(\mathcal{M}_x / f^*(m_0)\mathcal{M}_x) < \infty$, see

[Na] chap. IV, thm. 7. Then $\mathcal{M}_x$ is a CM $\mathcal{O}_0$-module, since $\mathcal{M}_x$ is a CM $\mathcal{O}_{(X,x)}$-module, see [Se, 2] chap. IV, prop. 11. Hence $\mathcal{M}_x$ is a finitely generated free $\mathcal{O}_0$-module, since Kdim $\mathcal{M}_x = n =$ Kdim $\mathcal{O}_0$ and $\mathcal{O}_0$ is a regular local ring, see [Se, 2] chap. IV, prop. 21. Let r be the rank of $\mathcal{M}_x$ as $\mathcal{O}_0$-module. We can choose open neighbourhoods U of x in X and V of o in $\mathbb{C}^n$ such that $f_*(\mathcal{M}_U)$ is a free sheaf of $\mathcal{O}_V$-modules of rank r. Let $\mathcal{O} = \mathcal{O}_V$. Then $\mathcal{M}_y$ is a free $\mathcal{O}_{f(y)}$-module, say of rank $r(y)$, for all $y \in U$. The direct sum $\bigoplus_{y \in f^{-1}(s)} \mathcal{M}_y$ is a free $\mathcal{O}_s$-module of rank r for all $s \in V$. Thus

$$\sum_{x \in f^{-1}(y)} \ell\left(\mathcal{M}_y / f^*(\mathfrak{m}_s)\mathcal{M}_y\right) = \sum_{y \in f^{-1}(s)} r(y) = r$$

and $\ell\left(\mathcal{M}_x / f^*(\mathfrak{m}_o)\mathcal{M}_x\right) = r$. This proves the proposition.

<u>Definition</u> (7.3) Let (Σ, o) be a germ of an analytic space in $(\mathbb{C}^m, o)$ defined by an ideal I in $\mathcal{O}$. A <u>deformation of (Σ, o)</u> consists of a germ of a flat map $G : (X, o) \longrightarrow (S, o)$ of analytic spaces together with an imbedding $i : (\Sigma, o) \to (X, o)$ such that $(i(\Sigma), o) \cong (G^{-1}(o), o)$. We can imbed (S, o) in $(\mathbb{C}^T, o)$ and (X, o) in $(\mathbb{C}^m \times \mathbb{C}^T, o)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
(\Sigma, o) & \xrightarrow{i} & (X, o) & \xrightarrow{G} & (S, o) \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{C}^m, o) & \xrightarrow{j} & (\mathbb{C}^m \times \mathbb{C}^T, o) & \xrightarrow{p} & (\mathbb{C}^T, o)
\end{array}$$

where $j : (\mathbb{C}^m, o) \longrightarrow (\mathbb{C}^m \times \mathbb{C}^T, o)$ is defined by $j(z) = (z, o)$ and $p : (\mathbb{C}^m \times \mathbb{C}^T, o) \longrightarrow (\mathbb{C}^T, o)$ is the projection on the second factor. A deformation is called <u>infinitesimal</u> is the maximal ideal $\mathfrak{m}_S$ of $\mathcal{O}_{(S,o)}$ is nilpotent, i.e. $\mathfrak{m}_S^N = 0$ for some $N \in \mathbb{N}$. In case $\mathfrak{m}_S^2 = 0$ we call it a <u>first order</u> deformation. See [Ar, 3] for more about deformations of singularities.

<u>Definition</u> (7.4) Let $G : (\underline{X}, 0) \to (S, 0)$ be a map of analytic spaces together with $i : (\Sigma, 0) \to (\underline{X}, 0)$ define a deformation of $(\Sigma, 0)$. Let $\tilde{\mathcal{O}}$ be the local ring $\mathcal{O}_{(\mathbb{C}^m \times S, 0)}$. Let $\tilde{I}$ be the ideal in $\tilde{\mathcal{O}}$ defining the germ $(\underline{X}, 0)$ considered as subspace of $(\mathbb{C}^m \times S, 0)$. A germ of an analytic map $F : (\mathbb{C}^m \times S, 0) \to (\mathbb{C}, 0)$ is called a <u>deformation of $(f, \Sigma, 0)$</u> if $(F)\tilde{\mathcal{O}} + J_F \subseteq \tilde{I}$ and $F(z, 0) = f(z)$, where $J_F = \left(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_m}\right)\tilde{\mathcal{O}}$ and $z_1, \ldots, z_m$ some local coördinates of $(\mathbb{C}^m, 0)$.

<u>Example</u> (7.6) Suppose $(\Sigma, 0)$ is a germ of a complete intersection in $(\mathbb{C}^m, 0)$ with an isolated singularity, defined by the ideal I in $\mathcal{O}$. Then I is generated by an $\mathcal{O}$-sequence $g_1, \ldots, g_n$. Suppose $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is a germ of an analytic ~~map~~ function such that $f \in \int I$ and $c_{I,e}(f) < \infty$. Then $f = \sum_{ij} h_{ij} g_i g_j$, since $\int I = I^2$ by (1.9). Let $\varphi_1, \ldots, \varphi_q$ be elements of $\mathcal{O}$ which project to a basis over $\mathbb{C}$ of the finite dimensional $\mathbb{C}$-vector space $\int I / \tau_{I,e}(f)$. Then $q = c_{I,e}(f)$. The germ $(\Sigma, 0)$ has a versal deformation $G : (\underline{X}, 0) \to (S, 0)$ where $(\underline{X}, 0) = (\mathbb{C}^m \times \mathbb{C}^p, 0)$ and $S = (\mathbb{C}^n \times \mathbb{C}^p, 0)$ and $G(z, u) = (G_1(z, u), \ldots, G_n(z, u), u)$ and $G_k(z, 0) = g_k(z)$, see [Lo] (6.5). We can write $\varphi_k = \sum_{ij} \varphi_{ijk} g_i g_j$ since $\varphi_i \in \int I = I^2$. Define

$$F(z, u, v) = \sum_{i,j} \left(h_{ij} + \sum_k \varphi_{ijk} v_k \right) G_i(z, u) \, G_j(z, u)$$

Then this defines a deformation of $(f, \Sigma, 0)$.

<u>Example</u> (7.7) Let $I = (yz, xz, xy)\mathcal{O}$ where $\mathcal{O} = \mathbb{C}\{x, y, z\}$. Let $\Sigma = V(I)$ then Σ is the union of the three coördinate lines in $\mathbb{C}^3$. We shall consider three different deformations of $(f, \Sigma, 0)$ (i) The miniversal deformation of $(\Sigma, 0)$ is obtained as follows. Let $S_1 = \mathbb{C}^3$ with coördinates s_1, s_2, s_3. The (2×2)-minors of the matrix

$$\begin{pmatrix} x & y & z \\ x + s_1 & 2y + s_2 & 3z + s_3 \end{pmatrix}$$

generate the ideal $\tilde{I}_1$, i.e.

$\tilde{I}_1 = (yz + s_3 y - s_2 z,\ 2xz + s_3 x - s_1 z,\ xy + s_2 x - s_1 y)\tilde{\mathcal{O}}_1$ where $\tilde{\mathcal{O}}_1 = \mathbb{C}\{x, y, z, s_1, s_2, s_3\}$.

Let $X_1 = V(\tilde{I}_1) \subseteq \mathbb{C}^3 \times S'_1$ and $p: \mathbb{C}^3 \times S'_1 \to S'_1$ the projection on the second factor. Let $G_1 = p|X_1$. Then $G_1 : X_1 \to S'_1$ is a miniversal deformation of $(\Sigma, 0)$. Let

$F_1(x,y,z,s_1,s_2,s_3) = (yz + s_3 y - s_2 z)^2 + (2xz + s_3 z - s_1 z)^2 + (xy + s_2 x - s_1 y)^2$

Then F_1 is a deformation of $(f, \Sigma, 0)$.

(ii) Let $S'_2 = \mathbb{C}$ and $X_2 = \Sigma \times S'_2$ and $G_2 : X_2 \to S'_2$ the projection on the second factor. Then G_2 is a trivial deformation of $(\Sigma, 0)$. But $F_2(x,y,z,s) = f(x,y,z) + sxyz$ defines a non-trivial deformation of $(f, \Sigma, 0)$.

(iii) Let S'_3 be defined by the ideal (t^2) in $\mathbb{C}\{t\}$. Let $\tilde{O}_3$ be the local ring of $(\mathbb{C}^3 \times S'_3, 0)$. Let $\tilde{I}_3 = (yz, xz, (x-t)y)\tilde{O}_3$ and $(X_3, 0)$ the germ of an analytic space defined by $\tilde{I}_3$. Let the map $G_3 : (X_3, 0) \to (S'_3, 0)$ be the projection on the second factor. Then G_3 defines an infinitesimal deformation of $(\Sigma, 0)$. There are no obstructions in lifting infinitesimal deformations of $(\Sigma, 0)$ to higher order. For instance, take $S' = \mathbb{C}$ and $\tilde{O} = \mathbb{C}\{x,y,z,t\}$ and $\tilde{I} = (yz, xz, (x-t)y)\tilde{O}$. Let $X = V(\tilde{I}) \subseteq \mathbb{C}^3 \times S'$ and $G : X \to S'$ the projection on S'. Then G defines a deformation of $(\Sigma, 0)$ which lifts G_3. But given the lift G of G_3 then it is not possible to lift F_3 to a deformation $F : (\mathbb{C}^3 \times S, 0) \to (\mathbb{C}, 0)$ of $(f, \Sigma, 0)$. Otherwise there exists an $H \in \tilde{O}$ such that

$F(x,y,z,t) = (yz)^2 + (2xz)^2 + (x-t)^2 y^2 + t^2 H(x,y,z,t)$. Then $(F)\tilde{O} + J_F \subseteq \tilde{I}$ implies $(H, \frac{\partial H}{\partial x}, \frac{\partial H}{\partial y} - z, \frac{\partial H}{\partial z}) \subseteq \tilde{I}$, since t is not a zero-divisor on O_X. A calculation shows that this is not possible.

Remark (7.8) One may wonder whether there exists a versal deformation of $(f, \Sigma, 0)$ in case $(\Sigma, 0)$ has an isolated singularity and $c_{I_\Sigma}(f) < \infty$. We think that (7.6) gives an example of such a versal deformation. But we did not work this out any further.

Definition (7.9) Let I be an ideal in O defining a germ $(\Sigma, 0)$ of a reduced analytic space in $(\mathbb{C}^m, 0)$. Let $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function such that $f \in \int I$ and $j_f < \infty$.

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Let $G : (X,0) \to (S,0)$ together with $i : (\Sigma,0) \to (X,0)$ define a deformation of $(\Sigma,0)$. Let $F : (\mathbb{C}^m \times S, 0) \to (\mathbb{C},0)$ be a deformation of $(f,\Sigma,0)$. Then the critical locus C_F of the deformation is the germ of the analytic space defined by the ideal J_F in $\tilde{\mathcal{O}}$.

Remark (7.10) Let $\tilde{F} : (\mathbb{C}^m \times S, 0) \to (\mathbb{C} \times S, 0)$ be defined by $\tilde{F}(z,s) = (F(z,s), s)$ and suppose $(S,0) \cong (\mathbb{C}^{\tau},0)$. Then C_F is the critical locus of the map $\tilde{F}$. The critical locus C_F always contains X by definition, since $(F)\tilde{\mathcal{O}} + J_F \subseteq \tilde{I}$. The support of the quotient of the two ideals $\tilde{I}/J_F$ is equal to $V(J_F : I)$ and $C_F = X \cup V(\tilde{J}_F : \tilde{I})$.

Definition (7.11) The residual critical locus of the deformation F of $(f,\Sigma,0)$ is by definition the germ of the analytic space in $(\mathbb{C}^m \times S, 0)$ defined by the ideal $(\tilde{J}_F : \tilde{I})$ in $\tilde{\mathcal{O}}$.

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Remark (7.13) The image of C_F under $\tilde{F}$ is usually called the discriminant $D_{\tilde{F}}$ of $\tilde{F}$. But to give $D_{\tilde{F}}$ a well defined analytic structure one uses for instance that $\tilde{F}/C_F : (C_F,0) \to (\mathbb{C} \times S, 0)$ is a finite map, which is not the case since $\Sigma \subseteq F^{-1}(0)$. But the map $F/RC_F : (RC_F, 0) \to (\mathbb{C} \times S, 0)$ is finite, if $j_f < \infty$. Hence $\tilde{I}/J_F$ is a finitely generated $\mathcal{O}_{(T,0)}$ - module, by [Na] IV thm.7.

Notation (7.12) Let $(T,0) = (\mathbb{C} \times S, 0)$ and denote the projection of T on the second factor S by π. By abuse of notation we shall denote $\pi \circ \tilde{F}$ also by π. Denote the projection of T on the first factor $\mathbb{C}$ by p. Hence we have the following commutative diagram of analytic maps.

$$\begin{tikzcd}
 & (\mathbb{C},0) \\
(\mathbb{C}^m \times S, 0) \arrow[ur, "F"] \arrow[r, "\tilde{F}"] \arrow[dr, "\pi"'] & (T,0) \arrow[u, "p"'] \arrow[d, "\pi"] \\
 & (S,0)
\end{tikzcd}$$

Definition (7.14) Let R be a Noetherian ring. Let M be a finitely generated R-module. Then M has a finite presentation $R^p \xrightarrow{\varphi} R^q \to M \to 0$. Let (φ_{ij}) be the matrix of φ.
The zeroth Fitting ideal $F_0(M)$, of M is the ideal in R generated by the $(q\times q)$-minors of the matrix of φ. Set $F_0(M) = (0)$ in case $p<q$. See [Lo](4.E) or [Te]. This definition does not depend on the chosen presentation of M.

Definition (7.15) Let F be a deformation of $(f,\Sigma,0)$. Then the residual discriminant locus RD_F, of F is the germ of an analytic space in $(T,0)$ defined by the Fitting ideal $F_0(\tilde{F}_*(\tilde{I}/J_F))$ in $\mathcal{O}_{(T,0)}$.

Proposition (7.16) Let $(\Sigma,0)$ be a germ of a reduced curve in $(\mathbb{C}^m,0)$ defined by an ideal I in $\mathcal{O}$. Let $f:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ be a germ of an analytic function such that $f\in\int I$ and $j_f<\infty$. Let (Σ,i,X,G,S) be a deformation of $(\Sigma,0)$ with a regular analytic base space $(S,0)$. Let $F:(\mathbb{C}^m\times S,0)\to(\mathbb{C},0)$ be a deformation of $(f,\Sigma,0)$. Let $f_s:(\mathbb{C}^m,0)\to(\mathbb{C},0)$ be defined by $f_s(z)=F(z,s)$. Then
(i) $\tilde{F}_*(\tilde{I}/J_F)$ is a CM $\mathcal{O}_{(T,0)}$-module
(ii) RD_F is a hypersurface in $(T,0)$
(iii) $\pi_*(\tilde{I}/J_F)$ is a CM $\mathcal{O}_{(S,0)}$-module
(iv) There exist representatives of all germs considered such that $j_f = \sum_{p\in f_s^{-1}(0)} j(f_{s,p})$ for all $s\in S$.

Proof The curve $(\Sigma,0)$ is reduced, hence $(\Sigma,0)$ is CM of codim $m-1$ in $(\mathbb{C}^m,0)$. So the ideal I is perfect of grade $m-1$. The ideal $\tilde{I}$ of $(X,0)$ in $\tilde{\mathcal{O}}$ is also perfect of grade $m-1$, since $(S,0)$ is regular and $(X,0)$ is a deformation of $(\Sigma,0)$.
The $\mathcal{O}$-module I/J_f has support in $\{0\}$. and $\tilde{I}/J_F \otimes_{\tilde{\mathcal{O}}} \mathcal{O}_S/m_0\mathcal{O}_S \cong I/J_f$
thus the support of the $\tilde{\mathcal{O}}$-module $\tilde{I}/J_F$ in $(\mathbb{C}^m\times S,0)$ has codim $\geq m$. The local ring $\tilde{\mathcal{O}}$ is regular, so grade $(\tilde{I}/J_F)\geq m$.

The ideal J_F is generated by m elements. We can apply theorem (4.3) of chapter II. So $\tilde{I}/J_F$ is a perfect $\tilde{\mathcal{O}}$-module of grade m. Let $M = \tilde{I}/J_F$. Denote $\mathcal{O}_{(S',0)}$ by $\mathcal{O}_{S'}$ and $\mathcal{O}_{(T,0)}$ by $\mathcal{O}_T$. Now $\pi_*(M)$ is a finitely generated $\mathcal{O}_{S'}$-module and a fortiori $\tilde{F}_*(M)$ is a finitely generated $\mathcal{O}_T$-module, since $M \otimes_{\tilde{\mathcal{O}}} \mathcal{O}_{S'}/m_0\mathcal{O}_{S'} \cong I/J_f$ is a finite dimensional $\mathbb{C}$-vector space and use [Na] chap. IV, thm. 7. The $\tilde{\mathcal{O}}$-module M is CM, hence $\pi_*(M)$ and $\tilde{F}_*(M)$ are CM $\mathcal{O}_{S'}$- resp $\mathcal{O}_T$-modules, see [Se, 2] IV, prop. 11. The $\mathcal{O}$-module M has grade m, so $\mathrm{Kdim}\, M = \mathrm{Kdim}\, \mathcal{O}_{S'}$ and $\mathrm{Kdim}\, \tilde{F}_*(M) = \mathrm{Kdim}\, \mathcal{O}_T - 1$. Thus $\mathrm{pd}_{\mathcal{O}_T}(\tilde{F}_*(M)) = 1$, see [Se, 2] IV, prop. 21, i.e. there exists an exact sequence of $\mathcal{O}_T$-modules

$$0 \longrightarrow \mathcal{O}_T^p \xrightarrow{\varphi} \mathcal{O}_T^q \longrightarrow \tilde{F}_*(M) \rightarrow 0$$

We must have $p \leq q$, since φ is injective and $p \geq q$ since $\mathrm{Kdim}\, \tilde{F}_*(M) < \mathrm{Kdim}\, \mathcal{O}_T$. So $F_0(\tilde{F}_*(M)) = (\det(\varphi_{ij}))\mathcal{O}_T$ where (φ_{ij}) is the matrix of φ. Hence RD_F is a hypersurface in $(T,0)$.

Part (iv) is a consequence of (iii) and proposition (7.2). This proves the proposition.

In the following lemma and two propositions we shall consider only the case of a germ of a <u>one dimensional reduced complete intersection</u> $(\Sigma,0)$ in $(\mathbb{C}^{n+1},0)$ defined by the ideal I in $\mathcal{O}$. Let $f: (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a germ of an analytic function such that $f \in \int I$ and $j_f < \infty$. Let I be generated by the $\mathcal{O}$-sequence $g_1, \ldots, g_n$. Then we can write $f = \sum_{i,j} h_{ij} g_i g_j$ with $h_{ij} = h_{ji}$, since $\int I = I^2$ by (1.9). The invariant δ_f is defined in (5.13). Let $\Delta = \det(h_{ij})$. Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$ be a decomposition of Σ in its branches. Define $\delta_f(k) = \dim_{\mathbb{C}}(\mathcal{O}_{\Sigma_k}/\Delta)$.

<u>Lemma</u> (7.17) In the situation above we have

(i) $\delta_f = \dim_{\mathbb{C}}(\mathcal{O}_\Sigma/\Delta) = \sum_k \delta_f(k)$

(ii) $\delta_f = 0$ if and only if (h_{ij}) is invertible if and only if there exist elements $\bar{g}_1, \ldots, \bar{g}_n$ such that $f = \bar{g}_1^2 + \cdots + \bar{g}_n^2$ and $(\bar{g}_1, \ldots, \bar{g}_n)\mathcal{O} = I$.

<u>Proof</u> (i) The ideal I is generated by the $\mathcal{O}$-sequence $g_1, \dots, g_n$. So I/I^2 is a free $\mathcal{O}_\Sigma$-module of rank n and $\mathrm{Hom}_{\mathcal{O}_\Sigma}(I/I^2, \mathcal{O}_\Sigma) \simeq \mathcal{O}_\Sigma$ The map h_f as defined in (5.9) has matrix (h_{ij}). Now $\delta_f = \dim_{\mathbb{C}}(\mathrm{Coker}\, h_f)$ by definition and $\delta_f = \dim_{\mathbb{C}}(\mathcal{O}_\Sigma/\Delta)$ since $\delta_f < \infty$. The proof of (i) goes by induction on r. The case $r=1$ we already proved. Suppose Σ has r branches and $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$ is a prime decomposition of I. Take $I_1 = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{r-1}$ and $I_2 = \mathfrak{p}_r$. Then we have the following exact and commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \mathcal{O}/I \arrow[r] \arrow[d, "\Delta"] & \mathcal{O}/I_1 \oplus \mathcal{O}/I_2 \arrow[r] \arrow[d, "\Delta\oplus\Delta"] & \mathcal{O}/(I_1+I_2) \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}/I \arrow[r] \arrow[d] & \mathcal{O}/I_1 \oplus \mathcal{O}/I_2 \arrow[r] \arrow[d] & \mathcal{O}/(I_1+I_2) \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}/(I+\Delta) \arrow[r] \arrow[d] & \mathcal{O}/(I_1+\Delta) \oplus \mathcal{O}/(I_2+\Delta) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where K and C are by definition the kernel resp. the cokernel of the map $\Delta : \mathcal{O}/(I_1+I_2) \to \mathcal{O}/(I_1+\Delta_2)$. Both K and C are finite dimensional over $\mathbb{C}$ and both have the same dimension since I_1+I_2 is an $\mathfrak{m}$-primary ideal. So we have by the snake lemma an exact sequence

$$0 \to K \to \mathcal{O}/(I+\Delta) \to \mathcal{O}/(I_1+\Delta) \oplus \mathcal{O}/(I_2+\Delta) \to C \to 0$$

of finite dimensional $\mathbb{C}$-vector spaces and $\dim_{\mathbb{C}} K = \dim_{\mathbb{C}} C$. The induction hypotheses gives $\dim_{\mathbb{C}} \mathcal{O}/(I_1+\Delta) = \sum_{k<r} \delta_f(k)$ and the above exact sequence gives $\delta_f = \sum_k \delta_f(k)$.

(ii) Suppose (h_{ij}) is invertible then an exercise in quadratic forms over $\mathcal{O}_\Sigma$ gives that $f = \bar{g}_1^2 + \dots + \bar{g}_n^2$ with $(\bar{g}_1, \dots, \bar{g}_n)\mathcal{O} = I$. The other implications are easy.
This proves the lemma.

