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Projective resolutions of the quotient of two ideals

by Ruud Pellikaan

Department of Mathematics and Computer Sciences, Free University, P.O. Box 7161, 1007 MC Amsterdam, the Netherlands

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INTRODUCTION

Consider two ideals I and J, with $J \subset I$, in a commutative Noetherian ring R with unit. We are concerned with constructing finite projective resultions of I/J.

Suppose J is generated by m elements and grade $(I/J) \geq m$ and suppose either I is generated by an R-sequence of length $n \leq m$ or R/I has projective dimension 2 or I is a perfect ideal of grade $m-1$. Then I/J is a perfect R-module of grade m.

This problem arises for instance if one studies germs of analytic functions $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with a one dimensional singular locus. Let J_f be the Jacobi ideal of f, i.e.

$$J_f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right) \text{ and } I = \text{rad } (I_f).$$

If $\dim_{\mathbb{C}}(I/J_f) < \infty$ then this number can be interpreted as the number of A_1 and D_∞ singularities in a generic deformation of f, using the results of this paper, see [11].

Section 1 gives a review of some multilinear algebra we need. In section 2 we construct a complex $\mathbb{K}(g, \phi)$ associated with a sequence g of n elements and a $(m \times n)$-matrix ϕ, in case $m \geq n$. This complex is acyclic in case the grade of the ideal $(\Sigma\, \phi_{ij} g_j)R + I_n(\phi)$ is at least m. In order to proof this result the bialgebra approach of Buchsbaum and Eisenbud [3] is used. The proof of Theorem 2.1

is analogous to Theorem 5.1 of [3]. In section 3 projective resolutions are constructed of I/J in the mentioned cases. In section 4 a length formula is proved. Section 5 contains applications. For basic definitions and results concerning commutative and homological algebra we refer to Matsumura [9] and Serre [14].

1. A REVIEW OF SOME MULTILINEAR ALGEBRA

We give a review of some multilinear algebra concerning the exterior and symmetric algebra's. For unproven assertions we refer to Buchsbaum and Eisenbud [3]. In this section R will be a Noetherian commutative ring with unit element.

If F is a free R-module we denote by ΛF the *exterior algebra* on F. It is the free graded commutative R-algebra, generated by elements of F in degree 1. The graded commutative law is:

$$f\cdot g=(-1)^{pq}g\cdot f \text{ where } p=\deg f \text{ and } q=\deg g,$$

$$f^2=0 \text{ if } \deg f \text{ is odd.}$$

The diagonal map $\Delta : F\rightarrow F\oplus F$ induces an algebra map $\Delta : \Lambda F\rightarrow \Lambda F\otimes \Lambda F$. If $f\in \Lambda F$ is of degree 1 then $\Delta(f)=f\otimes 1+1\otimes f$. The elements of degree 0 of ΛF form a ring isomorphic to R and projection into degree 0 is an algebra map $\varepsilon : \Lambda F\rightarrow R$. It is called the co-unit. ε and Δ satisfy a set of identities dual to those of the unit $\eta : R\rightarrow \Lambda F$ and multiplication $m : \Lambda F\otimes \Lambda F\rightarrow \Lambda F$. Thus F becomes a graded commutative, cocommutative bialgebra.

For an R-module M we will write $M^*=\mathrm{Hom}_R(M, R)$. Let $\mu\in M^*$ and $m\in M$, define $\langle \mu, m\rangle=\mu(m)$.

ΛF is a bialgebra, hence ΛF^* is a bialgebra too. The map $F^*\rightarrow(\Lambda F)^*$, dual to the projection $\Lambda F\rightarrow F$, induces a natural algebra map $\alpha : \Lambda F^*\rightarrow(\Lambda F)^*$, which is a map of bialgebra's. α is an isomorphism if F is a finitely generated free R-module. We shall identify ΛF^* with $(\Lambda F)^*$ via α from now on

SF is the *symmetric algebra*, which we will regard as the free graded commutative algebra generated by its elements of *degree* 2. $SF=\Sigma\, S_iF$. We will give the elements of S_iF the degree $2i$. We identify S_0F with R and S_1F with F. If F is an R-module generated by a basis $x_1, \ldots, x_n$ then SF is isomorphic to the polynomial ring $R[x_1, \ldots, x_n]$.