Proposition (7.10) Let I be an ideal in $\mathcal{O}$ defining a germ of a reduced curve $(\Sigma, 0)$ in $(\mathbb{C}^{n+1}, 0)$, which is a complete intersection and with branches $\Sigma_1, \dots, \Sigma_r$. Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function such that $f \in \int I$ and $j_f < \infty$. Let I be generated by the $\mathcal{O}$-sequence $g_1, \dots, g_n$. Let $\mathcal{S} = \mathbb{C}^{n^2} \times \mathbb{C}^{(n+1)n}$. Let $F: (\mathbb{C}^{n+1} \times \mathcal{S}, 0) \to (\mathbb{C}, 0)$ be the I-unfolding of f defined by

$$F(z, a, b) = f(z) + \sum_{k,l} a_{kl}\, g_k g_l + \sum_{i,k} b_{ik} z_i g_k^2$$

Let $s = (a, b)$ and $f_s(z) = F(z, s)$. Then there exist a dense subset V of $\mathcal{S}$ and an open neighbourhood U of 0 in $\mathbb{C}^{n+1}$ such that for all $s \in V$ sufficiently small

(i) f_s has only A_1 singularities in $U \setminus \Sigma$

(ii) f_s has only A_∞ and D_∞ singularities in $(U \cap \Sigma) \setminus \{0\}$ and $\delta_{f_s, 0} = 0$ and

(iii)
$$\delta_f(j) = \#\{\, D_\infty \text{ points of } f_s \text{ on } (U \cap \Sigma_j) \setminus \{0\} \,\}$$

$$\delta_f = \#\{\, D_\infty \text{ points of } f_s \text{ on } (U \cap \Sigma) \setminus \{0\} \,\}$$

$$j_f = \#\{\, A_1 \text{ points of } f_s \text{ on } (U \cap \Sigma) \setminus \{0\} \,\} + \delta_f + j_{f_s, 0}$$

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Proof (i) Consider the map $\Phi: (\mathbb{C}^{n+1} \times \mathcal{S}, 0) \to (\mathbb{C}^{n+1}, 0)$ defined by $\Phi = \left(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_n}\right)$, where $z_0, \dots, z_n$ are local coordinates of $(\mathbb{C}^{n+1}, 0)$. Then

$$\frac{\partial \Phi_i}{\partial a_{kk}} = 2 g_k \frac{\partial g_k}{\partial z_i} \qquad \text{and} \qquad \frac{\partial \Phi_i}{\partial b_{jk}} = 2 z_j g_k \frac{\partial g_k}{\partial z_i} + \delta_{ij} g_k^2$$

where δ_{ij} is Kronecker's delta.
Hence

$$\det\left(\frac{\partial \Phi}{\partial b_{0k}} - z_0 \frac{\partial \Phi}{\partial a_{kk}}, \dots, \frac{\partial \Phi}{\partial b_{nk}} - z_n \frac{\partial \Phi}{\partial a_{kk}}\right) = g_k^{2(n+1)}$$

Let $K = (g_1^{2(n+1)}, \dots, g_n^{2(n+1)})\mathcal{O}$ then $K \subseteq I_n(d\Phi)$ the ideal generated by the $(n \times n)$-minors of $d\Phi$. So $V(K)$ contains the singular locus of the map Φ, i.e. Φ is a submersion outside $\Sigma \times \mathcal{S}$. By Sard's theorem, see [St], there exists a dense subset V_0 of $\mathcal{S}$ such that Φ_s is a submersion outside Σ, for all $s \in V_0$, where $\Phi_s: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ is defined by $\Phi_s(z) = \Phi(z, s)$. Hence f_s has only A_1 singularities on a neighbourhood of 0 in $\mathbb{C}^{n+1}$ and outside Σ.

(ii) Take a parametrization $\gamma_j : (\mathbb{C},0) \to (\Sigma_j,0)$ of the branch $(\Sigma_j,0)$. Let $H_{kl}(z,a,b) = h_{kl}(z) + a_{kl} + \sum_i z_i b_{ik}\delta_{kl}$. Let I_j be the ideal defining Σ_j. Take

$\psi_{kl} = \gamma_j^*(h_{kl} + I_j) + a_{kl} + \sum_i \gamma_j^*(z_i + I_j)\, b_{ik}\delta_{kl}$ in $\mathbb{C}\{t\}[a,b]$

Then (ψ_{kl}) is a generic $(n\times n)$-matrix. Hence $\det(\psi_{kl}) = 0$ defines a hypersurface H_j in $\mathbb{C}\times S'$. The singular locus of H_j is defined by the $(n-1)\times(n-1)$-minors of (ψ_{kl}), hence this locus has codim 4 in $\mathbb{C}\times S'$, see [E-H]. In particular is H_j reduced and the projection p_j of H_j on S' is a finite map, since $\delta_f(j) = \dim_{\mathbb{C}} \mathcal{O}_{\Sigma_j}/\Delta) < \infty$. Hence the discriminant D_j of p_j, i.e. the projection of the singular locus of the map p_j, has codim ≥ 1. Let $V_j = S' \setminus D_j$ then V_j is an open dense subset of S' and the map $p_j : H_j \setminus p_j^{-1}(D_j) \longrightarrow V_j$ is an unramified covering of degree $\delta_f(j)$. Let $s = (a,b)$ and $\tilde{\Delta}(z,s) = \det(H_{kl}(z,s))$ and $\tilde{\Delta}_s(z) = \tilde{\Delta}(z,s)$. Then the hypersurface $\tilde{\Delta}_s(z) = 0$ in $\mathbb{C}^{n+1}$ intersects the curve Σ_j transversally in $\delta_f(j)$ points for all $s \in V_j$ and these intersection points on Σ_j correspond one-to-one to D_∞ points of f_s.

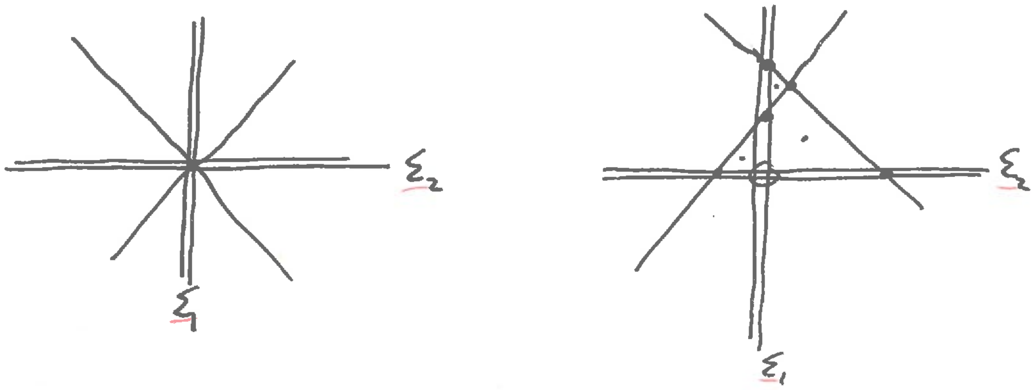
Let V_{k+1} be the complement of the hypersurface $\tilde{\Delta}(0,a,b) = 0$ in S'. Let $V = V_0 \cap V_1 \cap \dots \cap V_{k+1}$ then V is dense in S', since V_0 is dense in S' and $V_1, \dots, V_{k+1}$ are open dense in S'. Thus on a neighbourhood U of 0 in $\mathbb{C}^{n+1}$ we have that f_s has only A_1 points on $U \setminus \Sigma$ and only A_∞ and D_∞ points on $(U \cap \Sigma) \setminus \{0\}$ and the number of D_∞ points of f_s is $\delta_f(j)$ for all $s \in V$, moreover $\delta_{f_s,0} = 0$ since $s \notin V_{k+1}$.

(iii) We already proved the formula concerning the number of D_∞ points on $\Sigma_j \setminus \{0\}$. The second formula is a consequence of the first one and lemma (7.17)(i). The third formula is a consequence of proposition (7.16)(iv) since every A_1 point and every D_∞ point give a contribution 1 to j_f.

Example (7.19) Let $f_s(x,y) = (xy)^2(x-y+s)(x+y+2s)$. Let $\Sigma_1 = V(x)$ and $\Sigma_2 = V(y)$. Then $\Sigma = \Sigma_1 \cup \Sigma_2$ is the singular locus of $f = f_0$. Further $\#\{D_\infty \text{ points of } f_s \text{ on } \Sigma_j\} = \delta_f(j) = 2$ and

$\#\{A_1 \text{ points of } f_s \text{ on } \mathbb{C}^2 \setminus \Sigma\} = 4$

and $\delta_{f_{s,0}} = 0$ and $j(f_s,0) = 1$ for all $s \neq 0$.



Next we also deform Σ. Let I be an ideal in $\mathcal{O}$ defining a germ of a reduced curve $(\Sigma,0)$ in $(\mathbb{C}^{n+1},0)$ which is a complete intersection and with branches $\Sigma_1,\dots,\Sigma_r$. Let I be generated by the $\mathcal{O}$-sequence $g_1,\dots,g_n$. Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function such that $f\in\int I$ and $j_f<\infty$. Let the map $G:(\mathbb{C}^{n+1}\times\mathbb{C}^p,0)\longrightarrow(\mathbb{C}^n\times\mathbb{C}^p,0)$ be a versal deformation of $(\Sigma,0)$ with $G(z,v)=(G_1(z,v),\dots,G_n(z,v),v)$ and $G_j(z,0)=g_j(z)$, see [Lo](6.5). We can write $f=\sum_{k,l} h_{kl}\, g_k g_l$ with $h_{kl}=h_{lk}$. Define the ideal $\tilde{I}$ by $\tilde{I}=(G_1-u_1,\dots,G_n-u_n)\tilde{\mathcal{O}}$ where $\tilde{\mathcal{O}}$ is the local ring of germs of analytic functions on $(\mathbb{C}^{n+1}\times S,0)$ and $S'=\mathbb{C}^{n^2}\times\mathbb{C}^{(n+1)n}\times\mathbb{C}^n\times\mathbb{C}^p$ and let $X=V(\tilde{I})\subseteq\mathbb{C}^{n+1}\times S'$. The projection of X on S' defines a deformation of $(\Sigma,0)$ which is a trivial extension of the deformation G of $(\Sigma,0)$. Define the deformation F of $(f,\Sigma,0)$ by

$$F(z,a,b,u,v)=\Big(\sum_{k,l} h_{kl}(z)+a_{kl}+\sum_i b_{ik} z_i \delta_{kl}\Big)\big(G_k(z,v)-u_k\big)\big(G_l(z,v)-u_l\big).$$

, with $(a,b,u,v)\in S'$. Let $f_s:(\mathbb{C}^{n+1},0)\longrightarrow(\mathbb{C},0)$ be defined by $f_s(z)=F(z,s)$. Let $w=(u,v)\in\mathbb{C}^n\times\mathbb{C}^p$ and Σ_w be the curve in $\mathbb{C}^{n+1}$ defined by $G(z,v)=u$.

Proposition (7.20) There exist a dense subset V of S' and an open neighbourhood U of 0 in $\mathbb{C}^{n+1}$ such that for all $s=(a,b,u,v)\in S'$ and $w=(u,v)$ sufficiently small

(i) f_s has only A_1 singularities on $U\setminus\Sigma_w$

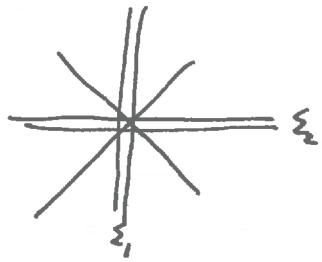
(ii) Σ_w is a non-singular curve and f_s has only A_∞s and D_∞s singularities on $U\cap\Sigma_w$

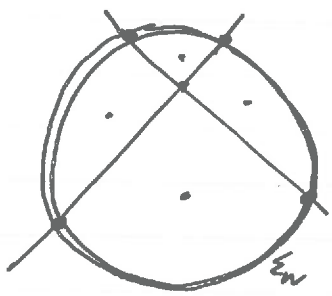
(iii) $\quad \delta_f = \#\{D_\infty \text{ points of } f_s \text{ on } U \cap \Sigma_w\}$

$\quad\; j_f = \#\{D_\infty \text{ points of } f_s \text{ on } U \cap \Sigma_w\} + \#\{A_1 \text{ points of } f_s \text{ on } U \setminus \Sigma_w\}$

Proof (i) This part is analogous to the proof of (i) in proposition (7.10)
(ii) The discriminant D_G of the map G is a hypersurface in $\mathbb{C}^n \times \mathbb{C}$, see [Lo](4.8), so for all $w = (u,v) \notin D_G$ is Σ_w a non-singular curve.
(iii) Let $H_{kl} = h_{kl} + a_{kl} + \sum_i b_{ik} z_i \delta_{kl}$ and $\tilde{\Delta} = \det(H_{kl})$. The projection $p: \ X \cap V(\tilde{\Delta}) \longrightarrow S$ is a finite map of degree δ_f. There exist points in S where $p^{-1}(s)$ consists of δ_f points by proposition (7.18) (ii). ~~Hence in a dense subset of S consists~~ Hence $p^{-1}(s)$ consists of δ_f distinct points for all s in a dense subset of S and these points correspond to the D_∞ points of f_s on Σ_w. ~~The function f_s has only A_∞ points on Σ_w and A_1 points outside Σ.~~
The formula for δ_f is already proved and for j_f goes with the help of proposition (7.18) (iv).

Example (7.21) Let $f(x,y) = x^2y^2(x-y)(x+y)$. Then
$\#\{D_\infty \text{ points of } f_s \text{ on } \Sigma_w\} = \delta_f = 4$ and $\#\{A_1 \text{ points of } f_s \text{ on } \mathbb{C}^2 \setminus \Sigma\} = 5$





Example (7.22) Let f be the function of example (7.7). The deformation f_{1s} has for all $s \in U_1$, with U_1 an open dense subset of S_1, six A_1 points outside Σ_s and four D_∞ singularities on Σ_s and Σ_s is nonsingular. The deformation F_2 has the property that for all $t \neq 0$, f_{2t} has four A_1 singularities outside Σ and six D_∞ points on $\Sigma \setminus \{0\}$ and $j(f_{2t}, 0) = 0$.

Remark (7.23) Proposition (7.20) together with proposition (3.10) prove a conjecture posed by Siersma [Si], in greater generality.

§8 Analytic functions with a one dimensional singular locus

Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma,0)$. Suppose in certain local coördinates $z_0,z_1,\dots,z_n$ of $(\mathbb{C}^{n+1},0)$ we have $\Sigma\cap\underline{V}(z_0)=\{0\}$ on a neighbourhood of 0. Then we shall prove that $\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}$ is an $\underline{\mathcal{O}}$- regular sequence.

The following two lemma's do not presuppose that f has a one dimensional singular locus and are well known.

Lemma (8.1) Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ and $f:U\to\mathbb{C}$ an analytic function on U. Then f is locally constant on $\underline{V}(J_f)$. In particular if $f(0)=0$ and f is singular in 0, then f is zero on $\underline{V}(J_f)$ on a neighbourhood of 0.

Proof In fact it is enough to prove the case that $f(0)=0$ and f is singular at 0. The inclusion $\underline{V}((f)+J_f)\subseteq\underline{V}(J_f)$ is obvious. Suppose that they are not equal, then there exists a germ of an analytic curve $\gamma:(\mathbb{C},0)\to(\underline{V}(J_f),0)$, such that $\gamma(t)\notin\underline{V}((f)+J_f)$ for all t small enough. Now $\frac{d}{dt}(f\circ\gamma)=\sum_i\left(\frac{\partial f}{\partial z_i}\circ\gamma\right)\frac{d\gamma_i}{dt}=0$. Thus f is constant on the image of γ, and therefore constant 0, since $f(\gamma(0))=0$, which is a contradiction. So $\underline{V}((f)+J_f)=\underline{V}(J_f)$. and we are done.

Lemma (8.2) Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ and $f:U\to\mathbb{C}$ an analytic function. Let $f_i=\frac{\partial f}{\partial z_i}$.
Then $\underline{V}(f,f_1,\dots,f_n)=\underline{V}(f_0,f_1,\dots,f_n)\cup\underline{V}(z_0,f_1,\dots,f_n)$.

Proof This proof has the same pattern as the one of (8.1). Let $\underline{\mathcal{O}}=\underline{\mathcal{O}}_{(\mathbb{C}^{n+1},0)}$.
A power of f is an element of $(f_0,f_1,\dots,f_n)\underline{\mathcal{O}}$ and of $(z_0,f_1,\dots,f_n)\underline{\mathcal{O}}$, by (8.1).
Thus $\underline{V}(f_0,f_1,\dots,f_n)\cup\underline{V}(z_0,f_1,\dots,f_n)\subseteq\underline{V}(f,f_1,\dots,f_n)$. Suppose they are not equal, then there exists a germ of a curve

$\gamma : (\mathbb{C},0) \longrightarrow V(f, f_1, \ldots, f_n)$, such that $\gamma(t) \notin V(f_0, f_1, \ldots, f_n) \cup V(z_0, f_1, \ldots, f_n)$ for all t sufficiently small. Now $f(\gamma(t)) = 0$ for all t small enough. So
$$0 = \frac{d}{dt} f(\gamma(t)) = \sum_{i=0}^{n} f_i(\gamma(t)) \cdot \frac{d\gamma_i}{dt} = f_0(\gamma(t)) \cdot \frac{d\gamma_0}{dt}.$$
Now $f_0(\gamma(t)) \neq 0$, so $\frac{d\gamma_0}{dt} = 0$ and γ_0 is constant and therefore equal to 0 for all t small enough. Hence $\gamma(t) \in V(z_0, f_1, \ldots, f_n)$, which is a contradiction and we are done.

Theorem (8.3) Let $X \subseteq U \subseteq \mathbb{C}^N$ be an analytic subset of an open set U of $\mathbb{C}^N$. Let $f : X \to \mathbb{C}$ be an analytic function. Let $x \in X$ and suppose $f(x) = 0$. Then, if $\varepsilon > 0$ is small enough and $\eta > 0$, $\varepsilon \gg \eta$ the mapping induced by f
$$\Psi_{\varepsilon,\eta} : B_\varepsilon \cap X \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\}$$
, where B_ε is the closed real ball in $\mathbb{C}^N$ at center x and radius $\varepsilon > 0$, D_η is the open disc of $\mathbb{C}$ centered at 0 and with radius $\eta > 0$, is a topological fibration.

Proof see [LE, 1].

Corollary (8.4) Let $(X,0)$ be a germ of a reduced irreducible analytic space and $f : (X,0) \to (\mathbb{C},0)$ a germ of an analytic function, which has a section $\sigma : (\mathbb{C},0) \to (X,0)$ such that $(f \circ \sigma) = id_{\mathbb{C}}$. Then there exists a representative $f : X \to \mathbb{C}$ of the germ, such that $f^{-1}(t)$ is connected for all $t \neq 0$ and sufficiently small.

Proof If $\dim X = 1$ then $(\sigma(\mathbb{C}), 0)$ is a closed subset of $(X,0)$ of dim 1 and must be equal to $(X,0)$, since $(X,0)$ is irreducible, and we are done. Suppose $\dim X \geq 2$, then $\dim f^{-1}(0) < \dim X$, since $f : (X,0) \to (\mathbb{C},0)$ is onto and $(X,0)$ is irreducible. Thus $X \setminus f^{-1}(0)$ is connected, again since X is irreducible. By theorem (8.3) we may suppose that we have a representative $f : X \to D$ of the given germ such that D is an open neighbourhood of 0 and $f : X \setminus f^{-1}(0) \longrightarrow D \setminus \{0\}$ is a topological fibration. The topological space $X \setminus f^{-1}(0)$ is connected and also pathwise connected, since X and $f^{-1}(0)$ are analytic sets.

Let $t \in D \setminus \{0\}$ and $x_0 = \sigma(t)$. and let x_1 be some other point in $f^{-1}(t)$. Then there exists a continuous path $\alpha : [0,1] \to \mathcal{X} \setminus f^{-1}(0)$ from x_0 to x_1. The map $f \circ \alpha : [0,1] \to D \setminus \{0\}$ defines a loop w in $D \setminus \{0\}$ with base point t. Let $\beta : [0,1] \to \mathcal{X} \setminus f^{-1}(0)$ be defined by $\beta(s) = (\sigma \circ f \circ \alpha)(1-s)$ for $0 \leq s \leq 1$. Then β is a loop in $\mathcal{X} \setminus f^{-1}(0)$ with base point x_0. and $f \circ \beta$ is the loop w^{-1}, since $f \circ \sigma = id_D$. So $\alpha \circ \beta$ is a path in $\mathcal{X} \setminus f^{-1}(0)$ from x_0 to x_1, which is a lift of $w \circ w^{-1}$ in $D \setminus \{0\}$. Thus $\alpha \circ \beta$ is homotopic to a curve γ from x_0 to x_1 in $f^{-1}(t)$. Hence $f^{-1}(t)$ is connected for all $t \in D \setminus \{0\}$. This proves the corollary.

Proposition (8.5) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a one dimensional singular locus Σ. Let $z_0, z_1, \ldots, z_n$ be local coördinates of $(\mathbb{C}^{n+1}, 0)$ and let $f_i = \frac{\partial f}{\partial z_i}$. If $\Sigma \cap V(z_0) = \{0\}$ then $V(f_1, \ldots, f_n)$ is one dimensional and $f_1, \ldots, f_n$ is an $\mathcal{O}$-regular sequence.

First we give the idea of the proof. Assume that $V(f_1, \ldots, f_n)$ is not one dimensional at 0. Then it contains a component V of dimension ≥ 2 such that V is not contained in $V(z_0)$. Call $x = z_0$ and let $y_i = z_i$ for $i = 1, \ldots, n$. Consider $f_t : \mathbb{C}^n \to \mathbb{C}$ defined by $f_t(y) = f(t, y)$. Suppose the projection $p : (V, 0) \to (\mathbb{C}, 0)$ on the first coördinate axis, has a section γ. Then $V_t = V \cap V(x - t)$ is connected for all t small enough by corollary (8.4). And f_t is locally constant on V_t by lemma (8.1), so f_t must be constant on V_t.
Let $g(x,y) = f(x,y) - f(\gamma(x))$, then g is zero on V. Thus
$V \subseteq V(g, g_1, \ldots, g_n) = V(g_0, g_1, \ldots, g_n) \cup V(x, g_1, \ldots, g_n)$, by lemma (8.2)
The irreducible closed set V is not contained in $V(x)$, hence $V \subseteq V(g_0, g_1, \ldots, g_n)$. We have $\frac{df}{dx} = \frac{dg}{dx} + \frac{\partial f}{\partial x} \circ \gamma + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i} \circ \gamma\right) \frac{d\gamma_i}{dx}$ is zero on $V \cap V(x)$. Hence

$$V \cap V(x) \subseteq V(f_0, f_1, \ldots, f_n) \cap V(x) = \Sigma \cap V(x) = \{0\}.$$

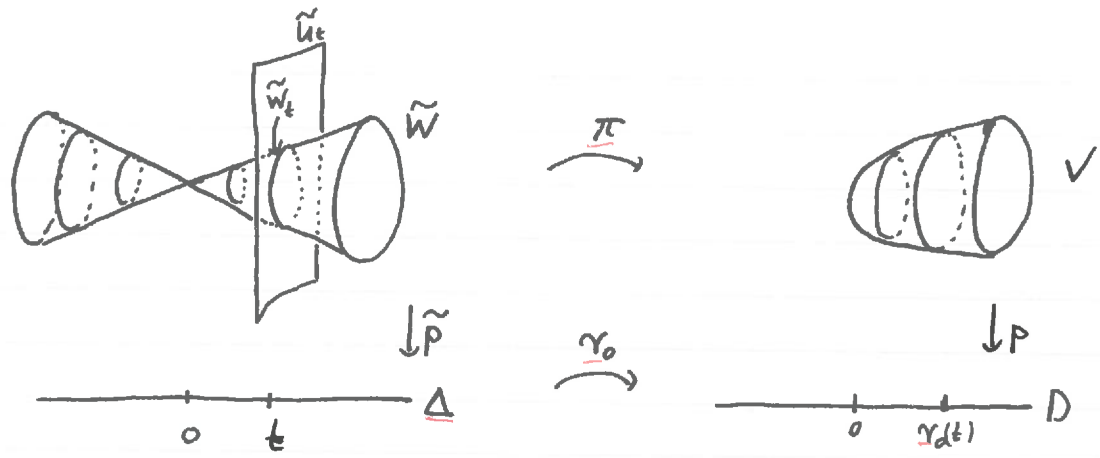
But $V \cap V(x)$ is at least one dimensional. This gives a contradiction.

We made the assumption that the projection p has a section γ. This need not be the case and then one has to consider a branched covering of $\mathbb{C}^{n+1}$, which makes the original idea of the proof less clear.

Proof (8.5) Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ and let $f: U \to \mathbb{C}$ be a representative of the given germ. Suppose $\underline{V}(f_1, \ldots, f_n)$ is not one dimensional at 0. Then $\underline{V}(f_1, \ldots, f_n)$ must have a component V of dimension ≥ 2. This component V cannot be in the hyperplane $\underline{V}(z_0)$, since otherwise $\underline{V}(f_0) \cap V \subseteq \underline{V}(f_0) \cap V \cap H \subseteq \Sigma_0 \cap \underline{V}(z_0) = \{0\}$ and $\underline{V}(f_0) \cap V$ is at least one dimensional at 0. Hence the projection $p: (V,0) \to (\mathbb{C},0)$ on the first coördinate is surjective on an open neighbourhood D of 0 in $\mathbb{C}$. One can find a germ of a curve $\gamma: (\mathbb{C},0) \to (V,0)$ such that the composition $(p \circ \gamma): (\mathbb{C},0) \to (\mathbb{C},0)$ is a finite map. Take representatives of the germs considered, i.e. there is an open neighbourhood Δ of 0 in $\mathbb{C}$ such that (after possibly shrinking D) $\gamma: \Delta \to V$ is an analytic map and $p \circ \gamma: \Delta \to D$ is a covering, branched at 0 in D. Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$, then $\gamma_0 = (p \circ \gamma)$. Let $\pi: \Delta \times \mathbb{C}^n \to \mathbb{C}^{n+1}$ be the branched covering defined by $\pi(x,y) = (\gamma_0(x), y)$. Let $\tilde{U} = \pi^{-1}(U)$ and $\tilde{f} = f \circ \pi$, then $\tilde{f}: \tilde{U} \to \mathbb{C}$ is an analytic map. Let $x, y_1, \ldots, y_n$ be local coördinates of $\tilde{U}$ and $\tilde{f}_0 = \frac{d\tilde{f}}{dx}$ and $\tilde{f}_i = \frac{d\tilde{f}}{dy_i}$ for $i = 1, \ldots, n$. Then we have the following commutative diagram.

$$\begin{tikzcd}
\Delta \arrow[r, "\tilde{\gamma}"] \arrow[rr, bend left, "\gamma"] \arrow[dr, equal] & \tilde{U} \arrow[r, "\pi"] \arrow[d, "\tilde{p}"] \arrow[rr, bend left, "\tilde{f}"] & U \arrow[r, "f"] \arrow[d, "p"] & \mathbb{C} \\
 & \Delta \arrow[r, "\gamma_0"] & D &
\end{tikzcd}$$

, where $\tilde{\gamma}(x,y) = (x, \gamma_1(x), \ldots, \gamma_n(x))$ and $\tilde{p}(x,y) = y$. Take $\tilde{V} = \pi^{-1}(V)$, then $\tilde{V} \subseteq \underline{V}(\tilde{f}_1, \ldots, \tilde{f}_n)$. Let $\tilde{\Gamma}$ be the image of $\tilde{\gamma}$ in $\tilde{V}$. Let $\tilde{W}$ be an irreducible component of $\tilde{V}$, which contains $\tilde{\Gamma}$. Then $\tilde{W}$ has dimension ≥ 2. Let $\tilde{U}_t = \tilde{p}^{-1}(t)$ and $\tilde{W}_t = \tilde{W} \cap \tilde{p}^{-1}(t)$. Define $\tilde{f}_t: \tilde{U}_t \to \mathbb{C}$ as the restriction of $\tilde{f}$ to $\tilde{U}_t$.



Now $\tilde{W}_t \subseteq V(J_{\tilde{f}_t})$. So $\tilde{f}_t$ is locally constant on $\tilde{W}_t$, by lemma (8.1). Let $\tilde{p}: \tilde{W} \to \Delta$ be the restriction of $\tilde{p}$ to $\tilde{W}$, by abuse of notation. Then $\tilde{p}: \tilde{W} \to \Delta$ has a section $\tilde{r}$. After possibly shrinking all representatives, we may assume that $\tilde{W}_t = \tilde{W} \cap \tilde{p}^{-1}(t)$ is connected for all $t \in \Delta \setminus \{0\}$, by corollary (8.4). Therefore $\tilde{f}_t$ is constant on $\tilde{W}_t$ and its value is $\tilde{f}(\tilde{r}(t))$. Then g is zero on $\tilde{W}$ and $\frac{\partial g}{\partial y_i} = \frac{\partial \tilde{f}}{\partial y_i}$. Hence $\tilde{W} \subseteq V(g, g_1, \dots, g_n)$.
Now $V(g, g_1, \dots, g_n) = V(g_0, g_1, \dots, g_n) \cup V(x, g_1, \dots, g_n)$ by lemma (8.2) and $\tilde{W}$ is a closed irreducible set, not contained in $V(x)$, since V is not contained in $V(z_0)$. Hence $\tilde{W}$ is contained in $V(g_0, g_1, \dots, g_n)$.
Let $\mathfrak{p}$ be the prime ideal in $\mathcal{O}_{(U,0)}$ defining the germ $(V,0)$ and $\tilde{\mathfrak{q}}$ the prime ideal in $\mathcal{O}_{(\tilde{U},0)}$ defining $(\tilde{W},0)$. Then

$$\frac{\partial g}{\partial x} = \frac{\partial \tilde{f}}{\partial x} - \frac{\partial}{\partial x}(\tilde{f} \circ \tilde{r} \circ \tilde{p}) = \pi^*\left(\frac{\partial f}{\partial z_0}\right) \cdot \frac{dr_0}{dx} - \left[(r \circ \tilde{p})^*\left(\frac{\partial f}{\partial z_0}\right)\frac{dr_0}{dx} + \sum_{i=1}^{n}(r \circ \tilde{p})^*\left(\frac{\partial f}{\partial z_i}\right)\frac{dr_i}{dx}\right]$$

$$\equiv \left[\pi^*\left(\frac{\partial f}{\partial z_0}\right) - (r \circ \tilde{p})^*\left(\frac{\partial f}{\partial z_0}\right)\right]\frac{dr_0}{dx} \mod \tilde{\mathfrak{q}},$$

since $(r \circ \tilde{p})^* \mathfrak{p} \subseteq \tilde{\mathfrak{q}}$ and $\frac{\partial f}{\partial z_i} \in \mathfrak{p}$ and since $\tilde{f} \circ \tilde{r} \circ \tilde{p} = f \circ r \circ \tilde{p}$.
Now $x \notin \tilde{\mathfrak{q}}$, since $\tilde{W}$ is not contained in $V(x)$. and $r_0(x) = u(x)x^k$ with $k \geq 1$ and $u(0) \neq 0$, so $\frac{dr_0}{dx} \notin \tilde{\mathfrak{q}}$. Moreover $\frac{\partial g}{\partial x} \in \tilde{\mathfrak{q}}$.
Hence $\pi^*\left(\frac{\partial f}{\partial z_0}\right) - (r \circ \tilde{p})^*\left(\frac{\partial f}{\partial z_0}\right) \in \tilde{\mathfrak{q}}$. Furthermore f is singular at 0 and $r(0) = 0$, so $(r \circ \tilde{p})^*\left(\frac{\partial f}{\partial z_0}\right) \in (x)\,\mathcal{O}_{(\tilde{U},0)}$. Thus $\pi^*\left(\frac{\partial f}{\partial z_0}\right) \in \tilde{\mathfrak{q}} + (x)$.
Also $\pi^*\left(\frac{\partial f}{\partial z_i}\right) \in \tilde{\mathfrak{q}} + (x)$ for $i \geq 1$, since $\tilde{f}_i = \pi^*\left(\frac{\partial f}{\partial z_i}\right)$ for $i \geq 1$.
Thus $\pi^*(f_0, \dots, f_n, z_0) \subseteq \tilde{\mathfrak{q}} + (x)$, so

$$\tilde{W} \cap V(x) \subseteq \pi^{-1}(V(f_0, \dots, f_n) \cap V(z_0)) = \pi^{-1}(\Sigma \cap V(z_0)) = \pi^{-1}(0) = \{0\}.$$

But $\tilde{W} \cap V(x)$ is at least one dimensional. This gives a contradiction and we have proved $V(f_1, \dots, f_n)$ is one dimensional, i.e. of codim n in $\mathcal{O}_{(U,0)}$, hence $f_1, \dots, f_n$ is a $\mathcal{O}_{(U,0)}$-regular sequence. This proves the proposition.

Remark (8.6) (i) A more elementary proof of proposition (8.5) is desirable.

(ii) Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$, with $\mathcal{O} = \mathcal{O}_{(\mathbb{C}^{n+1},0)}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the minimal prime ideals lying over J_f, i.e. $\Sigma_i = V(\mathfrak{p}_i)$ is a component of $\Sigma = V(J_f)$. Suppose Σ is one dimensional, then $\mathfrak{p}_i$ has height n, so $\mathfrak{p}_i \subsetneq \mathfrak{m}$. Thus $\mathfrak{m} \setminus (\mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r)$ is not empty, by [Ma] 1.B.

(iii) For any $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r)$ one can find local coördinates $z_0, z_1, \dots, z_n$ of $(\mathbb{C}^{n+1}, 0)$ such that $x = z_0$ and $\Sigma \cap V(x) = \{0\}$. And viceversa.

(iv) Let $f(0, y)$ have an isolated singularity at 0. Then $V(x)$ is called an <u>admissible hyperplane</u> by Yomdin [Lê, 2]. So $x, \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ is a regular $\mathcal{O}$-sequence, hence $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ is a regular $\mathcal{O}$-sequence.

Example (8.7) Let $f(x, y, z) = x(y^2 + z^2) + (yz)^2$, then $f(0, y, z) = (yz)^2$ has a non-isolated singularity. But $\Sigma = V(J_f) = V(y, z)$. and $\Sigma \cap V(x) = \{0\}$. Therefore $V(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is one dimensional. Thus the condition $\Sigma \cap V(x) = \{0\}$ is really weaker then the condition that $V(x)$ is an admissible hyperplane of f.

§9 The transversal type of f along a curve.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma, 0)$. We analise the transversal type of f along Σ, with respect to different equivalence relations for isolated singularities, i.e. with respect to R-, K- and μ-type. We shall give a necessary and sufficient condition in the case the transversal R-type of f does not vary along Σ.

Definition (9.1) Let $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be two germs of analytic functions, then f and g are called R (= right) equivalent if there exists a germ $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ of a local analytic isomorphism such that $g = f \circ h$. They are called K (= contact) equivalent if $(f^{-1}(0), 0) \cong$ and $(g^{-1}(0), 0)$ are isomorphic germs of analytic spaces.
If moreover f and g have an isolated singularity at 0, then they are called μ-homotopic, if there exists a family $\{f_t\}$, $t \in [0,1]$, of germs $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of analytic functions with an isolated singular point at 0, such that the Milnor number $\mu(f_t, 0)$ is constant, and $f_0 = f$, $f_1 = g$, and $F : \mathbb{C}^{n+1} \times [0,1] \to \mathbb{C}$, defined by $F(z,t) = f_t(z)$ is continuous.
V These relations define equivalence relations and the corresponding classes we call R-class, K-class resp. μ-class. or R-, K- resp. μ-type.

Definition (9.3) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a singular locus $(\Sigma, 0)$, such that $(\Sigma, 0)$ is a germ of a non singular curve. We say that the transversal R-type of f along Σ at 0 is constant, if for every choice of local coordinates $x, y_1, \ldots, y_n$ of $(\mathbb{C}^{n+1}, 0)$ such that $\Sigma = V(y_1, \ldots, y_n)$, the R-type of $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, with $f_t(y) = f(t, y)$, is the same for all t sufficiently small.
In the same way one defines what it means that the K-type (or μ-type) of f along Σ at 0 is constant.

<u>Proposition</u> (9.4) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma, 0)$. Let $(\Sigma_1, 0), \ldots, (\Sigma_r, 0)$ be the branches of $(\Sigma, 0)$. The transversal μ-type of f along Σ_i is constant at all points of a punctured neighbourhood of 0 in Σ_i.

<u>Proof</u> See [Y] or [Lê, 2] (1.3.1) and (1.3.2).

<u>Remark</u> (9.5) As a consequence we can associate to every branch $(\Sigma_i, 0)$ a well defined μ-class of an isolated singularity, which is the transversal μ-type of f along $\Sigma_i \setminus \{0\}$. We shall give some examples to show that the situation with respect to transversal R- and k-type is quite different.

<u>Example</u> (9.6) Let $f(x, y, z) = y^4 + xy^2z^2 + z^4$, then the singular locus Σ of f is $V(y, z)$. The transversal μ-type is $\tilde{E}_7$ with Milnor number 9. The transversal k- and R-type vary, since the cross ratio of the four lines of $f_x^{-1}(0)$ varies along Σ.

<u>Example</u> (9.7) Let $f(x, y, z) = y^5 + (1+x) y^2 z^2 + z^5$, then the singular locus Σ of f is the x-axis. The transversal μ-type and k-type are constant along Σ $(x \neq -1)$ and of type $T_{5,5,2}$. But the transversal R-type of f along Σ varies.

<u>Example</u> (9.8) Let $f(x, y, z) = y^6 + x y^3 z^3 + z^6$, then the singular locus of f is the x-axis. Take $\bar{x} = x - y$, then $\bar{f}(\bar{x}, y, z) = y^6 + (\bar{x} + y) y^3 z^3 + z^6$. The transversal k-type of f along Σ with respect to the coördinates x, y, z is quasi homogeneous, so $\mu(f_x, 0) = \tau(f_x, 0)$ the Tjurina number. But for the transversal k-type of f along Σ with respect to the coördinates $\bar{x}, y, z$ one has $\mu(\bar{f}_{\bar{x}}, 0) > \tau(\bar{f}_{\bar{x}}, 0)$. Hence the transversal k- (and R-) type of f depends on the chosen local coördinates.

Lemma (9.9) Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma,0)$ such that $(\Sigma,0)$ is a nonsingular curve. Let $x, y_1,\dots,y_n$ be local coördinates of $(\mathbb{C}^{n+1},0)$ such that $\Sigma = V(y_1,\dots,y_n)$. Let $f_x(y)=f(x,y)$. Suppose that the R-type of f_x is constant for all x sufficiently small. Then $\frac{\partial f}{\partial x}\in\left(\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_n}\right)\mathcal{O}$.

proof For all x sufficiently small is f_x R-equivalent with f_0, so there exist an $h_x:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$, a germ of a local analytic isomorphism such that $f(0,y)=f(x,h_x(y))$. By the finite determinacy theorem for isolated singularities

V_1 and

, we may assume that the map $H:(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}\times\mathbb{C}^n,0)$ defined by $H(x,y)=(x,h_x(y))$ is analytic. So

$$0=\frac{\partial}{\partial x}(f(0,y))=\frac{\partial}{\partial x}(f\circ H)=H^*\left(\frac{\partial f}{\partial x}\right)+\sum_{i=1}^{n}H^*\left(\frac{\partial f}{\partial y_i}\right)\cdot\frac{\partial H_i}{\partial x}.$$

Hence $H^*\left(\frac{\partial f}{\partial x}\right)\in H^*\left(\left(\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_n}\right)\mathcal{O}\right)$. So we have proved the lemma, since H^* is a local isomorphism.

Lemma (9.10) Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function. Let $x, y_1,\dots,y_n$ be local coördinates such that $f(x,y)=f(0,y)$. Suppose $z_0, z_1,\dots,z_n$ are some other local coördinates such that the hyperplane $V(z_0)$ intersects the curve $V(y_1,\dots,y_n)$ transversally at 0. Then $\frac{\partial f}{\partial z_0}\in\left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right)\mathcal{O}$.

proof Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$ and $f:U\to\mathbb{C}$ a representative of the given function germ. Let $x, y_1,\dots,y_n$ be local coordinates of U at 0, and $z_0,\dots,z_n$ some

V_2 other local coördinates such that $f(x,y)=f(0,y)$. and

Then, there is a local analytic isomorphism $h:U\to U$ such that $z_i=h_i$ in $\mathcal{O}_U$. Now

$$0=\frac{\partial}{\partial x}f(0,y)=\frac{\partial h_0}{\partial x}\cdot h^*\left(\frac{\partial f}{\partial z_0}\right)+\sum_{i=1}^{n}\frac{\partial h_i}{\partial x}\,h^*\left(\frac{\partial f}{\partial z_i}\right).$$

Let $\gamma:(\mathbb{C},0)\to(\mathbb{C}^{n+1},0)$ be given by $\gamma(x)=h(x,0)$. Then γ is a parametrization of Σ. Now $\frac{\partial h}{\partial x}(0)\neq 0$, since h is a local isomorphism. The tangent direction of Σ at 0 is $\frac{\partial h}{\partial x}(0)$

The hyperplane $V(z_0)$ intersects Σ transversally at 0. Hence $\frac{\partial h_0}{\partial x}(0) \neq 0$. So $\frac{\partial h_0}{\partial x}$ is a unit in $\mathbb{C}\{x,y\}$ and $h^*(\frac{\partial f}{\partial z_0}) \in h^*(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n})\mathcal{O}$. Hence $\frac{\partial f}{\partial z_0} \in (\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n})\mathcal{O}$, since h^* is a local isomorphism. This proves the lemma.

<u>Lemma</u> (9.11) Let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function and suppose $\frac{\partial f}{\partial z_0} \in (\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n})\mathcal{O}$, with $z_0, z_1, \dots, z_n$ some local coördinates of $(\mathbb{C}^{n+1},0)$. Then one can find local coördinates $x, y_1, \dots, y_n$ such that $f(x,y) = f(0,y)$.

<u>proof</u> The proof is standard after Moser and Thom-Levine. Let f be defined on an open neighbourhood U of 0 in $\mathbb{C}^{n+1}$, and suppose $\frac{\partial f}{\partial z_0} = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial z_i}$ on U, with $z_0, z_1, \dots, z_n$ some local coördinates in $(U,0)$. Consider the following differential equation for an analytic map $h:(\mathbb{C}^{n+1},0) \to (\mathbb{C}^n,0)$.

$$\begin{cases} \dfrac{\partial h_i}{\partial x} = -a_i(x, h(x,y)). & \text{for } i=1,\dots,n. \\ h(0,y) = y \end{cases}$$

It has a unique solution. Let $H:(\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ be defined by $H(x,y) = (x, h(x,y))$. Then H is an analytic local isomorphism and $f \circ H$ is independent of x, since

$$\frac{\partial}{\partial x}(f \circ H) = H^*\left(\frac{\partial f}{\partial z_0}\right) + \sum_{i=1}^{n} \frac{\partial h_i}{\partial x} H^*\left(\frac{\partial f}{\partial z_i}\right) = H^*\left(\frac{\partial f}{\partial z_0}\right) + \sum_{i=1}^{n} H^*(a_i) H^*\left(\frac{\partial f}{\partial z_i}\right)$$

$$= H^*\left(\frac{\partial f}{\partial z_0} - \sum_{i=1}^{n} a_i \frac{\partial f}{\partial z_i}\right) = 0.$$

Hence $f(x, h(x,y)) = f(0, h(0,y)) = f(0,y)$. This proves the lemma.

<u>Proposition</u> (9.12) Let $f:(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ, such that $(\Sigma,0)$ is a nonsingular curve. Then

a) Then the following statements are equivalent.

(i) There are local coördinates $x, y_1, \dots, y_n$ such that $f(x,y) = f(0,y)$ and $\Sigma = V(y_1,\dots,y_n)$.

(ii) There are local coördinates $\bar{x}, \bar{y}_1, \dots, \bar{y}_n$ such that $f_{\bar{x}}(\bar{y}) = f(\bar{x},\bar{y})$ and $\Sigma = V(\bar{y}_1,\dots,\bar{y}_n)$ and the R-type of $f_{\bar{x}}$ is constant for all $\bar{x}$ sufficiently small.

V (iii) For every choice of local coördinates $z_0, z_1, \dots, z_n$ such that $V(z_0)$ intersects Σ transversally at 0. Then $\frac{\partial f}{\partial z_0} \in (\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n})\mathcal{O}$

b) Moreover if either of a)(i),(ii) or (iii) holds then the transversal R-type of f along $\underline{\Sigma}$ at o is constant.