The diagonal map $\Delta : F\rightarrow F\oplus F$ induces an algebra map $\Delta : SF\rightarrow S(F\oplus F)\simeq \simeq SF\otimes SF$, with $\Delta(f)=f\otimes 1+1\otimes f$ for elements $f\in S_1F$. The projection onto $S_0F=R$ gives an algebra map $\varepsilon : SF\rightarrow R$. This makes SF into a graded commutative, cocommutative bialgebra. Hence $(SF)^*$ is an algebra with multiplication Δ^*.

Since SF is an infinite direct sum, we will work with the graded dual $(SF)^*_{gr}=\Sigma\,(S_iF)^*$, which is a subalgebra of $(SF)^*$.

The map $F^*\rightarrow(SF)^*_{gr}$ induces an algebra map $\alpha' : S(F^*)\rightarrow(SF)^*_{gr}$. But it is not an isomorphism, unless R contains the field of rationals. One has for instance for $\phi\in F^*$ and $f\in F$

$$\langle \alpha'(\phi^p), f^p\rangle=p!\langle \phi, f\rangle^p.$$

DF is the *divided power algebra* on F. It is the graded commutative algebra generated by elements $f^{(p)}$ of degree $2p$ called the p^{th} divided powers of f, where $f \in F$ is regarded as an element of degree 2 in DF. These divided powers satisfy certain conditions, see [3].

DF is a bialgebra and if F is a free R-module with basis $x_1, \ldots, x_n$ then D_pF is free on generators $\Pi x_i^{(p_i)}$ with $\Sigma p_i = p$. We can define a pairing

$$\mu : D(F^*) \otimes (SF) \to R$$

$$\mu(\phi^{(p)}, \Pi f_i^{p_i}) = \begin{cases} 0 & \text{if } \Sigma p_i \neq p. \\ \Pi \phi(f_i)^{p_i} & \text{if } \Sigma p_i = p. \end{cases}$$

If F is a free R-module with basis $x_1, \ldots, x_n$ and if $x_1^*, \ldots, x_n^*$ is the dual basis of F^* then

$$\alpha'(x_i^{*(p)})(x_i^p) = 1.$$

So μ is a perfect pairing in this case and induces isomorphisms $D(F^*) \to$ $\to (SF)_{gr}^*$ and $SF \to (D(F^*))_{gr}^*$ of bialgebras.

Hence SF is isomorphic to $(S(SF)_{gr}^*)_{gr}^*$ as bialgebra.

We can view ΛF as a ΛF^*-module and vice versa and we can consider $D(F^*)$ and SF as modules over each other. In general if $\alpha : B \to A_{gr}^*$ is a homogeneous bialgebra homomorphism. We define

$$s(\Delta) : A_{gr}^* \otimes A \to A$$

by $s(\Delta)(\gamma \otimes f) = \gamma(\Delta(f) \cdot \Delta(f)) \in A$, for $\gamma \in A_{gr}^*$ and $f \in A$, and define

$$n : B \otimes A \to A$$

by

$$n = s(\Delta)(\alpha \otimes 1).$$

We shall write $b(a)$ or ba for $n(b \otimes a)$.

We have $b(a) = \Sigma \langle \alpha(b), a_1^i \rangle\, a_2^i$ if $\Delta(a) = \Sigma\, a_1^i \otimes a_2^i$.

The map n makes A into a B-module.

The map $\alpha : B \to A_{gr}^*$ gives rise to an algebra map

$$A \to (A_{gr}^*)_{gr}^* \to B_{gr}^*.$$

Thus B is also an A-module.

From now on F will be a finitely generated free R-module. Then SF is a graded SF-module and F is a graded ΛF^*-module. We may regard $SF \otimes \Lambda F$ as a bigraded $SF \otimes \Lambda F^*$-module. The identity map $1 : F \to F$ gives, by the identification Hom $(F, F) = F \otimes F^*$ an element $c = c_F$ of $F \otimes F^* = S_1 F \otimes \Lambda^1 F$. We shall write

$$\partial_F : SF \otimes \Lambda F \to SF \otimes \Lambda F$$

for the $SF \otimes \Lambda F$-module map, given by multiplication by c. It is a map of bidegree $(2, -1)$, hence $\partial_F^2 = 0$.

Let $LF = \text{Ker } \partial_F$ then LF is a bigraded $SF \otimes \Lambda F^*$-module and its bihomogeneous components are $L_p^q F$, where

$$\partial_p^q : S_{p-1}F \otimes \Lambda^q F \to S_p F \otimes \Lambda^{q-1} F$$

$$L_p^q F = \text{Ker } \partial_{p+1}^{q-1}.$$

The usefullness of the bialgebra approach lies for instance in the fact that the map ∂_F is defined without referring to any basis of F and that the property $\partial_F^2 = 0$ is a direct consequence of the degree of the map. Although this approach is rather abstract one must recognize that the bialgebra $SF \otimes \Lambda F$ with the map ∂_F is nothing but the generic Koszul complex, i.e. let $x_1, \ldots, x_n$ be a basis of F then we have already noted that $SF \simeq R[x_1, \ldots, x_n]$. After this identification $S_{p-1}F \otimes \Lambda^q F$ has elements $b \otimes (x_{i_1} \wedge \ldots \wedge x_{i_q})$ with b a homogeneous polynomial of degree $p-1$ in $R[x_1, \ldots, x_n]$. The map $\partial_p^q : S_{p-1}F \otimes \otimes \Lambda^q F \to S_p F \otimes \Lambda^{q-1} F$ sends $b \otimes (x_{i_1} \wedge \ldots \wedge x_{i_q})$ to

$$\Sigma\,(-1)^{t+1} b x_{i_t} \otimes (x_{i_1} \wedge \ldots \wedge x_{i_{t-1}} \wedge x_{i_{t+1}} \wedge \ldots \wedge x_{i_q}).$$

PROPOSITION 1.1. Let F be a finitely generated free R-module of rank n. Then

a) $L_p^q F = \text{Ker } \partial_{p+1}^{q-1} = \text{Im } \partial_p^q = \text{Coker } \partial_{p-1}^{q+1}$ if $p+q \neq 1$.
b) $L_p^1 F = S_p F$ for all p, $L_1^q F = \Lambda^q F$ for all $q \neq 0$, $L_p^0 F = L_0^q F = 0$ for all $q \neq 1$ and all p, $L_p^q F = 0$ for all $q > n$.
c) $L_p^n F \simeq S_{p-1} F \otimes \Lambda^n F$.
d) $L_p^q F$ is free of rank $\binom{n+p-1}{p+q-1}\binom{q+p-2}{p-1}$.

Given a map between finitely generated free R-modules $\phi : F \to G$, there is an induced map $L\phi : LF \to LG$ with bihomogeneous components

$$L_p^q \phi : L_p^q F \to L_p^q G.$$

Suppose F and G are free R-modules of rank m and n respectively and $m \geq n$. Define for every pair (p, q), $q \geq 1$ a complex

$$\mathbb{L}_p^q(\phi) : 0 \to L_p^m F \otimes L_{m-n}^{n-q+1} G^* \to L_p^{m-1} F \otimes L_{m-n-1}^{n-q+1} G^* \to \ldots$$

$$\ldots \to L_p^{n+1} F \otimes L_1^{n-q+1} G^* \to L_p^q F \to L_p^q G.$$

Here and in what follows $L_s^t G^*$ means $(L_s^t G)^*$, thus for example $L_s^1 G^* = (S_s G)^* \simeq D_s(G^*)$, the s^{th} component of the divided power algebra on G^*. For graded algebra's A we shall from now on delete the sub-script gr in A_{gr}^* and denote it by A^*.

Note that the complexes $\mathbb{L}_p^q(\phi)$ have length $m-n+1$. The map d is given as follows. Since LF is an $SF \otimes \Lambda F^*$-module, we may consider it by the canonical map $\Lambda F^* \simeq R \otimes \Lambda F^* \subset SF \otimes \Lambda F^*$ as a ΛF^*-module. Similarly $LG^* = \text{Hom}_{gr}(LG, R)$ is an $SG \otimes \Lambda G^*$-module that we can consider as an SG-module. The element $\phi \in \text{Hom}(F, G)$ corresponds to an element c_ϕ of bidegree

(1,2) in $\Lambda F^*\otimes SG$, since we have the isomorphisms Hom $(F, G)\simeq F^*\otimes G\simeq$ $\simeq\Lambda^1F^*\otimes S_1G\subset\Lambda F^*\otimes SG$. The element c_ϕ has odd degree, so $c_\phi^2=0$. Thus multiplication by c_ϕ induces a differential on the quadruply graded module $LF\otimes LG^*$, which we call d. The maps we have labelled d in the complex $\mathbb{L}_p^q(\phi)$ are homogeneous components of this d. To define the map $d_1: L_p^{n+1}F\otimes$ $\otimes L_1^{n-q+1}G^*\to L_p^qF$, note first that LF is a ΛG^*-module via the map $\Lambda\phi^*:\Lambda G^*\to\Lambda F^*$. We also have $L_1^{n-q+1}G^*=\Lambda^{n-q+1}G^*$. We define d_1 to be the structure map of the ΛG^*-module LF.