Proof a) (i) $\Rightarrow$ (ii) is trivial; (i) $\Rightarrow$ (iii) is proved in (9.10); (iii) $\Rightarrow$ (i) is proved in (9.11); (ii) $\Rightarrow$ (i) follows from (9.9) and (9.11).
b) Let f be defined on an open neighbourhood of o in $\mathbb{C}^{n+1}$. Let $z_0, z_1, \ldots, z_n$ be local coördinates such that $V(z_0)$ intersects $\underline{\Sigma}$ transversally at o. From a)(i) we derive that there are local coördinates $x, y_1, \ldots, y_n$ such that $f(x,y) = g(y)$ for all x sufficiently small. There exist a local analytic isomorphism $h: (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ with $z_i = h_i(x,y)$. The hyperplane H, with $H = V(z_0)$, is transversal to $\underline{\Sigma}$ at o, hence $\frac{\partial h_0}{\partial x}(o) \neq 0$ and we can write $h_0(x,y) = U(x,y)(x - P(y))$, with U a unit, by Weierstrass preparation theorem, see [Na] II.2. Let $H' = V(x - P(y)) = h^{-1}H$. We have the following commutative diagram

$$\begin{tikzcd}
 & (H,0) \arrow[r, "f|H"] & (\mathbb{C},0) \arrow[dd, equal] \\
(\mathbb{C}^n,0) \arrow[ur, "j"] \arrow[dr, "i"'] \arrow[drr, "g"] & & \\
 & (H',0) \arrow[uu, "h|H'"] \arrow[r, "f|H'"] & (\mathbb{C},0)
\end{tikzcd}$$

, with $i(y) = (P(y), y)$ and $j = (h|H') \circ i$. The maps i, j and $h|H'$ are all local analytic isomorphisms.
Hence the transversal function of f with respect to the coördinates $z_0, z_1, \ldots, z_n$ is $f|H$, and is R-equivalent with g and with f_x, the transversal function of f with respect to the coördinates $x, y_1, \ldots, y_n$, for all x sufficiently small.
The same is true for every point p in a neighbourhood of o in $\underline{\Sigma}$.
Hence the transversal R-type of f along $\underline{\Sigma}$ ~~at~~ o is constant.
This proves the proposition.

Proposition (9.13) Let $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus $(\underline{\Sigma},0)$. Let $z_0, z_1, \ldots, z_n$ be local coördinates of $(\mathbb{C}^{n+1},0)$ such that $\underline{\Sigma} \cap V(z_0) = \{0\}$. Then the following two statements are equivalent
(i) $\dim_{\mathbb{C}} \left(\mathcal{O}/J_f + ((f_1, \ldots, f_n) : f_0) \right) < \infty$
(ii) The transversal R-type of f along $\underline{\Sigma}_i$ is constant at all points of a punctured neighbourhood of o in $\underline{\Sigma}_i$, for every branch $\underline{\Sigma}_i$ of $\underline{\Sigma}$.

proof Let U be an open neighbourhood of 0 in $\mathbb{C}^{n+1}$, such that Σ and f are defined on U and $\Sigma \setminus \{0\}$ is nonsingular. This is possible, since Σ is one dimensional. Let $\mathcal{O}$ be the sheaf of analytic functions on U. Then $\ell(\mathcal{O}/_{J_f} + ((f_1,\dots,f_n):f_0)) < \infty$ is equivalent with $V(J_f + ((f_1,\dots,f_n):f_0)) = \{0\}$, after possibly shrinking U. Further $V(J_f + ((f_1,\dots,f_n):f_0)) = \Sigma \cap V((f_1,\dots,f_n):f_0)$. The hyperplane $H_t = V(z_0 - t)$ intersects Σ transversally in finitely many points, for all $t \neq 0$ sufficiently small, since $\Sigma \cap V(z_0) = \{0\}$. Let $p \in \Sigma_i \cap H_t$, with $t \neq 0$. Then

V $\quad p \notin V((f_1,\dots,f_n):f_0) \qquad f_0 \in (f_1,\dots,f_n)\mathcal{O}_p$ if and only if the transversal A-type of f along Σ_i at p is constant, by (9.12).

Thus $\Sigma \cap V((f_1,\dots,f_n):f_0) = \{0\}$ is equivalent with (ii). Hence (i) and (ii) are equivalent. This proves the proposition.

We want to end with a question.

Question (9.14) Let $(\Sigma, 0)$ be a germ of a reduced curve in $(\mathbb{C}^{n+1}, 0)$, with branches $\Sigma_1, \dots, \Sigma_r$. Let $X_1, \dots, X_r$ be μ-types of isolated singularities. Does there exist a function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ such that the singular locus of f is Σ and the transversal μ-type of f along Σ_i is X_i ?

Remark (9.15) In case $X_1 = \dots = X_r = A_1$ there always exists such a function by proposition (5.16).

§10 Series of isolated singularities

Let $f:(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ define a germ of an analytic function with a one dimensional singular locus Σ. Let $z_0, z_1, \ldots, z_n$ be local coördinates of $(\mathbb{C}^{n+1},0)$ such that $\Sigma \cap V(z_0) = \{0\}$. In this section we show that $f + z_0^k$ defines an isolated singularity for $k \gg 0$. We shall give a formula for its Milnor number in case the transversal R-type of f along $\Sigma_i \setminus \{0\}$ is constant for every branch Σ_i of Σ. A more specific formula is given in case Σ is a complete intersection and f has transversally only A_1 singularities along $\Sigma \setminus \{0\}$. We shall also give a formula for the Euler characteristic of the Milnorfibre of f.

Proposition (10.1) Let $f:(\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ and z_0 a coördinate such that $\Sigma \cap V(z_0) = \{0\}$. Then $f + \frac{z_0^{k+1}}{k+1}$ defines an isolated singularity at 0 for $k \gg 0$.

Remark (10.2) The proof is the same as in [Lê, 2].
A stronger condition is required there, i.e. $V(z_0)$ must be an admissible hyperplane, see (8.6)(ii).

proof (10.1) Let $\mathcal{O} = \mathcal{O}_{(\mathbb{C}^{n+1},0)}$. Let $\Gamma = V(f_1, \ldots, f_n)$, then Γ is one dimensional. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal prime ideals in $\mathcal{O}$, lying over $(f_1, \ldots, f_n)\mathcal{O}$, with $\mathcal{O} = \mathcal{O}_{(\mathbb{C}^{n+1},0)}$. Let $\Gamma_i = V(\mathfrak{p}_i)$, then $\Gamma_1, \ldots, \Gamma_s$ are the branches of Γ. There is for every i at most one k such that $f_0 + z_0^k \in \mathfrak{p}_i$. Otherwise, there are k and l, $k<l$ such that $f_0 + z_0^k$, $f_0 + z_0^l \in \mathfrak{p}_i$. Hence $z_0^k(1 - z_0^{l-k}) \in \mathfrak{p}_i$, and $1 - z_0^{l-k}$ is a unit in $\mathcal{O}$. So $z_0^k \in \mathfrak{p}_i$ and therefore $f_0 \in \mathfrak{p}_i$. So $(z_0^k, f_0, f_1, \ldots, f_n) \subseteq \mathfrak{p}_i$. Thus $\Gamma_i \subseteq \Sigma \cap V(z_0) = \{0\}$. This is a contradiction. Therefore there exist a k_0 such that for all $k \geq k_0$ and all i $f_0 + z_0^k \notin \mathfrak{p}_0$. So $f_0 + z_0^k \notin \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_s = \cup \operatorname{Ass}(\mathcal{O}_{\Gamma,0})$. The Krulldimension of $\mathcal{O}_{\Gamma,0}$ is one hence $f_0 + z_0$ is not a zero-divisor on $\mathcal{O}_{\Gamma,0}$, so $V(f_0 + z_0^k, f_1, \ldots, f_n) = \{0\}$ and $f + \frac{z_0^{k+1}}{k+1}$

has an isolated singularity at o, for $k \geq k_0$. This proves the proposition.

V **Notation** (10.3) Let $\mathcal{O} = \mathcal{O}_{(\mathbb{C}^{n+1},0)}$ V Let $J = J_f$ = the Jacobi ideal of f in $\mathcal{O}$. Let J_k be the Jacobi ideal of $f + \frac{z_0^{k+1}}{k+1}$ and let μ_k be its Milnor number. For an $\mathcal{O}$-module M we denote the length of M by $l(M)$, i.e. $l(M) = \dim_{\mathbb{C}}(M)$. For example $\mu_k = l(\mathcal{O}/J_k)$. Let $J = q_0 \cap q_1 \cap \dots \cap q_k$ be a primary decomposition of J and let q_0 be an m-primary component of J, if J has one, and take $q_0 = \mathcal{O}$ otherwise. Let $I = q_1 \cap \dots \cap q_k$. Suppose the singular locus Σ of f is one dimensional. So $\mathcal{O}/I$ has Krull dimension one and no embedded component, hence $\Sigma = V(I)$. We give Σ the analytic structure defined by I, i.e. $\mathcal{O}_\Sigma = \mathcal{O}/I$.
Warning: $\mathcal{O}_\Sigma$ may be nonreduced and we need not to have $f \in \int I$.
The local ring $\mathcal{O}_\Sigma$ is CM.
Let $z_0, z_1, \dots, z_n$ be local coördinates of $(\mathbb{C}^{n+1},0)$ and let $f_i = \frac{\partial f}{\partial z_i}$.
Take $x = z_0$. Suppose $\Sigma \cap V(x) = \{0\}$. This implies that x is not a zero-divisor on $\mathcal{O}_\Sigma$. (In such a case is

$$e_x(\Sigma) := \text{multiplicity of } (\Sigma, 0) \text{ with respect to } x = l(\mathcal{O}_\Sigma/x)$$

and $\quad k\, e_x(\Sigma) = l(\mathcal{O}_\Sigma/(x^k))$.
In the situation above we have the following three lemma's.

Lemma (10.4) $\qquad \mu_k = k\, e_x(\Sigma) + l(I/I \cap J_k)$ for $k \gg 0$.

Proof The following sequence is exact

$$0 \to I+J_k/J_k \to \mathcal{O}/J_k \to \mathcal{O}/I+J_k \to 0$$

and consists for $k \gg 0$ of $\mathcal{O}$-modules of finite length, by proposition (10.1). Now $I/I\cap J_k \cong I+J_k/J_k$ and $\mu_k = l(\mathcal{O}/J_k)$. The local ring $\mathcal{O}_\Sigma$ is CM and $I+J_k = I+(x^k)$ therefore $l(\mathcal{O}/I+J_k) = k\, e_x(\Sigma)$.
This proves the lemma.

Lemma (10.5) Suppose $\ell\left(\mathcal{O}/I + ((f_1,\dots,f_n):f_0)\right) < \infty$, then
$$I \cap J_k \subseteq I(f_0) + (f_1,\dots,f_n)\mathcal{O} \quad \text{for } k \gg 0.$$

Proof Suppose $a \in I \cap J_k$, then $a = \sum_i a_i f_i + a_0 x^k \in I$. Now $J \subseteq I$, hence $a_0 x^k \in I$, so $a_0 \in I$, since x is not a zero-divisor on $\mathcal{O}/I$. Since $J = I \cap \mathfrak{q}_0$ and $\mathfrak{q}_0$ is an $\mathfrak{m}$-primary ideal or equal to $\mathcal{O}$, we have $\ell(I/J) < \infty$. So we can find $N_1 \in \mathbb{N}$ such that $\mathfrak{m}^{N_1} I \subseteq J$, in particular $x^{N_1} I \subseteq J$. For all $k \geq N_1$ is $a_0 x^k = \sum_i b_i x^{k-N_1} f_i$ for some $b_0, \dots, b_n \in \mathcal{O}$. We assumed $\ell\left(\mathcal{O}/I + ((f_1,\dots,f_n):f_0)\right) < \infty$, so there exists $N_2 \in \mathbb{N}$ such that $x^{N_2} \in I + ((f_1,\dots,f_n):f_0)$, hence $x^{N_2} f_0 \in I(f_0) + (f_1,\dots,f_n)\mathcal{O}$. Therefore $a_0 x^k \in I(f_0) + (f_1,\dots,f_n)\mathcal{O}$ for all $k \geq N_1 + N_2$. Thus $a \in I(f_0) + (f_1,\dots,f_n)\mathcal{O}$. This proves the lemma.

Lemma (10.6) Suppose $\ell\left(\mathcal{O}/I + ((f_1,\dots,f_n):f_0)\right) < \infty$, then
$$I \cdot x^k \subseteq J_k \quad \text{for } k \gg 0.$$

Proof Let $J' = I(f_0) + (f_1,\dots,f_n)\mathcal{O}$. Then $I \cap J_k \subseteq J' \subseteq I$, by (10.5). From (10.4) we derive $\ell(I/I\cap J_k) < \infty$ for $k \gg 0$. Hence $\ell(I/J') < \infty$ for $k \gg 0$. So there exists $N_3 \in \mathbb{N}$ such that $\mathfrak{m}^{N_3} I \subseteq J'$, in particular $x^k I \subseteq J'$ for all $k \geq N_3$.
Let $\mathcal{O}^q \xrightarrow{\delta} \mathcal{O}^p \xrightarrow{g} I \to 0$ be a finite presentation of the ideal I, where $g = (g_1,\dots,g_p)$. Then we can write $f_i = \sum_j \varphi_{ij} g_j$ for all $i = 1,\dots,n$. The following diagram is commutative and exact

$$\begin{tikzcd}
 & & \mathcal{O}^q \arrow[d, "\delta"] & & \\
 & \mathcal{O}^p \oplus \mathcal{O}^n \arrow[r, "\varphi"] \arrow[d] & \mathcal{O}^p \arrow[d, "g"] & & \\
0 \arrow[r] & J' \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & I/J' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

, where $\varphi = \left(\begin{array}{ccc|c} f_0 & & 0 & \\ & \ddots & & \varphi_{ij} \\ 0 & & f_0 & \end{array}\right)$.

Hence $\mathcal{O}^q \oplus \mathcal{O}^p \oplus \mathcal{O}^n \xrightarrow{\delta+\psi} \mathcal{O}^p \longrightarrow I/J' \to 0$ is a presentation of the $\mathcal{O}$-module I/J'.
Let $\mathbb{1}$ be the $(p\times p)$-identity matrix. For all $k \geq N_3$ we can find a $p\times(p+n)$-matrix λ and a $(p\times q)$-matrix μ such that $\mathbb{1}x^k = \lambda\psi + \mu\delta$.
The matrix λ consists of two submatrices β and γ of sizes $(p\times p)$ and $(p\times n)$ resp. and

$$\lambda\psi = \beta f_0 + \gamma\varphi$$

After possibly increasing N_3 with one, we may suppose that the entries of β are in $\mathfrak{m}$. Therefore $\mathbb{1}+\beta$ is invertible. Define $\alpha = (\mathbb{1}+\beta)^{-1}\beta$. Then $\alpha\beta = \beta\alpha$ and $\beta = (\mathbb{1}-\alpha)^{-1}\alpha$ and $(\mathbb{1}+\beta)^{-1} = (\mathbb{1}-\alpha)$. After multiplication with $(\mathbb{1}-\alpha)$ on the left of $\mathbb{1}x^k = \beta f_0 + \gamma\varphi + \mu\delta$
we get $(\mathbb{1}-\alpha)\, x^k = \alpha f_0 + (1-\alpha)\gamma\varphi + (1-\alpha)\mu\delta$
So $\mathbb{1}.x^k = \alpha(f_0+x^k) + \gamma'\varphi + \mu'\delta$
Hence $I\cdot x^k \subseteq (f_0+x^k, f_1,\dots,f_n)\mathcal{O}$. This proves the lemma.

Proposition (10.7) Suppose $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ defines a germ of an analytic function with a one dimensional singular locus Σ. Let $z_0, z_1,\dots,z_n$ be local coördinates of $(\mathbb{C}^{n+1},0)$ such that $\Sigma\cap V(z_0) = \{0\}$. Suppose the transversal R-type of f along a punctured neighbourhood of $(\Sigma_i,0)$ in Σ_i is constant, for every branch $(\Sigma_i,0)$ of $(\Sigma,0)$. Let $J_f = I\cap\mathcal{O}_{f_0}$ as before. Then, the Milnor number μ_k of $f+\frac{x^{k+1}}{k+1}$ for $k\gg 0$ is equal to

$$\mu_k = \ell(I/J_f) + k\cdot e_x(\Sigma) + \ell\big(\mathcal{O}/I + ((f_1,\dots,f_n):f_0)\mathcal{O}\big)$$

Proof The transversality condition implies by (9.13) $\ell\big(\mathcal{O}/I+((f_1,\dots,f_n):f_0)\mathcal{O}\big)<\infty$. So we can apply lemma's (10.5) and (10.6):

$$I\cap J_k \subseteq I\cdot(f_0) + (f_1,\dots,f_n)\mathcal{O} = I(f_0+x^k-x^k) + (f_1,\dots,f_n)\mathcal{O}$$
$$\subseteq I.(x^k) + I\cap J_k \subseteq I\cap J_k$$

Let $J' = I(f_0) + (f_1,\dots,f_n)\mathcal{O}$, then $I\cap J_k = J'$.
~~The following sequence is exact.~~

$$\cancel{0\to J/I\cap J_k \to I/I\cap J_k \to I/J \to 0}$$

~~And~~ $\cancel{J/I\cap J_k = J/J' \cong \mathcal{O}/I+((f_1,\dots,f_n):f_0)}$, since $(f_1,\dots,f_n)\mathcal{O}\subseteq J'$ is

~~J/J' generated by f_0, and one easily shows that the annihilator of J/J' is $I + ((f_1,\dots,f_n):f_0)$. Thus $J/I\cap J_k = J/J' \cong \mathcal{O}/I+((f_1,\dots,f_n))$~~
~~$\mu_k = \ell(I/J_f) + k.e_x(\Sigma) + \ell(\mathcal{O}/I+((f_1,\dots,f_n):f_0))$. This proves the proposition.~~

Proposition (10.8) Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ. Let $I = \mathrm{rad}(J_f)$ and suppose $\ell(I/J_f)<\infty$ and I is generated by a $\mathcal{O}$-regular sequence. Let $z_0, z_1,\dots,z_n$ be local coördinates such that $\Sigma\cap V(z_0)=\{0\}$. Let $x=z_0$ then

$$\ell(\mathcal{O}/I+((f_1,\dots,f_n):f_0)) = \delta_f + \mu(\Sigma,0)+e_x(\Sigma)-1$$

and

$$\mu_k = j_f+\delta_f+\mu(\Sigma,0)+(k+1)e_x(\Sigma)-1$$

for $k\gg 0$.

V_1

Proof of (10.8). The second formula follows from the first one, by proposition (10.7), since the transversal R-type is A_1 on every branch of Σ. The length of I/J_f we called j_f in case I is radical.
V_2 Now $f\in\int I = I^2$. V_2 $f=\sum_{k,l} h_{kl}g_kg_l$ where $(g_1,\dots,g_n)\mathcal{O}=I$ and $h_{kl}=h_{lk}$. Further $f_i=\sum_k \varphi_{ik}g_k$ for $i=0,1,\dots,n$ where $\varphi_{ik}\equiv 2\sum_l h_{kl}\frac{\partial g_l}{\partial z_i} \pmod I$. Let φ be the $(n+1)\times n$-matrix with entries φ_{ik} and φ^i the $(n\times n)$-matrix obtained from φ by deleting the $(i+1)^{th}$ row, for $i=0,\dots,n$. Let $\Delta_i=(-1)^i\det\varphi^i$, then $\sum_i\Delta_i f_i=0$ and $((f_1,\dots,f_n):f_0) = (\Delta_0)\mathcal{O}+(f_1,\dots,f_n)\mathcal{O}$, see chapter II, (4.2). So $I+((f_1,\dots,f_n):f_0)=I+(\Delta_0)\mathcal{O}$. The derivative dg is an $(n+1)\times n$-matrix. Let $(dg)^0$ be the $(n\times n)$-matrix obtained from dg by deleting the first row. Let h be the $(n\times n)$-matrix with entries h_{kl}. Then $\varphi^0=2h\,(dg)^0$, so
$\Delta_0=2^n\det(h).\det(dg)^0$.
Thus $\ell(\mathcal{O}/I+\Delta_0)=\ell(\mathcal{O}/I+\det(h))+\ell(\mathcal{O}/I+\det(dg)^0)$.
Further $\ell(\mathcal{O}/I+\det(h))=\delta_f$, by (7.17). The map $\tilde g=(x,g_1,\dots,g_n)$
$\tilde g:(\mathbb{C}^{n+1},0)\to(\mathbb{C}^{n+1},0)$ defines a zero dimensional complete intersection singularity and $\ell(\mathcal{O}/I+\det(dg)^0)=\mu(g)+\mu(\tilde g)=\mu(g)+e_x(\Sigma)-1$, by [Lo] (5.11).
Hence $\ell(\mathcal{O}/I+((f_1,\dots,f_n):f_0))=\delta_f+\mu(\Sigma,0)+e_x(\Sigma)-1$.
This proves the proposition.

Proposition (10.10) Let I be an ideal in $\mathcal{O}$ defining a germ $(\Sigma,0)$ which is a reduced complete intersection curve in $(\mathbb{C}^{n+1},0)$. Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function such that $f\in\int I$ and $j_f<\infty$. Let F be the Milnor fibre of f and F_k the Milnor fibre of $f+\frac{z_0^{k+1}}{k+1}$. Let $v(x)$ be an admissible hyperplane then for $k>>0$.

$$\chi(F)=\chi(F_k)+(-1)^{n+1}(k+1).e_X(\Sigma).$$, where $\chi(F)$ is the Euler characteristic of F

Proof see [Y] or [Lê, 2] (2.2.2). A general theorem is proved by Yomdin for functions with a one dimensional singular locus Σ, without the condition that Σ is a complete intersection and with arbitrary transversal singularities.

Proposition (10.11) Let I be an ideal in $\mathcal{O}$ defining a germ $(\Sigma,0)$ which is a reduced complete intersection curve in $(\mathbb{C}^{n+1},0)$. Let $f:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ be a germ of an analytic function such that $f\in\int I$ and $j_f<\infty$. Let F be the Milnor fibre of f Then

$$\chi(F)=1+(-1)^n[j_f+\delta_f+\mu(\Sigma,0)-1]$$

Proof The Milnor fibre of an isolated singularity is a bouquet of μ n-spheres, with μ the Milnor number. So $\chi(F_k)=1+(-1)^n\mu_k$, Combining the formula for μ_k in proposition (10.8) and for $\chi(F)$ in proposition (10.10) we get the desired result.

Remark (10.12) In the situation of (10.11) f has a deformation such that f_t has Σ_t as singular locus, where Σ_t is the Milnor fibre of Σ, and with only A_1 singularities outside Σ_t and only A_∞ and D_∞ singularities on Σ_t. Moreover.

$$\delta_f=\#\{D_\infty \text{ points of } f_t \text{ on } \Sigma_t\}.$$

and $j_f=\#\{D_\infty \text{ points of } f_t \text{ on } \Sigma_t\}+\#\{A_1 \text{ points of } f_t \text{ outside } \Sigma_t\}$.