LEMMA 1.2. In the above situation we have

$$d^2=0,\ d_1d=0 \text{ and } (L_p^q\phi)d_1=0.$$

So $\mathbb{L}_p^q(\phi)$ is a complex.

THEOREM 1.3. Let R be a Noetherian ring and suppose that $\phi: F\to G$ is an R-linear map between free R-modules of rank m and n resp. with $m\geq n$. If grade $I_n(\phi)=m-n+1$ then $\mathbb{L}_p^q(\phi)$ is a free resolution of Coker $(L_p^q\phi: L_p^qF\to L_p^qG)$. If moreover $(R, \mathfrak{m})$ is a local ring with maximal ideal $\mathfrak{m}$ and $\phi(F)\subset\mathfrak{m}G$, then $\mathbb{L}_p^q(\phi)$ is a minimal resolution.

REMARK. $I_n(\phi)$ is the ideal generated by the $(n\times n)$-minors of a matrix of ϕ. Under the above assumptions we have: in case $p=1: \mathbb{L}_1^q(\phi)$ is a resolution of Coker $(\Lambda^q\phi)$ for all q, and for $q=1: \mathbb{L}_p^1(\phi)$ is a resolution of Coker $(S_p\phi)$ for all p.

2. THE COMPLEX $\mathbb{K}(\beta, \phi)$

Let R be a ring. Let I and J be two ideals in R. Suppose J is generated by m elements $\alpha_1, \ldots, \alpha_m$ in I and suppose I is generated by $\beta_1, \ldots, \beta_n$, then we can write

$$\alpha_i=\Sigma\ \phi_{ij}\beta_j.$$

Let $F=R^m$ and $G=R^n$ and let $\alpha: F\to R$ and $\beta: G\to R$ be maps with components $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ resp. Then we have the following commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\ \phi\ } & G \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ R & = & R \end{array}$$

where ϕ is the map with matrix (ϕ_{ij}).

We shall construct a complex $\mathbb{K}(\beta, \phi)$ in case $m\geq n$, which is a resolution of I/J when $\beta_1, \ldots, \beta_n$ is an R-sequence and if moreover grade $(J+I_n(\phi))\geq m$. Buchsbaum and Eisenbud [3] constructed a complex $\mathbb{K}(\phi, a)$ associated with an

$(m\times n)$-matrix ϕ and a $(m\times 1)$-sequence a in case $m\geq n$, which is a resolution of $R/J(\phi,a)$, where $J(\phi,a)=(b_1,\dots,b_n)+I_n(\phi)$ with

$$b_j=\Sigma\, a_i\phi_{ij}$$

generalizing a result of Herzog [7] who considered the cases $m=n$ and $m=n+1$ only.

Their situation corresponds to the commutative diagram

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ F & \xrightarrow{\phi} & G \end{array}$$

which is in a sense "half dual" to the former diagram.

Suppose we are in the firsft mentioned situation, i.e. $\alpha\in F^*$, $\beta\in G^*$ and $\phi^*(\beta)=\alpha$. Then the following diagram commutes:

$$(*)\qquad \begin{array}{ccc} \Lambda^{n-q+2}F & \xrightarrow{\Lambda^{n-q+2}\phi} & \Lambda^{n-q+2}G \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ \Lambda^{n-q+1}F & \xrightarrow{\Lambda^{n-q+1}\phi} & \Lambda^{n-q+1}G \end{array}$$

where β is the action on the ΛG^*-module ΛG, with $\beta\in\Lambda G^*$, and α is the action on the ΛF^*-module ΛF with $\alpha\in\Lambda F^*$ and $\Lambda\phi^*(\beta)=\alpha$.

We shall define maps of complexes:

$$\begin{array}{c} \mathbb{L}_1^{n-q+2}(\phi) \\ \downarrow{\scriptstyle v^q} \\ \mathbb{L}_1^{n-q+1}(\phi) \end{array}$$

which are given by (*) in degrees 0 and 1 and which makes the complexes $\mathbb{L}_1^{n-q+1}(\phi)$ into the rows of the double complex (**) of figure 1.
We define $\mathbb{K}(\beta,\phi)$ to be the total complex associated to the double complex (**).

The convention of the degrees in the double complex (**) is as follows. The components of the bottom row have bidegree $(r,0)$ and the components of the right hand column have bidegree $(0,s)$. So, the components in the right bottom and left upper corner have bidegrees $(0,0)$ and $(m-n+1,n-1)$ resp.

We shall now define the double complex (**) in detail. The differential d of the rows $\mathbb{L}_1^{n-q+1}(\phi)$ is defined in § 1, i.e. d is induced by the action of $1\otimes c_\phi\otimes 1$ in $SF\otimes\Lambda F^*\otimes SG\otimes\Lambda G^*$ on $LF\otimes LG^*\subset SF\otimes\Lambda F\otimes D(G^*)\otimes\Lambda G^*$. We shall define a second operator as follows. The element $1\otimes 1\otimes 1\otimes\beta\in SF\otimes\otimes\Lambda F^*\otimes SG\otimes\Lambda G^*$ acts on $SF\otimes\Lambda F\otimes D(G^*)\otimes\Lambda G^*$ and has quadruple

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & L^1_{m-n}G^*\otimes\Lambda^mF\to\cdots\to & & L^1_2G^*\otimes\Lambda^{n+2}F & \to & L^1_1G^*\otimes\Lambda^{n+1}F & \to & \Lambda^nF & \to & \Lambda^nG \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & L^{q-1}_{m-n}G^*\otimes\Lambda^mF\to\cdots\to & & L^{q-1}_2G^*\otimes\Lambda^{n+2}F & \to & L^{q-1}_1G^*\otimes\Lambda^{n+1}F & \to & \Lambda^{n-q+2}F & \to & \Lambda^{n-q+2}G \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & L^q_{m-n}G^*\otimes\Lambda^mF\to\cdots\to & & L^q_2G^*\otimes\Lambda^{n+2}F & \to & L^q_1G^*\otimes\Lambda^{n+1}F & \to & \Lambda^{n-q+1}F & \to & \Lambda^{n-q+1}G \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0\to & L^n_{m-n}G^*\otimes\Lambda^mF\to\cdots\to & & L^n_2G^*\otimes\Lambda^{n+2}F & \to & L^n_1G^*\otimes\Lambda^{n+1}F & \to & F & \to & G
\end{array}
$$

Figure 1

degree $(0,0,0,1)$ and commutes with $1\otimes c_\phi\otimes 1$. We call υ the induced action of β on $LF\otimes LG^*$. Hence υ and d commute and $\upsilon\upsilon=0$. Call υ^q_p the bihomogeneous component of υ, i.e. for $0\le q\le n-1$

$$\upsilon^q_p : L^{n-q+1}_{p-1}G^*\otimes L^{n+p-1}_1F\to L^{n-q}_{p-1}G^*\otimes L^{n+p-1}_1F \text{ for } 2\le p\le m-n+1,$$

$$\upsilon^q_1=\alpha:\Lambda^{q+2}F\to\Lambda^{q+1}F$$

$$\upsilon^q_0=\beta:\Lambda^{q+2}G\to\Lambda^{q+1}G$$

We shall show now that υ also commutes with d_1. The map d_1 is given by $d_1(\gamma\otimes a)=\gamma(a)$ for $\gamma\otimes a\in\Lambda^qG^*\otimes\Lambda^{n+1}F=L^q_1G^*\otimes\Lambda^{n+1}F$. The following diagram commutes

$$
\begin{array}{ccc}
\Lambda^{q-1}G^*\otimes\Lambda^{n+1}F & \xrightarrow{\ d_1\ } & \Lambda^{n-q+2}F \\
{\scriptstyle\beta\otimes 1}\downarrow & & \downarrow \\
\Lambda^qG^*\otimes\Lambda^{n+1}F & \xrightarrow{\ d_1\ } & \Lambda^{n-q+1}F
\end{array}
$$

since

$$d_1(\beta\otimes 1)(\gamma\otimes a)=d_1((\beta\wedge\gamma)\otimes a)=(\beta\wedge\gamma)(a)=\beta(\gamma(a))$$
$$=\alpha(\gamma(a))=\alpha(d_1(\gamma\otimes a))=\alpha d_1(\gamma\otimes a).$$

We used that the action of β on ΛF is the action of α on ΛF, since $\Lambda\phi^*(\beta)=\alpha$.

This proves that (**) is a double complex.