, see (7.20). It would be nice to build up the Milnor fibre of f out of these data, as Siersma has done for the case that $(\Sigma,0)$ is a line [Si] or a plane curve, to say something about the homotopy type of the Milnor fibre. We can rephrase proposition (10.11) and say

$$\chi(F) = 1 + (-1)^n \{ \# \{A_1 \text{ points}\} + 2\#\{D_\infty \text{ points}\} + \mu(\Sigma_0) - 1 \}.$$

Remark (10.13) We did not give a definition of "series of singularities", since we do not know any. Let us quote a remark of V.I. Arnold [An], page 153:
"..... Although the series undoubtedly exist, it is not at all clear what a series of singularities is.", and on page 154:
"... However a general definition of series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicity (for example $D \sim x^2y$, $T \sim xyz$), so that the hiearchy of series reflects the hiearchy of non-isolated singularities".
Although we did not succeed in giving a definition of a series, it was one of the main motives of this thesis.
A point which needs to be solved in this context is the following problem posed by Wall [W] and made more concrete by Siersma.

Let $(\Sigma,0)$ be a germ of a reduced curve in $(\mathbb{C}^{n+1},0)$, defined by the ideal I in $\mathcal{O}$. Let $(\Sigma_1,0),\dots,(\Sigma_r,0)$ be the branches of $(\Sigma,0)$. For every $\varphi \in \mathcal{O}$ we can associate a valuation $v_i(\varphi)$ as follows. Take a parametrization $\gamma_i : (\mathbb{C},0) \to (\Sigma_i,0)$ then $\gamma_i^*(\varphi) = u \cdot t^{v_i(\varphi)}$ with $u(0) \neq 0$. Then $v_i(\varphi) = \ell(\mathcal{O}_{\Sigma_i}/(\varphi))$. Let $v(\varphi) = \sum_i v_i(\varphi)$, then $v(\varphi) = \ell(\mathcal{O}_\Sigma/(\varphi))$.
Let $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function such that $f \in \int I$ and $j_f < \infty$.

~~Question (10.14) Does there exist a constant $C(f)$ only depending on the right equivalence type of f such that $f+\varphi$ has an isolated singularity with Milnor number $\mu(f+\varphi) = C(f) + v(\varphi)$ for $v(\varphi) \gg 0$ and $o(\varphi) \gg 0$?~~

Question (10.14) Does there exist a constant $C(f)$ only depending on the right equivalence type of f such that $f+\varphi$ has an isolated singularity with Milnor number $\mu(f+\varphi) = C(f) + v(\varphi)$ for all $\varphi \in \mathcal{O}$, where $v(\varphi) \gg 0$ and $o(\varphi) \gg 0$ and $\varphi \neq 0$?

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CHAPTER II THE DEPTH AND PROJECTIVE DIMENSION OF THE QUOTIENT OF TWO IDEALS

INTRODUCTION

Motivated by the theory of analytic functions with non-isolated singularities, see chapter I section 7 and 10, we are led to consider modules I/J, where I and J are ideals in a ring R such that $J \subseteq I$. Questions such as: is I/J Cohen-Macaulay or perfect ?, come up. Such questions were dealt with by Artin and Nagata [A-N] and later by Huneke [Hu] who detected and repaired a gap in [A-N]. These two papers deal with questions concerning the depth of R/J, I/J and $R/(J:I)$ under certain circumstances. We shall give their results in §1. This is a rather technical section, which is logically independent of the rest of this chapter and it contains no new results.
One is naturally lead to ask whether R/J, I/J and $R/(J:I)$ have finite projective dimension. This is treated in sections 3 and 4. The main new results are the following.
In section 3: we construct a complex $K.(g,\varphi)$ which is a finite free resolution of I/J in case grade $I/J \geq m \geqslant n$, where m is the number of generators of J and I is generated by an R-sequence. The module I/J is perfect as a result.
In section 4: we construct a finite projective resolution of I/J in case I is a perfect ideal of grade n and J is an ideal generated by $n+1$ elements and grade $(I/J) \geqslant n+1$.
Again, the module I/J is perfect as a result.
Section 2 gives a review of some multilinear algebra we need.

For basic definitions and results in commutative algebra we refer to [Ma]. R will denote a commutative Noetherian ring with unit. In section 1 we assume R to be a Cohen-Macaulay local ring.
Let M be a finitely generated R-module. Let I and J be

ideals in R. Then we use the following notations. $\operatorname{Spec} R$ is the collection of prime ideals in R. Let $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\}$ and $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq (0)\}$. The annihilator of M is denoted by $\operatorname{ann} M$ and we have $V(\operatorname{ann} M) = \operatorname{Supp} M$.

Let $\operatorname{pd} M$ = projective dimension of M as an R-module.

$\operatorname{dp}_I M$ = I-depth of M as an R-module, i.e. the maximal length of an M-regular sequence in I.

$\operatorname{dp} M = \operatorname{dp}_{\mathfrak{m}} M$ in case R is a local ring with maximal ideal $\mathfrak{m}$

The following equality is due to (Serre,) Auslander and Buchsbaum

$$\operatorname{dp} M + \operatorname{pd} M = \operatorname{dp} R \quad \text{, where } R \text{ is a local ring.}$$

Let $\ell(M)$ = length of M as an R-module

$(J : I) = \{r \in R \mid rI \subseteq J\}$

~~$\operatorname{ht} I$ = height I~~

$\operatorname{grade} M = \min\{i \mid \operatorname{Ext}^i_R(M,R) \neq (0)\}$

V_1 V_1 abuse of notation. If $I = \operatorname{ann} M$ then $\operatorname{grade} M = \operatorname{dp}_I R$

The Krull dimension of a ring R is denoted by $\dim R$ and $\dim M = \dim R/\operatorname{ann} M$. We denote the Krull dimension of a closed set V in $\operatorname{Spec} R$ by $\dim V$ too. Then $\dim R/I = \dim V(I)$. We call $\dim \operatorname{Spec} R - \dim V$ the codimension of V in $\operatorname{Spec} R$ and denote it by $\operatorname{codim} V$.

V_2 A local ring R is called V_2 CM if $\dim R = \operatorname{dp} R$ and a module M over a local ring R is called CM if $\dim M = \operatorname{dp} M$.

The following inequalities always hold:

$$\operatorname{grade} I \leq \operatorname{ht} I \leq \operatorname{codim} V(I).$$

If R is a local CM ring then $\operatorname{grade} I = \operatorname{ht} I = \operatorname{codim} V(I)$ and $\operatorname{grade} M = \operatorname{codim} \operatorname{Supp} M$.

We always have the inequality $\operatorname{grade} M \leq \operatorname{pd} M$ and M is called perfect if $\operatorname{grade} M = \operatorname{pd} M$. A perfect module is CM and conversely a CM module finite projective

V_3 dimension V_3 is perfect. In particular, if R is a regular local ring then the concepts CM and perfect coincide.

If R is a local ring and we have an exact sequence of finitely generated R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ then $\operatorname{pd} M_2 \leq \max\{\operatorname{pd} M_1, \operatorname{pd} M_3\}$ and $\operatorname{pd} M_3 = \operatorname{pd} M_1 + 1$ in case of inequality, analogously $\operatorname{dp} M_2 \geq \min\{\operatorname{dp} M_1, \operatorname{dp} M_3\}$ and $\operatorname{dp} M_1 = \operatorname{dp} M_3 + 1$ in case of inequality.

Let φ be a matrix with entries in R then $I_n(\varphi)$ is the ideal generated by the $(n\times n)$-minors of φ.
For an R-module M we define $M^* = \mathrm{Hom}_R(M,R)$.
For an ideal I in R we denote the minimal number of generators of I by $\mu(I)$.
For basic definitions and results of multilinear algebra we refer to [B-E].

This chapter is a corrected and expanded version of the preprint: Residual Jacobi ideals of hypersurface singularities, preprint 357, dec. 1984, Dep. of Math Un. of Utrecht.

§1 RESIDUAL INTERSECTIONS AND LINKAGE

Let R be a CM local ring and I and J two ideals in R such that $J \subseteq I$. We shall give some variations on a result of Artin and Nagata [A-N], which was corrected by Huneke [Hu], concerning the depth of R/J, $R/(J:I)$ and I/J. We start with some definitions and facts about linkage. Throughout this section R shall denote a CM local ring and if we say ring we mean a CM local ring.

<u>Definition</u> (1.1) Let R be a ring and $\underline{a}$ an $\underline{b}$ two ideals both of pure height r, i.e. all the associated primes have the same height r, in R. Then $\underline{a}$ and $\underline{b}$ are called <u>directly geometrically linked</u> if they have no associated primes in common and if $\underline{a} \cap \underline{b}$ is generated by an R-sequence. In geometric terms: $V(\underline{a})$ and $V(\underline{b})$ have no components in common and $V(\underline{a}) \cup V(\underline{b})$ is a complete intersection. If there is a chain of ideals $\underline{a}_1, \ldots, \underline{a}_n$ such that $\underline{a}_1 = \underline{a}$ and $\underline{a}_n = \underline{b}$ and $\underline{a}_i$ is directly geometrically linked to $\underline{a}_{i+1}$ for all $1 \leq i < n$ then $\underline{a}$ and $\underline{b}$ are called <u>geometrically linked</u>.

<u>Remark</u> (1.2) (i) If $\underline{a}$ and $\underline{b}$ are directly geometrically linked then $\mathrm{dp}(R/\underline{b})$ and $\mathrm{pd}(R/\underline{b})$ depend only on $\underline{a}$, see [A-N] theorem (1.1). If moreover $\underline{a}$ is generated by an R-sequence and $\underline{b} \neq R$ then $R/\underline{b}$ is CM, see [A-N] (1.2).
(ii) If $\underline{a}$ and $\underline{b}$ are geometrically linked and $\underline{a}$ is generated by an R-sequence and $\underline{b} \neq R$ then $R/\underline{b}$ is CM, see [A-N] (1.3).

<u>Definition</u> (1.3) An ideal $\underline{a}$ of pure height r in a ring R is called <u>relatively CM</u> if for some ideal $\underline{b}$ in R with $\underline{b} \neq R$ the ideals $\underline{a}$ and $\underline{b}$ are directly geometrically linked and $R/\underline{b}$ is CM.

<u>Remark</u> (1.4) If $\underline{a}$ is relatively CM and $\underline{b}'$ is another ideal in R with $\underline{b}' \neq R$ and $\underline{a}$ and $\underline{b}'$ are directly geometrically linked then $R/\underline{b}'$ is CM, by (1.2) (i).

The definition of geometric linkage was generalised by Peskine and Szpiro to pairs of ideals $\mathfrak{a}$ and $\mathfrak{b}$ which may have associated primes in common, see [P-S, 2].

<u>Definition</u> (1.5) Let R be a ring and $\mathfrak{a}$ and $\mathfrak{b}$ two ideals in R. Then $\mathfrak{a}$ and $\mathfrak{b}$ are called <u>directly (algebraically) linked</u> if there exists an R-sequence $x_1, \dots, x_r$ in $\mathfrak{a} \cap \mathfrak{b}$ such that $((x_1, \dots, x_r) : \mathfrak{a}) = \mathfrak{b}$ and $((x_1, \dots, x_r) : \mathfrak{b}) = \mathfrak{a}$. If there is a chain of ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ in R such that $\mathfrak{a} = \mathfrak{a}_1$ and $\mathfrak{b} = \mathfrak{a}_n$ and $\mathfrak{a}_i$ and $\mathfrak{a}_{i+1}$ are directly linked, then $\mathfrak{a}$ and $\mathfrak{b}$ are called <u>(algebraically) linked</u>.

<u>Remark</u> (1.6) If $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically linked then they are algebraically linked.

<u>Proposition</u> (1.7) Let R be a Gorenstein local ring. If $\mathfrak{a}$ and $\mathfrak{b}$ are linked and $\mathfrak{a}$ is a CM ideal then $\mathfrak{b}$ is a CM ideal.

<u>Proof</u>, see [P-S, 2].

<u>Corollary</u> (1.8) If $\mathfrak{a}$ is an ideal in a Gorenstein local ring R. Then $\mathfrak{a}$ is relatively CM if and only if $\mathfrak{a}$ is CM.

<u>Example</u> (1.9) Let k be a field and let $R = k[[X_1, X_2, X_3, X_4, Y_1, \dots, Y_r]]/(X_1X_3, X_1X_4, X_2X_4)$. Then R is a CM local ring and $Y_1, \dots, Y_r$ is an R-sequence. Take $\mathfrak{a} = (X_2X_3, Y_1, \dots, Y_r)R$ and $\mathfrak{b} = (X_1, X_4, Y_1, \dots, Y_r)R$. Then $(Y_1, \dots, Y_r)R = \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of pure height r in R and have no associated primes in common. So $\mathfrak{a}$ and $\mathfrak{b}$ are directly geometrically linked. Further $\mathfrak{b}$ is a CM ideal and $\mathfrak{a}$ is not CM. Hence $\mathfrak{a}$ is an example of a relatively CM ideal which is not CM itself.

<u>Definition</u> (1.10) Let M be an R-module. Let $\mu(M)$ be the minimal number of generators of M and $\mu_{\mathfrak{p}}(M)$ the minimal number of generators of $M_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-module. Define $C_i = C_i(\underline{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mu_{\mathfrak{p}}(I) \geq i\}$. Then C_i is a closed set in the Zariski topology of $\operatorname{Spec} R$. Define $e_i = e_i(I) = \operatorname{codim} C_i(I)$ for $i > 1$ and $e_1 = e_1(I) = \operatorname{codim} C_1(I) \cap V(I)$, where $\operatorname{codim} V = \operatorname{Kdim}(\operatorname{Spec} R) - \operatorname{Kdim} V$ for a closed set V in R and $\operatorname{codim} \phi = \infty$.

<u>Definition</u> (1.11) We say an ideal $\underline{a}$ <u>satisfies G_s</u> if $e_i(\underline{a}) \geq i$ for $i = 1, \ldots, s$ and satisfies G_∞ if $e_i \geq i$ for all positive integers i.

<u>Remark</u> (1.12) (i) If $\underline{a}$ is an ideal of height r then $e_i = r$ for $i = 1, \ldots, r$, hence $\underline{a}$ satisfies G_r. If $\underline{a}$ is generated by an R-sequence then $e_i = \infty$ for $i \geq r+1$, hence $\underline{a}$ satisfies G_∞.

(ii) A conceptually easier way of understanding the property G_s is given by the following characterization, see [Hu] (a.5). The ideal $\underline{a}$ satisfies G_s if and only if for every prime ideal $\mathfrak{p}$ over $\underline{a}$ of height smaller than s we have $\mu_{\mathfrak{p}}(I) \leq \operatorname{height} \mathfrak{p}$.

The following result of Artin and Nagata [A-N] is not true in this generality as was shown by Huneke [Hu].

<u>"Theorem"</u> (1.13) Let R be a CM local ring of dimension d. Let I be a relatively CM ideal of height n in R. Let m be an integer bigger than n and let $J = (f_1, \ldots, f_m)R$ be an ideal generated by m elements in R. Suppose $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ at every prime ideal $\mathfrak{p}$ in R of height $\mathfrak{p} < m$. Suppose I satisfies G_{m-1}. Then

a) $\operatorname{dp}(R/J) = d - m$

b) the ideal J can be written in the form $J = I \cap K$ where $\operatorname{height} K = m$ and no primary component of K contains I and R/K is CM.

V1 **Example** (1.15) This counterexample to "theorem" (1.13) is due to Huneke, see [Hu] (3.7). Let X and Y be generic (2×2)-matrices over a field k. Let $R = k[[X, Y]]$, i.e. R is the ring of formal power series in the entries X_{ij} and Y_{kl} of X and Y with coëfficients in k. Let $I = I_1(XY) + I_2(X) + I_2(Y)$, where $I_n(\varphi)$ is the ideal generated by the $(n\times n)$-minors of a matrix φ. Then I is CM of height 3, since it is equal to $I_2(\varphi)$ where φ is the generic matrix

$$\varphi = \begin{pmatrix} X_{11} & X_{21} & -Y_{21} & Y_{22} \\ X_{12} & X_{22} & Y_{11} & Y_{12} \end{pmatrix}$$

Thus I is also relatively CM since R is regular, by (1.9). Furthermore $I \cap I_1(X) \cap I_1(Y) = I_1(XY)$ and $I_1(XY)$ is generated by four elements. Take $J = I_1(XY)$ and $m = 4$. The ideal I satisfies G_3, since height $I = 3$. Further $J_{\mathfrak p} = I_{\mathfrak p}$ for all prime ideals $\mathfrak p$ of height $\mathfrak p < 4$, since height$(I_1(X) \cap I_1(Y)) = 4$. The ideal I satisfies G_3, since height $I = 3$. Moreover height$(I + I_1(X) \cap I_1(Y)) = 5$. Hence the decomposition of J in $J = I \cap K$ where K
V2 is an ideal of height $h \geq m$, V2 is unique, by (1.14). So $K = I_1(X) \cap I_1(Y)$. But K is not a CM ideal. This contradicts (1.13) b.

Remark (1.16) The proof of [A-N] goes by induction on m and starts at $m = n$, which is the situation of linkage. They just do not check that the ideal K they construct has no primary components containing I, but this is essentially used in the induction process. The proof remains correct if we restrict ourselves to the case $m = n+1$ and weaken assertion b of (1.13). The condition G_{m-1} is trivially satisfied, since I has height $n = m-1$.

We state in this case a version of (1.13) which has been proven by [A-N].

Theorem (1.17) Let R be a CM local ring of dimension d. Let I be a relatively CM ideal of height n. Let J be an ideal in R generated by $n+1$ elements in I. Suppose $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ of height $\mathfrak{p} < n+1$. Then

a) $\operatorname{dp}(R/J) = d-(n+1)$

b) the ideal J can be written in the form $J = I \cap K$ where $\operatorname{height} K = n+1$ and K is a CM ideal.

c) $\operatorname{dp}(I/J) = \min\{d-(n+1),\ \operatorname{dp}(R/I)+1\}$

Proof. Parts a) and b) are proved by [A-N] as remarked in (1.16). Part c) follows from the exact sequence $0 \to I/J \to R/J \to R/I \to 0$, which implies $\operatorname{dp}(R/J) \geq \min\{\operatorname{dp}(I/J), \operatorname{dp}(R/I)\}$ and $\operatorname{dp}(I/J) = \operatorname{dp}(R/I)+1$ in case of inequality.

Remark (1.18) In particular I/J is CM in case $\operatorname{dp}(R/I) \geq d-n-2$, e.g. in case I is a CM ideal. We shall give in (4.3) an analogous theorem in case I is a perfect ideal of grade n in a local ring R, which need not to be CM, and show that $\operatorname{pd}(R/J) = n+1$ and I/J is a perfect module of grade $n+1$.

X_{ij} Y hoofdletters

Example (1.19) Let's have another look at example (1.15). We have seen that $I_1(X) \cap I_1(Y)$ cannot be ideal K of theorem (1.13) b), since it is not CM. Let f_1, f_2, f_3, f_4 be the four entries of the (2×2)-matrix XY, i.e.

$$XY = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}.$$

We follow the proof of [A-N]. The ideal $(f_1, f_2, f_3)R$ is equal to $I \cap (X_{11}, X_{12}, f_3)R \cap (Y_{11}, Y_{21}, f_2)R$ and has height 3. Hence f_1, f_2, f_3 is an R-sequence. From their proof follows $J = I \cap K$ with $K = (X_{11}, X_{12}, f_3)R \cap (Y_{11}, Y_{21}, f_3)R + (f_4)R$ and K a CM ideal of height 4. One checks that

$$K = I_1(X) \cap I_1(Y) \cap (X_{11}, X_{12}, f_3, f_4, I_2(Y))R \cap (Y_{11}, Y_{21}, f_2, f_4, I_2(X))R.$$

Thus K is the intersection of four primary components, two of which contain I.

Example (1.20) We shall give an example which shows that the weakened version (1.17) of (1.13) cannot be true in case $m > n+1$. Let X, Y and R be as in example (1.15) and take again $I = I_1(XY) + I_2(X) + I_2(Y)$. But let $J = I_1(XY) + (X_{11}X_{22} - X_{12}X_{22} + Y_{11}Y_{22} - Y_{12}Y_{21})R$ and $m = 5$. Let $\mathfrak{m}$ be the maximal ideal of R. A computation shows that $\mathfrak{m} I \subseteq J \subseteq I$. Thus J has an $\mathfrak{m}$-primary $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all prime ideals of height $\mathfrak{p} < 8$, since height $\mathfrak{m} = 8$. Hence $dp(R/J) = 0 \neq d - m$, since $d = 8$ and $m = 5$. So part a of theorem (1.17) is not fullfilled. One can show that I satisfies G_∞.

Before we state Huneke's version of "theorem" (1.13) we need a definition.

Definition (1.21) Let R be a ring. An ideal I in R is called strongly CM if all the Koszul homology modules $H_i(g_1, \ldots, g_n; R)$ are CM or zero for a set of generators $g_1, \ldots, g_n$ of I. Sometimes one also requires I to be generically a complete intersection, i.e. for every minimal prime ideal $\mathfrak{p}$ over I we have $I_{\mathfrak{p}}$ is generated by an $R_{\mathfrak{p}}$-sequence, but this condition is not essential in this context.

Remark (1.22) This definition does not depend on the chosen generators of I, see [Hu](.). Ideals which are linked to an ideal generated by an R-sequence, are strongly CM, see [Hu](.).

Theorem (1.23) Let R be a CM local ring of dimension d. Let I be a strongly CM ideal. Let J be an ideal generated by $m > n$ elements in I. Suppose $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ of height $\mathfrak{p} < m$. Suppose I satisfies G_{m-1} then

a) $dp(R/J) = d - m$

b) the ideal J can be written in the form $J = I \cap K$ where K is a CM ideal of height $K = m$

c) I/J is a CM module of depth $d-m$

d) ~~if~~ if moreover height$(I+K) \geq m+1$ then $(J:I)$ is a CM ideal of height m and $J = I \cap (J:I)$

Remark (1.24)(i) This Theorem (1.23) is proved in [Hu](3.1) under the stronger assumptions that I satisfies G_m instead of G_{m-1} and that $J_{\mathfrak{p}} = I_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ of height $\mathfrak{p} \leq m$. But one can prove this theorem too.

(ii) One can weaken the assumption that I is strongly CM by requiring that I is an ideal of sliding depth, see [H-V V].

(iii) We could not find an example which shows that the condition that I satisfies G_{m-1} is necessary.

(iv) One may conclude that this is a technically and unsatisfactory state of affairs and far from being completely understood.

§2 A review of some multilinear algebra

We give a review of some multilinear algebra concerning the exterior and symmetric algebra's. For unproven assertions we refer to [B+]. In this section R will be a Noetherian commutative ring with unit element.

If F is a free R-module we denote by ΛF the *exterior algebra* on F. It is the free graded commutative R-algebra, generated by elements of F in degree 1.
The graded commutative law is:

$$f.g = (-1)^{pq}\, g.f \quad \text{where } p = \deg f \text{ and } q = \deg g.$$
$$f^2 = 0$$

The diagonal map $\Delta: F \to F \oplus F$ induces an algebra map $\Delta: \Lambda F \to \Lambda F \otimes \Lambda F$. If $f \in \Lambda F$ is of degree 1 then $\Delta(f) = f \otimes 1 + 1 \otimes f$. The elements of degree 0 of ΛF form a ring isomorphic to R and projection into degree 0 is an algebra map $\varepsilon: \Lambda F \to R$. It is called the counit. ε and Δ satisfy a set of identities dual to those of the unit $\eta: R \to \Lambda F$ and multiplication $m: \Lambda F \otimes \Lambda F \to \Lambda F$. Thus ΛF becomes a graded commutative, cocommutative bialgebra.

For a R-module M we will write $M^* = \mathrm{Hom}_R(M, R)$
let $\mu \in M^*$ and $m \in M$, define $\langle \mu, m \rangle = \mu(m)$.

ΛF is a bialgebra, hence ΛF^* is a bialgebra too. The map $F^* \to (\Lambda F)^*$, dual to the projection $\Lambda F \to F$, induces a natural algebra map $d: \Lambda F^* \to (\Lambda F)^*$, which is a map of bialgebra's. d is an isomorphism if F is a finitely generated free R-module. We shall identify ΛF^* with $(\Lambda F)^*$ via d from now on.

SF is the *symmetric algebra*, which we will regard as the free graded commutative algebra generated by its elements of degree 2. $SF = \sum_{i \geq 0} S_i F$. We will give the elements of $S_i F$ the degree $2i$. This terminology is not

standard, but it turns out to be the right one in this context. We identify S_0F with R and S_1F with F. If F is an R-module generated by a basis $x_1, \ldots, x_n$ then SF is isomorphic to the polynomial ring $R[x_1, \ldots, x_n]$.
The diagonal map $\Delta : F \to F \oplus F$ induces an algebra map $\Delta : SF \to S(F \oplus F) \cong SF \otimes SF$, with $\Delta(f) = f \otimes 1 + 1 \otimes f$ for elements $f \in S_1F$. The projection onto $S_0F = R$ gives an algebra map $\varepsilon : SF \to R$. This makes SF into a graded commutative, cocommutative bialgebra.

Since SF is an infinite direct sum, we will work with the graded dual $(SF)^*_{gr} = \sum_{i \geq 0} (S_iF)^*$.
The map $F^* \to (SF)^*_{gr}$ induces an algebra map $\alpha' : S(F^*) \to (SF)^*_{gr}$. But it is not an isomorphism, unless R contains the field of rationals. One has for instance for $\varphi \in F^*$ and $f \in F$

$$\langle \alpha'(\varphi^p), f^p \rangle = p! \langle \varphi, f \rangle^p$$

DF is the divided power algebra on F. It is the graded commutative algebra generated by elements $f^{(p)}$ called the p^{th} divided powers of f, where $f \in F$ is regarded as an element of degree 2 in DF. These divided powers satisfy certain conditions, see [B-E] page 251.
DF is a bialgebra and if F is a free R-module with basis $x_1, \ldots, x_n$ then D_pF is free on generators $\prod_i x_i^{(p_i)}$ with $\sum_i p_i = p$.
We can define an algebra map:

$$\alpha : D(F^*) \to (SF)^*_{gr} \quad \text{by the formula.}$$

$$\alpha(\varphi^{(p)})\left(\prod_i f_i^{p_i}\right) = \begin{cases} 0 & \text{if } \sum_i p_i \neq p \\ \prod_i \varphi(f_i)^{p_i} & \text{if } \sum_i p_i = p \end{cases}$$

If F is a free R-module with basis $x_1, \ldots, x_n$ and if $\xi_1, \ldots, \xi_n$ is the dual basis of F^* then

$$\alpha(\xi_i^{(p)})(x_i^p) = 1$$

So α is an isomorphism in this case.
Moreover $((SF)^*_{gr})^*_{gr} \cong SF$ as algebras.
So $(D(F^*))^*_{gr} \cong SF$ as algebras.

We can view ΛF as a ΛF^* - module and vice versa and we can consider $D(F^*)$ and SF as modules over each other. In general if $\alpha : B \to A^*_{gr}$ is a homogeneous bialgebra homomorphism. We define

$$S(\Delta) : A^*_{gr} \otimes A \longrightarrow A$$

by $S(\Delta)(\gamma \otimes f) = \gamma(\Delta(f) \cdot \Delta(f)) \in A$, for $\gamma \in A^*_{gr}$ and $f \in A$

and define $n : B \otimes A \longrightarrow A$

by $n = S(\Delta)(\alpha \otimes 1)$

We shall write $b(a)$ or ba for $n(b \otimes a)$.

We have $b(a) = \sum_i \langle \alpha(b), a_1^i \rangle a_2^i$ if $\Delta(a) = \sum_i a_1^i \otimes a_2^i$. The map n makes A into a B-module.

The map $\alpha : B \to A^*_{gr}$ gives rise to an algebra map

$$A \to (A^*_{gr})^*_{gr} \longrightarrow B^*_{gr}$$

Thus B is also an A-module.

From now on F will be a finitely generated free R-module. Then SF is a graded SF-module and ΛF is a graded ΛF^* - module. We may regard $SF \otimes \Lambda F$ as a bigraded $SF \otimes \Lambda F^*$ - module. The identity map $1 : F \to F$ gives, by the identification $\mathrm{Hom}(F, F) \cong F \otimes F^*$ an element $c = c_F$ of $F \otimes F^* = S_1 F \otimes \Lambda^1 F^*$. We shall write

$d_F : SF \otimes \Lambda F \longrightarrow SF \otimes \Lambda F$ for the $SF \otimes \Lambda F^*$-module map, given by multiplication by c. It is a map of bidegree $(2, -1)$, hence $d_F^2 = 0$.

Let $LF = \ker d_F$ then LF is a bigraded $SF \otimes \Lambda F^*$ -module and its bihomogeneous components are $L_p^q F$, where

$$d_p^q : S_{p-1} F \otimes \Lambda^q F \longrightarrow S_p F \otimes \Lambda^{q-1} F$$

$$L_p^q F = \ker d_{p+1}^{q-1}$$

The usefullness of the bialgebra approach lies for instance in the fact that the map d_F is defined without referring to any basis of F and that the property $d_F^2 = 0$ is a direct consequence of the degree of the map. Although this approach is rather abstract one must recognize that the bialgebra $SF \otimes \Lambda F$ with the map d_F is nothing but the <u>generic Koszul complex</u>, i.e. let $x_1, \ldots, x_n$ be a basis of F

then we have already noted that $SF \cong R[X_1, \dots, X_n]$
After this identification $S_pF \otimes \Lambda^q F$ has elements
$b \otimes (X_{i_1} \wedge \dots \wedge X_{i_q})$ with b a homogeneous polynomial
of degree p in $R[X_1, \dots, X_n]$.
The map $d_p^q : S_{p-1} F \otimes \Lambda^q F \longrightarrow S_p F \otimes \Lambda^{q-1} F$ is given by

$$b \otimes (X_{i_1} \wedge \dots \wedge X_{i_q}) \longmapsto \sum_{t=0}^{q} (-1)^{t+1} b X_{i_t} \otimes (X_{i_1} \wedge \dots \wedge X_{i_{t-1}} \wedge X_{i_{t+1}} \wedge \dots \wedge X_{i_q})$$

Proposition (2.1) Let F be a finitely generated free R-module of rank n then

a) $L_p^q F = \ker d_{p+1}^{q-1} = \operatorname{Im} d_p^q = \operatorname{Coker} d_{p-1}^{q+1}$ if $p+q \neq 1$

b) $L_p^1 F = S_p F$ for all p, $L_1^q F = \Lambda^q F$ for all $q \neq 0$
$L_p^0 F = L_0^q F = 0$ for all $q \neq 1$ and all p.
$L_p^q F = 0$ for all $q > n$

c) $L_p^n F \cong S_{p-1} F \otimes \Lambda^n F$

d) $L_p^q F$ is free of rank $\binom{n+p-1}{q+p-1}\binom{q+p-2}{p-1}$

Given a map between finitely generated free R-modules $\varphi : F \to G$, there is an induced map
$L\varphi : LF \to LG$ with bihomogeneous components
$L_p^q \varphi : L_p^q F \to L_p^q G$

Suppose F and G are free R-modules of rank m and n respectively. Define for every pair (p,q), $q \geq 1$ a complex

$$L_p^q(\varphi)_\cdot \; : \; 0 \to L_p^m(F) \otimes L_{m-n}^{n-q+1} G^* \xrightarrow{d} L_p^{m-1} F \otimes L_{m-n-1}^{n-q+1} G^* \xrightarrow{d} \dots$$

$$\dots \xrightarrow{d} L_p^{n+1}(F) \otimes L_1^{n-q+1} G^* \xrightarrow{d_1} L_p^q F \xrightarrow{L_p^q \varphi} L_p^q G$$

Here and in what follows $L_s^t G^*$ means $(L_s^t G)^*$, thus for example $L_s^1 G^* = (S_s G)^* \cong D_s(G^*)$, the sth component of the divided power algebra on G^*. For graded algebra's A we shall from now on delete the subscript gr in A^*_{gr} and denote it by A^*.

Note that the complexes $L_p^q(\varphi)_\cdot$ have lenth $m-n+1$. The map d is given as follows. Since LF is an $SF \otimes \Lambda F^*$-module, we may consider it, by the canonical map $\Lambda F^* \cong R \otimes \Lambda F^* \subseteq SF \otimes \Lambda F^*$ as a ΛF^*-module. Similarly $LG^* = \operatorname{Hom}_R(LG, R)$ is an

$SG \otimes \Lambda G^*$ - module that we can consider as an SG-module. The element $\varphi \in \mathrm{Hom}(F,G)$ corresponds to an element c_φ of bidegree $(1,2)$ in $\Lambda F^* \otimes SG$, since we have the isomorphisms $\mathrm{Hom}(F,G) \cong F^* \otimes G \cong \Lambda^1 F^* \otimes S_1 G \subseteq \Lambda F^* \otimes SG$. The element c_φ has odd degree, so $c_\varphi^2 = 0$. Thus multiplication by c_φ induces a differential on the quadruply graded module $LF \otimes LG^*$, which we call d. The maps we have labelled d in the complex $L_p{}^q(\varphi)$. are homogeneous components of this d. To define the map $d_1 : L_p{}^{n+1} F \otimes L_1{}^{n-q+1} G^* \longrightarrow L_p{}^q F$, note first that LF is a ΛG^*-module via the map $\Lambda\varphi^* : \Lambda G^* \longrightarrow \Lambda F^*$. We also have $L_1{}^{n-q+1} G^* = \Lambda^{n-q+1} G^*$. We define d_1 to be the structure map of the ΛG^*-module LF.

<u>Lemma</u> (2.2) In the above situation we have

$$d^2 = 0, \quad d_1 d = 0 \quad \text{and} \quad (L_p{}^q\varphi)\, d_1 = 0.$$

So $L_p{}^q(\varphi)$. is a complex.

<u>Theorem</u> (2.3) Let R be a Noetherian ring and suppose that $\varphi : F \to G$ is an R-linear map between free R-modules of rank m and n resp. with $m \geq n$. If grade $I_n(\varphi) = m-n+1$ then $L_p{}^q(\varphi)$. is a free resolution of $\mathrm{Coker}(L_p{}^q\varphi : L_p{}^q F \longrightarrow L_p{}^q G)$. If moreover $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$ and $\varphi(F) \subseteq \mathfrak{m} G$, then $L_p{}^q(\varphi)$. is a minimal resolution.

<u>Remark</u> $I_n(\varphi)$ is the ideal generated by the $(n \times n)$-minors of a matrix of φ.

Under the above assumptions we have:
in case $p=1$: $L_1{}^q(\varphi)$. is a resolution of $\mathrm{Coker}(\Lambda^q\varphi)$ for all q,
and for $q=1$: $L_p{}^1(\varphi)$. is a resolution of $\mathrm{Coker}(S_p\varphi)$ for all p.

§3 The complex $K.(\beta,\varphi)$

Let R be a ring. Let I and J be two ideals in R. Suppose J is generated by m elements $\alpha_1,\dots,\alpha_m$ in I and suppose I is generated by $\beta_1,\dots,\beta_n$. Then we can write

$$\alpha_i = \sum_j \varphi_{ij}\beta_j$$

Let $F = R^m$ and $G = R^n$ and let $\alpha: F\to R$ and $\beta: G\to R$ be maps with components $\alpha_1,\dots,\alpha_m$ and $\beta_1,\dots,\beta_n$ resp. Then we have the following commutative diagram:

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ \alpha\downarrow & & \downarrow\beta \\ R & = & R \end{array}$$

where φ is the map with matrix (φ_{ij}).

We shall construct a complex $K.(\beta,\varphi)$ in case $m\geq n$, which is a resolution of I/J when $\beta_1,\dots,\beta_n$ is an R-sequence and if moreover grade $(J + I_n(\varphi)) \geq m$.

Buchsbaum and Eisenbud [B-E] constructed a complex $K.(\varphi,a)$ associated with a $(m\times n)$-matrix φ and a $(m\times 1)$-sequence a in case $m\geq n$, which is a resolution of $R/J(\varphi,a)$, where $J(\varphi,a) = (b_1,\dots,b_n) + I_n(\varphi)$ with

$$b_j = \sum_i a_i\varphi_{ij}$$

, generalizing a result of Herzog [He], who considered the cases $m=n$ and $m=n+1$ only.

Their situation corresponds to the commutative diagram

$$\begin{array}{ccc} R & = & R \\ a\downarrow & & \downarrow b \\ F & \xrightarrow{\varphi} & G \end{array}$$

which is in a sense "half dual" to the former diagram.

Suppose we are in the first mentioned situation, i.e. $\alpha\in F^*$, $\beta\in G^*$ and $\varphi^*(\beta) = \alpha$.

Then the following diagram commutes:

$$(*)\qquad \begin{tikzcd} \Lambda^{n-q+2}F \arrow[r, "\Lambda^{n-q+2}\varphi"] \arrow[d, "\alpha"'] & \Lambda^{n-q+2}G \arrow[d, "\beta"] \\ \Lambda^{n-q+1}F \arrow[r, "\Lambda^{n-q+1}\varphi"] & \Lambda^{n-q+1}G \end{tikzcd}$$

where β is the action on the ΛG^*-module ΛG, with $\beta \in \Lambda G^*$, and α is the action on the ΛF^*-module ΛF with $\alpha \in \Lambda F^*$ and $\Lambda\varphi^*(\beta) = \alpha$. We shall define maps of complexes:

$$\begin{tikzcd} L_1^{n-q+2}(\varphi). \arrow[d, "v^q"] \\ L_1^{n-q+1}(\varphi). \end{tikzcd}$$

which are given by (*) in degrees 0 and 1 and which makes the complexes $L_1^{n-q+1}(\varphi).$ into the rows of the following double complex (**):

$$\begin{tikzcd}[column sep=small] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & L^1_{m-n}G^*\otimes\Lambda^m F \arrow[r, "d"] \arrow[d] & \cdots \arrow[r] & L^1_2G^*\otimes\Lambda^{n+2}F \arrow[r, "d"] \arrow[d] & L^1_1G^*\otimes\Lambda^{n+1}F \arrow[r, "d_1"] \arrow[d] & \Lambda^n F \arrow[r, "\Lambda^n\varphi"] \arrow[d] & \Lambda^n G \arrow[d] \\ & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ 0 \arrow[r] & L^{q-1}_{m-n}G^*\otimes\Lambda^m F \arrow[r, "d"] \arrow[d, "v^{n-q}_{m-n+1}"] & \cdots \arrow[r] & L^{q-1}_2G^*\otimes\Lambda^{n+2}F \arrow[r, "d"] \arrow[d, "v_3^{n-q}"] & L^{q-1}_1G^*\otimes\Lambda^{n+1}F \arrow[r, "d_1"] \arrow[d, "v_2^{n-q}"] & \Lambda^{n-q+2}F \arrow[r, "\Lambda^{n-q+2}\varphi"] \arrow[d, "\alpha"'] & \Lambda^{n-q+2}G \arrow[d, "\beta"] \\ 0 \arrow[r] & L^{q}_{m-n}G^*\otimes\Lambda^m F \arrow[r, "d"] \arrow[d] & \cdots \arrow[r] & L^{q}_2G^*\otimes\Lambda^{n+2}F \arrow[r, "d"] \arrow[d] & L^{q}_1G^*\otimes\Lambda^{n+1}F \arrow[r, "d_1"] \arrow[d] & \Lambda^{n-q+1}F \arrow[r, "\Lambda^{n-q+1}\varphi"] \arrow[d] & \Lambda^{n-q+1}G \arrow[d] \\ & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ 0 \arrow[r] & L^{n}_{m-n}G^*\otimes\Lambda^m F \arrow[r, "d"] & \cdots \arrow[r] & L^{n}_2G^*\otimes\Lambda^{n+2}F \arrow[r, "d"] & L^{n}_1G^*\otimes\Lambda^{n+1}F \arrow[r, "d_1"] & F \arrow[r, "\varphi"] & G \end{tikzcd}$$

We define $K.(\beta,\varphi)$ to be the total complex associated to the double complex (**).

The convention of the degrees in the double complex $(**)$ is as follows. The components of the bottom row have bidegree $(p,0)$ and the components of the right hand column have bidegree $(0,q)$. So, the components in the right bottom and left upper corner have bidegrees $(0,0)$ and $(m-n+1, n-1)$ resp.

We shall now define the double complex $(**)$ in detail. The differential d of the rows $L^{n-q+1}(\varphi)$ is defined in §2, i.e. d is induced by the action of $1 \otimes c_\varphi \otimes 1$ in $SF \otimes \Lambda F^* \otimes SG \otimes \Lambda G^*$ on $LF \otimes LG^* \subseteq SF \otimes \Lambda F \otimes D(G^*) \otimes \Lambda G^*$.
We shall define a second operator as follows. The element $1 \otimes 1 \otimes 1 \otimes \beta \in SF \otimes \Lambda F^* \otimes SG \otimes \Lambda G^*$ acts on $SF \otimes \Lambda F \otimes D(G^*) \otimes \Lambda G^*$ and has quadruple degree $(0,0,0,1)$ and commutes with $1 \otimes c_\varphi \otimes 1$. We call v the induced action of β on $LF \otimes LG^*$. Hence v and d commute and $vv = 0$. Call v_p^q the bihomogeneous components of v, i.e. for $0 \le q \le n-1$.

$$v_p^q : L_{p-1}^{n-q-1} G^* \otimes L_1^{n+p-1} F \longrightarrow L_{p-1}^{n-q} G^* \otimes L_1^{n+p-1} F \quad \text{for } 2 \le p \le m-n+1$$
$$v_1^q = \alpha \; : \; \Lambda^{q+2} F \longrightarrow \Lambda^{q+1} F$$
$$v_0^q = \beta \; : \; \Lambda^{q+2} G \longrightarrow \Lambda^{q+1} G$$

We shall show now that v also commutes with d_1. The map d_1 is given by $d_1(\gamma \otimes a) = \gamma(a)$ for $\gamma \otimes a \in \Lambda^q G^* \otimes \Lambda^{n+1} F = L_1^q G^* \otimes \Lambda^{n+1} F$.
The following diagram commutes

$$\begin{array}{ccc} \Lambda^{q-1} G^* \otimes \Lambda^{n+1} F & \xrightarrow{d_1} & \Lambda^{n-q+2} F \\ \beta\otimes 1 \downarrow & & \downarrow \alpha \\ \Lambda^q G^* \otimes \Lambda^{n+1} F & \xrightarrow{d_1} & \Lambda^{n-q+1} F \end{array}$$

Since
$$(d_1(\beta \otimes 1))(\gamma \otimes a) = d_1((\beta \wedge \gamma) \otimes a) = (\beta \wedge \gamma)(a) = \beta(\gamma(a))$$
$$= \alpha(\gamma(a)) = \alpha(d_1(\gamma \otimes a)) = (\alpha d_1)(\gamma \otimes a)$$

We used that the action of β on ΛF is the action of α on ΛF, since $\Lambda \varphi^*(\beta) = \alpha$.
This proves that $(**)$ is a double complex.

<u>Remark</u> In case $m = n$, we have only two columns in the double complex $(**)$. The first column is the Koszul complex $(\Lambda G, \beta)$

with $\Lambda^0 G$ deleted. The second column is the koszulcomplex $(\Lambda F, \alpha)$ with $\Lambda^0 F$ and $\Lambda^q F$ for $q>n$ deleted.

In case $m=n+1$ we have only three columns in the double complex (**). The first two are as mentioned before and the third is identified with ~~the dual~~ koszulcomplex $(\Lambda G^*, \beta)$ with $\Lambda^0 G^*$ deleted, since $\Lambda^{n+1}F \cong R$ and $L_1^q G^* = \Lambda^q G^*$. The map v_E^q is just $\beta\otimes 1$, where the map β is multiplication (on the left) on ΛG^*, i.e. $\gamma \mapsto \beta\wedge\gamma$

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The double complex (**) becomes in this case

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & G^*\otimes\Lambda^{n+1}F \arrow[r] \arrow[d] & \Lambda^n F \arrow[r, "\Lambda^n\varphi"] \arrow[d] & \Lambda^n G \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \Lambda^{q-1}G^*\otimes\Lambda^{n+1}F \arrow[r] \arrow[d, "\beta\otimes 1"'] & \Lambda^{n-q+2}F \arrow[r] \arrow[d, "\alpha"] & \Lambda^{n-q+2}G \arrow[d, "\beta"] \\
0 \arrow[r] & \Lambda^{q}G^*\otimes\Lambda^{n+1}F \arrow[r] \arrow[d] & \Lambda^{n-q+1}F \arrow[r] \arrow[d] & \Lambda^{n-q+1}G \arrow[d] \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & \Lambda^{n}G^*\otimes\Lambda^{n+1}F \arrow[r] & F \arrow[r, "\varphi"] & G
\end{tikzcd}$$

This double complex is selfdual if we identify $\Lambda^q F^*$ with $\Lambda^{n-q+1}F$. In particular the maps $\Lambda^q G^* \longrightarrow \Lambda^{n-q+1}F \cong \Lambda^q F^*$ is dual to the map $\Lambda^q F \xrightarrow{\Lambda^q\varphi} \Lambda^q G$. and the bottom row can be identified with

$$0 \longrightarrow R \xrightarrow{\Delta} R^{n+1} \xrightarrow{\varphi} R^n$$

where the components $\Delta_1, \dots, \Delta_{n+1}$ of Δ are the $(n\times n)$-minors of φ. i.e. Δ_i is the determinant of the matrix obtained from φ by deleting the ith column.

The selfduality of the double complex (**) is a special feature of the cases $m=n$ and $m=n+1$.

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The same rows of (**) appear in the doublecomplex of [B-E] in their construction of $k.(\varphi, a)$. The arrows in the columns are reversed. The definition of their maps in the columns is very laborious.

probeer (3.1) op een bladzijde 3-5 te krijgen

Theorem (3.1). Let φ: $F \to G$ be a map of free R-modules with rank $F = m \geqslant$ rank $G = n$. Let α: $F \to R$, β: $G \to R$ and $\alpha = \beta\varphi$, let $K.(\beta,\varphi)$ be the total complex associated to the double complex (**). Let $J(\beta,\varphi)$ be the ideal $I_n(\varphi) + \mathrm{Im}(\beta)$ and $I_n(\varphi)$ the ideal generated by the $(n \times n)$-minors of the $(m \times n)$-matrix of φ. Then:

(1) The homology of $K.(\beta,\varphi)$ is annihilated by $J(\beta,\varphi)$.

(2) If R is a local ring with maximal ideal $\mathfrak{m}$ and if $\varphi(F) \subset \mathfrak{m}G$ and $\mathrm{Im}(\beta) \subset \mathfrak{m}$ then $K.(\beta,\varphi)$ is a minimal complex.

(3) grade $J(\beta,\varphi) \leqslant m$ and moreover $K.(\beta,\varphi)$ is exact if grade $J(\beta,\varphi) = m$.

(4) If $\beta = (\beta_1,\ldots,\beta_n)$ is an R-sequence then $H_0(K.(\beta,\varphi)) = \mathrm{Im}(\beta)/\mathrm{Im}(\alpha)$.

In the generic case these conditions are actually satsified, we have (see [B-E] theorem (5.2) for the analogon):

Theorem (3.2). Let S be any commutative regular Noetherian ring and let $R = S[X_{ij},Y_k]_{1\leqslant i\leqslant m,1\leqslant j,k\leqslant n}$ be the polynomial ring in $mn + n$ indeterminates with $m \geqslant n$. Let φ: $R^m \to R^n$ be the map with matrix (X_{ij}), and let β: $R^n \to R$ be the map with matrix $(Y_1,\ldots,Y_n)$. Then $J(\beta,\varphi)$ is an ideal of grade $\geqslant m$ and $K.(\beta,\varphi)$ is a resolution of $\mathrm{Im}(\beta)/\mathrm{Im}(\alpha)$.

Proof of theorem (3.1)

Part (2) follows from the construction of $K.(\beta,\varphi)$.

Part (4). Suppose $(\beta_1,\ldots,\beta_n)$ is an R-sequence, then we have the following commutative diagram:

$$\begin{tikzcd}
 & & \Lambda^2 G \arrow[d, "\beta"] & & \\
 & F \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & G \arrow[d, "\beta"] & & \\
0 \arrow[r] & \mathrm{Im}(\alpha) \arrow[r] \arrow[d] & \mathrm{Im}\ \beta \arrow[r] \arrow[d] & \mathrm{Im}(\beta)/\mathrm{Im}(\alpha) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact columns and rows. This implies that the sequence

$$F \oplus \Lambda^2 G \xrightarrow{\varphi+\beta} G \longrightarrow Im(\beta)/Im(\alpha) \longrightarrow 0$$

is exact. Further $K_1(\beta,\varphi) = F \oplus \Lambda^2 G$ and $K_0(\beta,\varphi) = G$, hence

$$H_0(K.(\beta,\varphi)) \cong \mathrm{Coker}(\varphi,\beta) \cong \mathrm{Im}(\beta)/\mathrm{Im}(\alpha).$$

And one sees that $J(\beta,\varphi)$ annihilates $H_0(K.(\beta,\varphi))$, since $\mathrm{Im}(\alpha) \subset \mathrm{ann}(\mathrm{Im}(\beta)/\mathrm{Im}(\alpha))$ and $I_n(\varphi) \subset \mathrm{ann}(\mathrm{Coker}(\alpha)) \subset \mathrm{ann}(\mathrm{Coker}(\varphi+\beta))$.
Parts (1) and (3) follow from theorem (3.2) and the following:

Lemma (3.3). ("Lemme d'acyclicité" [P-S,1]).
Let R be a Noetherian ring with 1.
If $L_\bullet : 0 \to L_k \to L_{k-1} \to \dots \to L_1 \to L_0$ is a complex of finitely generated free R-modules, and J is an ideal of R which annihilates the homology of L., and grade(J) $\geq$ k, then L. is exact.

We repeat the argument from [B-E] page 291 to derive theorem (3.1) part (1) and (3) from theorem (3.2).
Let $S = \mathbb{Z}$ be the ring of integers, and $R_0 = S[X_{ij}, Y_k]$, let $\varphi_0 : R_0^m \to R_0^n$ be given by the matrix (X_{ij}) and let $\beta_0 : R_0^n \to R_0$ be given by the matrix $(Y_1,\dots,Y_n)$ as in theorem (3.2).
Then we know that $K.(\beta_0,\varphi_0)$ is a resolution of $\mathrm{Im}(\beta_0)/\mathrm{Im}(\alpha_0)$ with $\alpha_0 = \beta_0\varphi_0 : R_0^m \to R_0$. If R, φ and β are as in theorem (3.1) then $K.(\beta,\varphi)$ is a specialization of $K.(\beta_0,\varphi_0)$, that is there is a unique homorphism $\zeta: R_0 \to R$ of rings, such that $K.(\beta,\varphi) = K.(\beta_0,\varphi_0) \otimes_{R_0} R$. If $r \in J(\beta_0,\varphi_0)$ then the map $h: K.(\beta_0,\varphi_0) \to K.(\beta_0,\varphi_0)$, induced by multiplication with r, induces 0 on $\mathrm{Im}(\beta_0)/\mathrm{Im}(\alpha_0)$, and thus is homotopic to 0 by some homotopy s. But then $s \otimes_{R_0} R$ is a homotopy on $K.(\beta,\varphi)$, which shows that multiplication by $\zeta(r)$ is homotopic to zero on $K.(\beta,\varphi)$. Thus $\zeta(r)$ annihilates the homology of $K.(\beta,\varphi)$, since $J(\beta,\varphi) = R\zeta(J(\beta_0,\varphi_0))$, part (1) is proven.
As for part (3) we make use of lemma (3.3), which shows that if the homology of $K.(\beta,\varphi)$ is annihilated by an ideal of grade $\geq$ m, then $K.(\beta,\varphi)$ is

exact, since $K.(\beta,\varphi)$ has length m.

Thus if grade $J(\beta,\varphi) \geq m$ then $K.(\beta,\varphi)$ is exact and $\mathrm{pd}(H_0) \leq m$ with $H_0 = H_0(K.(\beta,\varphi))$.

However $J(\beta,\varphi) \subset \mathrm{ann}(H_0)$, hence:

$$\operatorname{grade} J(\beta,\varphi) \leq \operatorname{grade}(\mathrm{ann}(H_0)) = \operatorname{grade}(H_0) \leq \mathrm{pd}(H_0) \leq m.$$

So one concludes grade $J(\beta,\varphi) = m$.

Proof of theorem (3.2).

If R,φ and β are as in theorem (3.2), then it is well known that grade $I_n(\varphi) = m - n + 1$, see [E-N]. Thus we may apply the following lemma (analogous to lemma (5.5) of [B-E]):

Lemma (3.4). In the set-up of theorem (3.1), suppose that grade $I_n(\varphi) = m - n + 1$. Then some power of the ideal $J(\beta,\varphi)$ annihilates the homology of $K.(\beta,\varphi)$.

Proof We must give a proof without the help of theorem (3.2). By [B-E] the rows of (**) are exact under the hypothesis of the lemma. By the spectral sequence of the double complex (**), the homology of $K.(\beta,\varphi)$ is the same as the homology of the complex of cokernels of the maps $\Lambda^q\varphi$ for $q = 1,\dots,n$. That is, if we let $C_{p-1} = \operatorname{coker}(\Lambda^p\varphi)$ the maps in the diagram:

$$\begin{array}{ccccccccccc}
\Lambda^n F & \to & \cdots & \to & \Lambda^p F & \to & \cdots & \to & \Lambda^2 F & \to & F \\
\downarrow{\scriptstyle \Lambda^n\varphi} & & & & \downarrow{\scriptstyle \Lambda^p\varphi} & & & & \downarrow{\scriptstyle \Lambda^2\varphi} & & \downarrow{\scriptstyle \varphi} \\
\Lambda^n G & \to & \cdots & \to & \Lambda^p G & \to & \cdots & \to & \Lambda^2 G & \to & G \\
\downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
C_{n-1} & & & & C_{p-1} & & & & C_1 & & C_0 \\
\downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
0 & & & & 0 & & & & 0 & & 0
\end{array}$$

induce maps $C_p \to C_{p-1}$, which make them into a complex $C.$, having the same homology as $K.(\beta,\varphi)$. By proposition (1.5) of [B-E], the C_p are themselves annihilated by $I_n(\varphi)$, so the same goes for the homology of $K.(\beta,\varphi)$. Thus we need only to show that some power of $\mathrm{Im}(\alpha)$ kills $H.(K.(\beta,\varphi))$. To this end, let $\alpha = (\alpha_1,\ldots,\alpha_m) \in R^{m*} = F^*$ and let T be a localization of R at the multiplicatively closed set generated by one of the α_i's, we must show that $T \otimes K.(\beta,\varphi)$ is exact.

Let $x_1,\ldots,x_m$ be a basis for F and $y_1,\ldots,y_n$ for G and $\hat{x}_1,\ldots,\hat{x}_m$ a dual basis for F^* and $\hat{y}_1,\ldots,\hat{y}_n$ for G^*. We have $\alpha = \sum \alpha_i \hat{x}_i$ and $\beta = \sum_j \beta_j \hat{y}_j$. Now $(\alpha_1,\ldots,\alpha_m)T = T$ hence there exist $u_1,\ldots,u_m \in T$ such that $\sum_i u_i\alpha_i = 1$. Take $u = \sum_i u_i x_i \in T \otimes F \subset \Lambda(T\otimes F)$ then $1 = \sum_i u_i\alpha_i = \sum_{i,j} u_i\varphi_{ij}\beta_j = \sum_j v_j\beta_j$ with $v_j = \sum_i u_i\varphi_{ij}$. Take $v = \sum_j v_j y_j \in T\otimes G \subset \Lambda(T\otimes G)$ then $\varphi^*(\beta) = \alpha$. Moreover $\varphi(u) = v$ and $u(\alpha) = 1$ so $v(\beta) = 1$. For $x \in \Lambda F$ one has by corollary (1.2) of [B-E], since $\deg u = 1$:

$$x = u(\alpha)(x) = u \wedge (\alpha(x)) + (-1)^{1+\deg\alpha}\alpha(u\wedge x) = u\wedge(\alpha(x)) + \alpha(u \wedge x).$$

Thus $\mathrm{id}_{\Lambda(T\otimes F)} = u\alpha + \alpha u$, i.e.
$\mathrm{id}_{\Lambda(T\otimes F)}$ is homotopic to 0 on the koszul complex $(\Lambda(T\otimes F),\alpha)$.
In the same way $\mathrm{id}_{\Lambda(T\otimes G)} = v\beta + \beta v$, i.e. $\mathrm{id}_{\Lambda(T\otimes G)}$ is homotopic to 0 on the Koszul complex $(\Lambda(T\otimes G),\beta)$.
In the following diagram, the second and third row are the Koszul complex $(\Lambda(T\otimes F),\alpha)$ and $(\Lambda(T\otimes G),\beta)$ resp., which are both exact, since $\mathrm{Im}(\alpha)T = \mathrm{Im}(\beta)T = T$.

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^m(T\otimes F) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \Lambda^{n+1}(T\otimes F) \arrow[r] \arrow[d, equal] & 0 \arrow[d] & & & \\
0 \arrow[r] & \Lambda^m(T\otimes F) \arrow[r, "\alpha"] & \cdots \arrow[r] & \Lambda^{n+1}(T\otimes F) \arrow[r, "\alpha"] \arrow[d] & \Lambda^n(T\otimes F) \arrow[r, "\alpha"] \arrow[d, "\Lambda^n\varphi"] & \cdots \arrow[r, "\alpha"] & T\otimes F \arrow[r] \arrow[d, "\varphi"] & T \arrow[d, equal] \\
 & & & 0 \arrow[r] & \Lambda^n(T\otimes G) \arrow[r, "\beta"] \arrow[d] & \cdots \arrow[r, "\beta"] & T\otimes G \arrow[r] \arrow[d] & T \\
 & & & 0 \arrow[r] & T\otimes C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & T\otimes C_0 \arrow[r] \arrow[d] & 0 \\
 & & & & 0 & & 0 &
\end{tikzcd}$$

The homotopy's of both complexes are compatible, since $\varphi^*(\beta) = \alpha$ and $\varphi(u) = v$, so they induce a homotopy of $\mathrm{id}(T \otimes C.)$ to 0, so that $T \otimes H(C.) = 0$, thus some power of α_i annihilates $H(K.(\beta,\varphi))$. Since i was arbitrary, the lemma is proven.

To finish the proof of theorem (3.2) it is enough to show *the exactness of* $K.(\beta,\varphi)$. By lemma (3.3) and (3.4) it suffices to show that in this case grade $J(\beta,\varphi) \geqslant m$. We have indeed the following lemma (analogous to lemma (5.6) of [B-E]).

Lemma (3.5). Let β and φ be as in theorem (3.2). Then grade $J(\beta,\varphi) \geqslant m$.

Proof. Suppose grade $J(\beta,\varphi) < m$, Then there is a prime ideal p in R such that $J(\beta,\varphi) \subset p$ and $\mathrm{grade}_{R_p}(pR_p) < m$.

Suppose $Y_j \notin p$ for some $j = 1,\dots,n$, then Y_j is a unit of R_{Y_j}.

Let $f_i' = \sum_{k \neq j} X_{ik} Y_k Y_j^{-1} + X_{ij}$ *then* $f_i = Y_j f_i'$

Hence the following sequence $Y_1,\dots,Y_{j-1},Y_{j+1},\dots,Y_n,f_1,\dots,f_m$ is R_{Y_j}-regular *Now* $R \to R_{Y_j} \to R_p$ are localisations, so it is also an R_p-regular sequence, R_p is a local ring, hence $f_1,\dots,f_m$ is an R_p-regular sequence in pR_p, thus $\mathrm{grade}_{R_p}(pR_p) \geqslant m$ *which gives a contradiction.*

Thus $Y_j \in p$ for all $j = 1,\dots,n$ so $I_n(\varphi) + (Y_1,\dots,Y_n) \subset p$.

Let $\bar{R} = (R/(Y_1,\dots,Y_n)R \cong S[X_{ij}]$, we have $\mathrm{grade}_{\bar{R}}(I(\bar{\varphi})) = m - n + 1$, *by [E-N].*

So $\mathrm{grade}_{R_p}(pR_p) \geqslant \mathrm{grade}\ p \geqslant \mathrm{grade}\ I_n(\varphi) + (Y_1,\dots,Y_n)R \geqslant m + 1$. *This is a contradiction.* Thus grade $J(\beta,\varphi) \geqslant m$. This finishes the proof of lemma (3.5) and theorem (3.2).

§4 Resolutions of quotients of ideals.

Let I and J be ideals in a ring R. Suppose $J \subseteq I$ and:

a) J is generated by m elements and $\operatorname{grade}(I/J) \geq m$

b) one of the following three conditions is satisfied:

(i) I is a perfect ideal of grade $m-1$.

(ii) $\operatorname{pd}(R/I) = 2$

(iii) I is generated by an R-sequence of length $n \leq m$

Then we give a finite projective resolution of the quotient I/J and prove that I/J is a perfect R-module of grade m. See propositions (4.3), (4.5) and (4.9) for (i), (ii) and (iii) respectively.

We show that the ideal $J(g, \varphi)$ as defined in §3 is an invariant of the pair of ideals $J \subseteq I$ in case I is generated by an R-sequence.

Let R be a ring and $f = (f_1, \ldots, f_m)$ a sequence of elements in R. Then we denote by $H_p(f, R)$ the p^{th} koszul homology of the sequence f.

Suppose J is an ideal in R and $f_1, \ldots, f_m$ generate J. Suppose $\operatorname{grade} J = n$ and $a_1, \ldots, a_n$ is an R-sequence in J.

Let (a) be the ideal in R generated by $a_1, \ldots, a_n$.

Then the following two isomorphisms are well known:

$$\operatorname{Ext}^n_R(R/J, R) \cong \operatorname{Hom}_R(R/J, R/(a)), \text{ see [R] (3.1),}$$

$$H_{m-n}(f, R) \cong \operatorname{Hom}_R(R/J, R/(a)), \text{ see [A-B] (1.7).}$$

Both isomorphisms can be proved by induction on n. We include the following form of this fact, since we could not find an explicite isomorphism in the litterature.

Lemma (4.1) Let R be a ring and J an ideal in R generated by $f_1, \ldots, f_m$. Suppose $\operatorname{grade} J = n$. Let $(G_\cdot, d)$ be a projective resolution of R-modules of R/J. Take $F = R^m$ and let $f: F \to R$ be the map with matrix $(f_1, \ldots, f_m)$. Then there exists a map of complexes $\varphi: (\Lambda F, f) \to (G_\cdot, d)$ which extends the identity map $1: R/J \to R/J$ such that the map induced by φ^* gives the following isomorphism:

$$\operatorname{Ext}^n_R(R/J, R) \cong H^n(G_\cdot^*, d^*) \xrightarrow{\varphi^n} H^n(\Lambda F^*, f^*) \cong H_{m-n}(f, R)$$

Proof The complex $(G_\cdot, d)$ is a projective resolution of R/J. Hence there exists a map of complexes $\varphi_\cdot : (\Lambda F, f) \to (G_\cdot, d)$ which extends the identity map $1: R/J \to R/J$. There exists an R-sequence $a_1, \dots, a_n$ in J, since grade $J = n$. Let $A = R^n$ and let the map $a: A \to R$ be defined by the matrix $(a_1, \dots, a_n)$. Then $(\Lambda A, a)$ is a free resolution of $R/(a)$ and $(\Lambda A \otimes G_\cdot, a \otimes d)$ is a resolution of $R/(a) \otimes R/J \cong R/J$. Let $\pi: (\Lambda A \otimes G_\cdot, a \otimes d) \to (G_\cdot, d)$ be a map of complexes which extends the isomorphism $R/(a) \otimes R/J \to R/J$. Let $i: (\Lambda F, f) \to (\Lambda A \otimes \Lambda F, a \otimes f)$ be the map of complexes defined by $i(v) = 1 \otimes v$ where 1 is the unit in $R = \Lambda^0 A$. Then $\pi_\cdot \circ (1 \otimes \varphi_\cdot) \circ i_\cdot$ and $\varphi_\cdot$ are both maps of complexes from $(\Lambda F_\cdot, f)$ to $(G_\cdot, d)$ which extend the identity map $1: R/J \to R/J$. Hence both maps are homotopy equivalent. So we have a commutative diagram

$$\begin{array}{ccc} H^n((\Lambda A \otimes G_\cdot)^*) & \xrightarrow{(1\otimes\varphi)^n} & H^n((\Lambda A \otimes \Lambda F)^*) \\ \uparrow{\scriptstyle \pi^n} & & \downarrow{\scriptstyle i^n} \\ H^n(G_\cdot^*) & \xrightarrow{\varphi^n} & H^n(\Lambda F^*) \end{array}$$

The map π^n is an isomorphism, since $\Lambda A \otimes G_\cdot$ and $G_\cdot$ are both projective resolutions of R/J. The map i^n is an isomorphism since $a_1, \dots, a_n$ is an R-sequence in J. Thus to prove that φ^n is an isomorphism it is enough to prove it for $(1 \otimes \varphi)^n$ and this follows from a spectral sequence argument. Consider the double complex $E_0^{pq} = (\Lambda^p A \otimes \Lambda^q F)^*$. The E_1 term is $E_1^{pq} = H^p((\Lambda A \otimes \Lambda^q F)^*)$ which is zero unless $p = n$, since $a_1, \dots, a_n$ is an R-sequence. Thus the spectral sequence degenerates at E_2 and $E_2^{n0} \cong \ker(R/(a) \otimes \Lambda^0 F^* \to R/(a) \otimes \Lambda^1 F^*) \cong \operatorname{Hom}_R(R/J, R/(a))$. In the same way the double complex $\bar{E}_0^{pq} = (\Lambda^p A \otimes G_q)^*$ has a spectral sequence $\{\bar{E}_r\}$ which degenerates at $\bar{E}_2$ and with $\bar{E}_1$ terms $\bar{E}_1^{pq} = 0$ unless $p = n$. Thus $\bar{E}_2^{n0} \cong \ker(R/(a) \otimes G_0^* \to R/(a) \otimes G_1^*) \cong \operatorname{Hom}_R(R/J, R/(a))$. So the isomorphism between E_2 and $\bar{E}_2$ is induced by $(1 \otimes \varphi_\cdot)^*$

Hence $(1\otimes\varphi)^n : H^n((\Lambda A\otimes G.)^*) \longrightarrow H^n((\Lambda A\otimes \Lambda F)^*)$ is an isomorphism. Moreover $\mathrm{Ext}^n_R(R/J, R) \cong H^n(G.^*)$ and $H_{m-n}(f, R) \cong H^n(\Lambda F^*)$. This proves the lemma.

Construction (4.2) Let R be a ring. Let I and J be two ideals in R. Suppose I is a perfect ideal of grade n and let J be generated by $n+1$ elements $f_1, \dots, f_{n+1}$ in I. Suppose $\mathrm{grade}(I/J) \geq n+1$. Then grade $J = n$. There is a finite projective resolution $(G., d)$ of R/I as R-module, of length n, since I is perfect of grade n. Let $F = R^{n+1}$ and let $f : F \to R$ be the map given by the components $f_1, \dots, f_{n+1}$. Since $\mathrm{grade}(I/J) \geq n+1$, we have that the map $\mathrm{Ext}^n_R(R/I, R) \to \mathrm{Ext}^n_R(R/J, R)$ induced by the map $R/J \to R/I$, is an isomorphism. Let the map of complexes $\varphi. : (\Lambda F, f) \to (G., d)$ be an extension of the quotient map $R/J \to R/I$. From lemma (4.1) and the remarks above we obtain that the map $\varphi^n : H^n(G.^*, d^*) \longrightarrow H^n(\Lambda F^*, f^*)$ is an isomorphism. Consider the following diagram:

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & G_1^* \otimes \Lambda^{n+1} F & \xrightarrow{\psi_n} & \Lambda^n F & \xrightarrow{\varphi_n} & G_n \\
 & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & G_{n-q}^* \otimes \Lambda^{n+1} F & \xrightarrow{\psi_{q+1}} & \Lambda^{q+1} F & \xrightarrow{\varphi_{q+1}} & G_{q+1} \\
 & {\scriptstyle (-1)^{n-q} d_{n-q}^* \otimes 1} \downarrow & & \downarrow {\scriptstyle f} & & \downarrow {\scriptstyle d_q} \\
0 \to & G_{n-q+1}^* \otimes \Lambda^{n+1} F & \xrightarrow{\psi_q} & \Lambda^q F & \xrightarrow{\varphi_q} & G_q \\
 & \vdots & & \downarrow & & \downarrow \\
 & & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & G_n^* \otimes \Lambda^{n+1} F & \xrightarrow{\psi_1} & F & \xrightarrow{\varphi_1} & G
\end{array}
\qquad (*)
$$

The right-hand column is the complex $(G_\cdot, d)$ with G_0 deleted. The column in the middle is the complex $(\Lambda F, f)$ with $\Lambda^0 F$ and $\Lambda^{n+1}F$ deleted. The left-hand column is, up to signs, the dual (G^*, d^*) of the complex (G, d) tensored with $\Lambda^{n+1}F$ and with $G_0^* \otimes \Lambda^{n+1}F$ deleted. The map between the middle and the right-hand column is given by the map of complexes $\varphi_\cdot$ with φ_0 and φ_{n+1} deleted. The maps $\{\psi_q\}$ between the left-hand and middle column is given as follows. The dual of φ_q is the map $\varphi_q^* : G_q^* \longrightarrow \Lambda^q F^*$. Let $\psi_q : G^*_{n-q+1} \otimes \Lambda^{n+1}F \longrightarrow \Lambda^q F$ be given by $\psi_q(\gamma \otimes v) = (\varphi^*_{n-q+1}(\gamma))(v)$ for $\gamma \in G^*_{n-q+1}$ and $v \in \Lambda^{n+1}F$. Remember from §2 that $\varphi^*_{n-q+1}(\gamma)$ is an element of $\Lambda^{n-q+1}F^*$ and acts on ΛF.

Proposition (4.3) Let R be a ring. Suppose I is a perfect ideal of grade n in R. Let J be an ideal in R generated by $f_1, \dots, f_{n+1}$ in I. Then the above diagram $(*)$ is commutative. Let $k_\cdot$ be the associated complex of $(*)$. If $\operatorname{grade}(I/J) \geq n+1$ then $k_\cdot$ is a projective resolution of I/J and I/J is a perfect R-module of grade $n+1$ and $\operatorname{pd}(R/J) = n+1$.

Proof. The right-hand squares of $(*)$ are commutative by the construction of $\varphi_\cdot$. The commutativity of the left-hand squares of $(*)$ follows from the commutativity of the diagram:

$$\begin{tikzcd}
G^*_{n-q} \otimes \Lambda^{n+1}F \arrow[r, "\varphi^*_{n-q}\otimes 1"] \arrow[d, "(-1)^{n-q} d^*_{n-q} \otimes 1"'] & \Lambda^{n-q}F^* \otimes \Lambda^{n+1}F \arrow[r, "m_{q+1}"] \arrow[d, "(-1)^{n-q} f^* \otimes 1"'] & \Lambda^{q+1}F \arrow[d, "f"] \\
G^*_{n-q+1} \otimes \Lambda^{n+1}F \arrow[r, "\varphi^*_{n-q+1}\otimes 1"] & \Lambda^{n-q+1}F^* \otimes \Lambda^{n+1}F \arrow[r, "m_q"] & \Lambda^q F
\end{tikzcd}$$

The left-hand square is commutative, since it is the dual of a commutative square. The right-hand square is commutative, since f is an element of $F^* \subseteq \Lambda F^*$ and acts on ΛF, thus f^* acts on $(\Lambda F)^* \cong \Lambda F^*$. But f also acts on ΛF^* by left multiplication. Both actions of f are the same up to a sign, see [B-E] (1.4).

...on. What we have proved is that the action of $'d^*$ and d on ΛF^* differ by a sign. (see also [B-E]. Cor. 1.4.).

Next we prove the exactness of the complex $K.$ associated with the diagram $(*)$. First, extend the diagram $(*)$, by adjoining $G_0^* \otimes \Lambda^{n+1} F$ in the left upper corner, $\Lambda^{n+1}F$ and $\Lambda^0 F$ in the middle column and G_0 in the right bottom corner, together with the appropriate maps ψ_{n+1} and φ_0. Call this diagram $(**)$ and its associated complex $\tilde{K}.$ We may suppose that $G_0 = R$ and thus φ_0 and ψ_{n+1} are isomorphisms. Hence $K.$ and $\tilde{K}.$ have the same homology. The right-hand row of the diagram $(**)$ has zero homology exept at the place G_0, where its homology is R/I, since $G.$ is a resolution of R/I. The middle row of $(**)$ is the koszul complex of a sequence of $n+1$ generators of the ideal J of grade n. Hence this row has only non-zero homology at the places $\Lambda^1 F$ and $\Lambda^0 F$, which are $H_1(\Lambda F, f)$ and R/J. The left-hand column has only non-zero homology at the place $G_n^* \otimes \Lambda^{n+1} F$. Since this row is the dual of $G.^*$, up to signs and tensoring with a trivial factor, and since $G.$ is a resolution of a perfect R-module of grade n i.e. $H^i(G.^*) \cong \mathrm{Ext}_R^i(R/I, R) = 0$ for all $i<n$.

The cohomology $H^n(G.^* \otimes \Lambda^{n+1}F)$ is isomorphic with $H_1(\Lambda F, f)$ by the isomorphism induced by ψ_1, by the remarks in the beginning of (4.2). What we have computed is the E^1 term of the filtered complex $\tilde{K}.$

$$E^1_{p,q} \cong \begin{cases} R/I & \text{if } p=0 \text{ and } q=0 \\ R/J & \text{if } p=1 \text{ and } q=0 \\ H_1(\Lambda F, f) & \text{if } p=1 \text{ and } q=1. \\ H^n(G.^* \otimes \Lambda^{n+1}F) & \text{if } p=2 \text{ and } q=1 \end{cases}$$

And $E^1_{2,1} \xrightarrow{d^1} E^1_{1,1}$ is an isomosphism. Hence the E^2 term has only zeros exept at one place: $E^2_{1,0} \cong \ker(R/J \to R/I) \cong I/J$.

Thus $K.$ is exact and is a resolution of I/J of length $n+1$.

From $n+1 \leq \mathrm{grade}(I/J) \leq \mathrm{pd}(I/J) \leq n+1$ one concludes that I/J is perfect of grade $n+1$

From the exactness of the sequence $0 \to I/J \to R/J \to R/I \to 0$ one concludes that $pd(R/J) = n+1$, since $pd(R/I) = n$ and $pd(I/J) = n+1$.

Let R be a ring. Let I be an ideal in R generated by an R-sequence $g_1, \dots, g_n$. Let J be an ideal generated by $n+1$ elements $f_1, \dots, f_{n+1}$. Suppose $f_i = \sum_j \varphi_{ij} g_j$. Let φ be the $(n+1) \times n$-matrix with entries φ_{ij}. Let φ^i be the matrix obtained from φ by deleting the i^{th} row. Let $\Delta_i = (-1)^i \det \varphi^i$.

Proposition (4.4) The following holds in the situation above.

(i) $\sum_i \Delta_i f_i = 0$

(ii) if $\operatorname{grade}(I/J) \geq n+1$ and $f_1, \dots, f_n$ is an R-sequence then $((f_1, \dots, f_n) : f_{n+1}) = (f_1, \dots, f_n, \Delta_{n+1})R$.

Proof We have seen the diagram (*) of this section, up to signs, already in §3 on page , in the case that I is generated by an R-sequence. The map ψ_1 has matrix $(\Delta_1, \dots, \Delta_{n+1})$. Thus $\sum_i \Delta_i f_i = 0$. Let f be the sequence $(f_1, \dots, f_{n+1})$. If $\operatorname{grade}(I/J) \geq n+1$ and $f_1, \dots, f_n$ is an R-sequence then $\operatorname{grade}(f_1, \dots, f_n)R = n$ so

$H_1(f, R) \cong \operatorname{Hom}_R(R/(f_1, \dots, f_n)R, R/(f_1, \dots, f_{n+1})R) \cong ((f_1, \dots, f_n) : f_{n+1})/(f_1, \dots, f_n)R$

by [R] (3.1). Moreover the map ψ_2 induces an isomorphism $H^n((\Lambda G, g)^*) \longrightarrow H_1(f, R)$, where $(\Lambda G, g)$ is the Koszul complex of the sequence $g = (g_1, \dots, g_n)$. Further $R/I \cong \operatorname{Ext}_R^n(R/I, R) \cong H^n((\Lambda G, g)^*)$, since I is generated by the R-sequence $g_1, \dots, g_n$. Hence R/I is isomorphic with $((f_1, \dots, f_n) : f_{n+1})/(f_1, \dots, f_n)R$ by the isomorphism $r + I \longmapsto r\Delta_{n+1} + (f_1, \dots, f_n)R$. Thus $((f_1, \dots, f_n) : f_{n+1}) = (f_1, \dots, f_n, \Delta_{n+1})$. This proves the proposition.

Proposition (4.5) Let R be a local ring and I an ideal in R such that $pd(R/I) = 2$. Let J be an ideal in R generated by m elements. Suppose $J \subseteq I$ and $\operatorname{grade}(I/J) \geq m$. Then I/J is a perfect R-module of grade m and $pd(R/J) \leq m$

Proof I is an ideal in the ring R such that $pd(R/I) = 2$
We have a free resolution of length 2

$$0 \to R^{n-1} \xrightarrow{\beta} R^n \xrightarrow{g} R \to R/I \to 0$$

since R is a local ring. the ideal J is generated by m elements $f_1, \dots, f_m$ in I so $f_i = \sum_j \varphi_{ij} g_j$. Let φ be the map with matrix (φ_{ij}). We have the following commutative exact diagram

$$\begin{tikzcd}
 & & R^{n-1} \arrow[d, "\beta"] & & \\
 & R^m \arrow[r, "\varphi"] \arrow[d] & R^n \arrow[d, "g"] & & \\
0 \arrow[r] & J \arrow[r] & I \arrow[r] & I/J \arrow[r] & 0
\end{tikzcd}$$

Hence we get a presentation

$$R^m \oplus R^{n-1} \xrightarrow{\psi} R^n \to I/J \to 0$$

where $\psi = \varphi + \beta$. Further $\mathrm{rad}(J:I) = \mathrm{rad}\, I_n(\psi)$, so grade $I_n(\psi) \geq m$, since grade$(I/J) \geq m$. The matrix ψ has size $(m+n-1) \times n$ and $m = (m+n-1) - n + 1$. So I/J is a perfect R-module of grade m, by [E-N]. Consider the exact sequence $0 \to I/J \to R/J \to R/I \to 0$ we have that $pd(R/I) = 2$ and $pd(I/J) = m$, hence $pd(R/J) \leq \max\{2, m\}$, so $pd(R/J) \leq m$ in case $m \geq 2$. Now suppose $m = 1$, then there exists an exact sequence $0 \to R^p \to R \to R \to R/J \to 0$. Looking at the ranks one concludes that $p = 0$, so $pd(R/J) = 1$
This proves the proposition.

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Let I and J be ideals in the ring R and $J \subseteq I$. Suppose I is generated by $g_1, \dots, g_n$ and let J be generated by $f_1, \dots, f_m$. Then $f_i = \sum_j \varphi_{ij} g_j$. Define the ideal $J(g,\varphi) := J + I_n(\varphi)$. as in (3.1). Take $I_n(\varphi) = (0)$ in case $m < n$. We have the following lemma.

__Lemma__ (4.5) Let I be an ideal in R generated by an R-sequence $g_1, \dots, g_n$ and suppose $f_i = \sum_j \varphi_{ij} g_j$ and $J = (f_1, \dots, f_m)R$. Then $J(g,\varphi)$ does not depend on the generators of J, nor on the matrix φ, neither on the R-sequence $g_1, \dots, g_n$ generating I, but only on the pair $J \subseteq I$.

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__Proof.__ The proof consists of five steps.

(i) Let $J = (f_1, \dots, f_m)R$ and suppose $f_{m+1} \in J$, then $f_{m+1} = \sum_{i=1}^{m} a_i f_i = \sum_{i=1}^{m} \sum_j a_i \varphi_{ij} g_j$. Define the $(m+1) \times n$ matrix φ^1 by adjoining a new column to φ with entries $\sum_{i=1}^{m} a_i \varphi_{ij}$. Then $I_n(\varphi) = I_n(\varphi^1)$ and we have $J(g,\varphi) = J(g,\varphi^1)$.

(ii) If $J = (f_1, \dots, f_m)R = (h_1, \dots, h_k)R$ then we can repeat the step before and ajoin $h_1, \dots, h_k$ successively to $f_1, \dots, f_m$ and we get $J(g,\varphi) = J(g,\varphi^1) = \dots = J(g,\varphi^k)$. In the same way we get $J(g,\psi) = J(g,\psi^1) = \dots = J(g,\psi^m)$ after adjoining $f_1, \dots, f_m$ successively to $h_1, \dots, h_k$, with $h_i = \sum_j \psi_{ij} g_j$.

(iii) So we may suppose $k = m$. Now let $J = (f_1, \dots, f_m)R = (h_1, \dots, h_m)R$. Then it is enough to prove $J(g,\varphi)R_p = J(g,\psi)R_p$ for every $p \in \operatorname{Spec} R$, that is, we may assume R to be local. We have $h_i = \sum_j \alpha_{ij} f_j$ and $f_i = \sum_j \beta_{ij} h_j$ and also $f_i = \sum_j \varphi_{ij} g_j$ and $h_i = \sum_j \psi_{ij} g_j$. Let α and β be the R-lineair maps from R^m to R^m with matrices (α_{ij}) and (β_{ij}) resp. Let k be the residue field of the local ring R and let α_0 and β_0 the induced maps from k^m to k^m. There is a k-lineair map $\gamma_0 : k^m \to k^m$ such that $\mu_0 := \gamma_0(1 - \alpha_0\beta_0) + \beta_0$ is invertible. This is an exercise in linear algebra, see [G] p. 146. Take an R-linear map $\gamma : R^m \to R^m$ which lifts γ_0 and

define $\mu := \gamma(1-\alpha\beta)+\beta$. Now μ is invertible and $f_i = \sum_j \mu_{ij} h_j$.
Take $\eta_{ik} := \sum_j \mu_{ij}\psi_{jk}$. Then

$$\sum_k \varphi_{ik} g_k = f_i = \sum_j \mu_{ij} h_j = \sum_{j,k}\mu_{ij}\psi_{jk}g_k = \sum_k \eta_{ik} g_k.$$

and $J(g,\psi) = J(g,\eta)$, since μ is invertible. Nieuwe regel (iv) So we may assume $f_i = \sum_j \varphi_{ij} g_j$ and $f_i = h_i = \sum_j \psi_{ij} g_j$. We need to prove $J(g,\varphi) = J(g,\psi)$. Since $g_1,\dots,g_n$ is an R-sequence and $\sum \varphi_{ij} g_j = \sum_j \psi_{ij} g_j$ we can find elements α_{ijk} in R such that $\alpha_{ijk}+\alpha_{ikj}=0$ and $\alpha_{ijj}=0$ and $\psi_{ij} = \varphi_{ij} + \sum_k \alpha_{ijk} g_k$. One gets the matrix (ψ_{ij}) out of (φ_{ij}) by performing the following operation repeatedly

$P(q,r,s;\alpha)$: $\varphi_{ij} \mapsto \varphi_{ij} + \alpha\,\delta_{iq}(\delta_{jr} g_s - \delta_{js} g_r)$, with $1 \le q \le m$ and $1 \le r < s \le n$ and $\alpha \in R$. δ_{ij} is Kronecker's delta. An exercise with determinants shows that the ideal $J(g,\varphi)$ remains the same under the operation $P(q,r,s;\alpha)$.

Thus $J(g,\psi) = J(g,\varphi)$ and $J(g,\varphi)$ does not depend on the chosen generators $f_1,\dots,f_m$ of J and not on the matrix (φ_{ij}).

(v) If $g_1',\dots,g_n'$ is another R-sequence generating I ~~(every R-sequence generating I has the same length)~~, then as before, there is an invertible matrix (γ_{jk}) such that $g_j' = \sum_k \gamma_{jk} g_k$. Let $f_i = \sum_j \varphi_{ij} g_j$ and $f_i = \sum_j \varphi'_{ij} g'_j$ then

$f_i = \sum_{j,k} \varphi'_{ij}\gamma_{jk} g_k = \sum_k \psi_{ik} g_k$, with $\psi_{ik} = \sum \varphi'_{ij}\gamma_{jk}$. So $J(g',\varphi') = J(g,\psi)$ since γ is invertible. And we are in the situation already dealt with. This proves the lemma.

<u>Remark (4.7)</u> The assumption that I is generated by an R-sequence is essential in lemma (4.6) as one sees in the following example. Take $R = k[x,y,z]$ with k a field containing $\frac{1}{2}$. Let $I = (yz, zx, xy)R$ and $J = (x(y^2+z^2), y(x^2+z^2), z(x^2+y^2))R$. Define the matrices φ and ψ by

$$\varphi = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} x & z-y & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$$

Then $J(g,\varphi) = (xyz)R + J \subseteq I$ and $J(g,\psi) \equiv -x^3 \pmod{I}$. So $J(g,\varphi) \neq J(g,\psi)$.

<u>Lemma</u> (4.8) Let R be a ring containing two ideals I and J. Suppose I contains J and is generated by an R-sequence. Then $\underline{V}(J(g,\varphi)) = \underline{V}(J:I)$.

<u>Proof</u> The ideal I is generated by an R-sequence $g_1,\dots,g_n$. Let J be generated by $f_1,\dots,f_m$. We may assume $m \geq n$ after adjoining zero's. Then $f_i = \sum_j \varphi_{ij} g_j$, since $J \subseteq I$. Theorem (3.1) gives $J(g,\varphi) \subseteq (J:I)$. So $\underline{V}(J:I) \subseteq \underline{V}(J(g,\varphi))$ and $I_n(\varphi).I \subseteq J$. One always has $\underline{V}(J) = \underline{V}(I) \cup \underline{V}(J:I)$. One easily verifies that $\underline{V}(J) = \underline{V}(I) \cup \underline{V}(J(g,\varphi))$. Now we want to prove $\underline{V}(J(g,\varphi)) \subseteq \underline{V}(J:I)$. Suppose $\underline{p} \notin \underline{V}(J:I)$. If $\underline{p} \notin \underline{V}(I)$ then $\underline{p} \notin \underline{V}(J(g,\varphi))$ by the equalities above. If $\underline{p} \in \underline{V}(I)$ then $I \subseteq \underline{p}$ and $IR_{\underline{p}}$ is generated by an $R_{\underline{p}}$-sequence. Further $JR_{\underline{p}} = IR_{\underline{p}}$, since $\underline{p} \notin \underline{V}(J:I)$. Thus $J(g,\varphi)R_{\underline{p}} = J(g,1)R_{\underline{p}} = R_{\underline{p}}$, by lemma (4.6). So $\underline{p} \notin \underline{V}(J(g,\varphi))$. Hence we have proved $\underline{V}(J(g,\varphi)) = \underline{V}(J:I)$.

<u>Proposition</u> (4.9) Suppose I is an ideal in the ring R generated by an R-sequence. Let J be an ideal in R generated by m elements. Suppose $J \subseteq I$ and grade$(I/J) \geq m \geq n$. Then I/J is a perfect R-module of grade m and pd$(R/J) \leq m$.

<u>Remark</u> (4.10) Propositions (4.5) and (4.9) do not assume R to be Cohen-Macaulay as theorem (1.26) does. The conclusions of (4.5) and (4.9) are stronger than (1.26) a) and c). Remember that height k = grade k and $\mathrm{dp}\, M = \mathrm{Kdim}\, R - \mathrm{pd}\, M$ for a Cohen-Macaulay ring R and an ideal k in R and a finitely generated R-module M of finite projective dimension. Further if $J = I \cap k$ for some ideal k in R then grade $k \geq m$ then grade$(I/J) \geq m$, since $k \subseteq (J:I)$.

<u>Proof</u> of (4.9). The ideal I is generated by an R-sequence $g_1,\dots,g_n$ and J is generated by m elements $f_1,\dots,f_m$ in I. We can write $f_i = \sum_j \varphi_{ij} g_j$. Lemma (4.6) implies $\mathrm{rad}(J:I) = \mathrm{rad}\, J(g,\varphi)$. By assumption is grade$(I/J) \geq m$, hence grade $J(g,\varphi) \geq m$. Now $m \geq n$, so we can apply theorem (3.1) and we get a

finite free resolution of I/J of length m. Hence $m \leq \text{grade}(I/J) \leq \text{pd}(I/J) \leq m$. So I/J is a perfect R-module of grade m. From the exact sequence $0 \to I/J \to R/J \to R/I$ and $\text{pd}(R/I) = n \leq m = \text{pd}(I/J)$ we derive $\text{pd}(R/J) \leq m$. This proves the proposition.

<u>Remark</u> (4.11) We can give a resolution of the non-perfect ideal J in the situation of proposition (4.9) in the following way. Delete the right-hand column in the double complex (**) on page of section 3. and call its associated complex $\bar{L}.(g,\varphi)$. This complex is a resolution of the ideal J under the assumptions of proposition (4.9). Let $L.(g,\varphi) = (\bar{L}.(g,\varphi) \xrightarrow{f} R)$, then $L.(g,\varphi)$ is a resolution of R/J of length m.

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Ik ben op 30 maart 1953 geboren te Djakarta in Indonesië. Het lager onderwijs heb ik gevolg in Curaçao en in Den Haag. De middelbare school bezocht ik te Den Haag en Amsterdam. Het eindexamen H.B.S.-B heb ik in 1970 afgelegd op het Ir. Lely lyceum te Amsterdam. Mijn wiskunde leraar was J. Donkers. Aan de Universiteit van Amsterdam heb ik eerst natuurkunde gestudeerd en ben later met wiskunde verder gegaan. Mijn kandidaats examen wiskunde behaalde ik in 1974. Ik was kandidaats assistent van 1975 tot 1977. Bij prof. dr. F. Oort heb ik seminaria in de algebraïsche meetkunde gevolgd. Mijn doctoraal examen behaalde ik in maart 1981 onder leiding van prof. dr. W.T. van Est in de algebraïsche topologie. Van april 1981 tot april 1985 was ik als wetenschappelijk medewerker verbonden aan het landelijke Z.W.O. Singularitieten project onder leiding van prof. dr. E. Looijenga, prof. dr. D. Siersma en prof. dr. J.H.M. Steenbrink. Van april tot en met december 1985 werk ik op het Mathematisch Instituut te Utrecht.