The same rows of (**) appear in the double complex of Buchsbaum and Eisenbud [3] in their construction of $\mathbb{K}(\phi, a)$. The arrows in the columns are reversed.

THEOREM 2.1. Let $\phi : F \to G$ be a map of free R-modules with rank $F = m \geq$ $\geq$ rank $G = n$. Let $\alpha : F \to R$ and $\beta : G \to R$ be R linear maps such that $\alpha = \beta\phi$, let $\mathbb{K}(\beta, \phi)$ be the total complex associated to the double complex (**). Let $I_n(\phi)$ be the ideal generated by the $(n \times n)$-minors of the $(m \times n)$-matrix of ϕ. Let $J(\beta, \phi) = I_n(\phi) + \text{Im }(\alpha)$. Then

(1) The homology of $\mathbb{K}(\beta, \phi)$ is annihilated by $J(\beta, \phi)$.

(2) If R is a local ring with maximal ideal $\mathfrak{m}$ and if $\phi(F) \subset \mathfrak{m}G$ and $\text{Im }(\beta) \subset \mathfrak{m}$ then $\mathbb{K}(\beta, \phi)$ is a minimal complex.

(3) grade $J(\beta, \phi) \leq m$ and moreover $\mathbb{K}(\beta, \phi)$ is exact if and only if grade $J(\beta, \phi) = m$.

(4) If $\beta = (\beta_1, \ldots, \beta_n)$ is an R-sequence then $H_0(\mathbb{K}(\beta, \phi)) = \text{Im }(\beta)/\text{Im }(\alpha)$.

In the generic case these conditions are actually satisfied, we have (see [3] theorem (5.2) for the analogon):

THEOREM 2.2. Let S be any commutative regular Noetherian ring and let $R = S[X_{ij}, Y_k;\ 1 \leq i \leq m,\ 1 \leq j, k \leq n]$ be the polynomial ring in $mn + n$ indeterminates with coefficients in S and $m \geq n$. Let $\phi : R^m \to R^n$ be the map with matrix (X_{ij}), and let $\beta : R^n \to R$ be the map with matrix $(Y_1, \ldots, Y_n)$. Then $J(\beta, \phi)$ is an ideal of grade m and $\mathbb{K}(\beta, \phi)$ is a resolution of $\text{Im }(\beta)/\text{Im }(\alpha)$.

PROOF OF THEOREM 2.1.

Part (2) follows from the construction of $\mathbb{K}(\beta, \phi)$. Part (4). Suppose $(\beta_1, \ldots, \beta_n)$ is an R-sequence, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \Lambda^2 G & & & & \\
 & & & & \downarrow{\scriptstyle \beta} & & & & \\
 & & F & \xrightarrow{\phi} & G & & & & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \\
0 & \longrightarrow & \text{Im }(\alpha) & \longrightarrow & \text{Im }(\beta) & \longrightarrow & \text{Im }(\beta)/\text{Im }(\alpha) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact columns and rows. This implies that the sequence

$$F \oplus \Lambda^2 G \xrightarrow{\phi + \beta} G \longrightarrow \text{Im }(\beta)/\text{Im }(\alpha) \longrightarrow 0$$

is exact. Further $\mathbb{K}_1(\beta,\phi)=F\oplus\Lambda^2G$ and $\mathbb{K}_0(\beta,\phi)=G$, hence

$$H_0(\mathbb{K}(\beta,\phi))\simeq \text{Coker }(\phi+\beta)\simeq \text{Im }(\beta)/\text{Im }(\alpha).$$

So one sees that $J(\beta,\phi)$ annihilates $H_0(\mathbb{K}(\beta,\phi))$, since Im $(\alpha)\subset$ ann(Im (β)/ Im (α)) and $I_n(\phi)\subset$ ann(Coker (ϕ)) $\subset$ ann(Coker $(\phi+\beta)$). Parts (1) and (3) follow from Theorem 2.2 and the following:

LEMMA 2.3. ("Lemma d'Acyclicité" [12]).

Let R be a Noetherian ring with 1.

If $\mathbb{L}: 0\to L_k\to L_{k-1}\to\cdots\to L_1\to L_0$ is a complex of finitely generated free R-modules, and J is an ideal of R which annihilates the homology of $\mathbb{L}$, and grade $(J)\geq k$, then $\mathbb{L}$ is exact.

We repeat the argument from [3] to derive Theorem 2.1 part (1) and (3) from Theorem 2.2. Let $S=\mathbb{Z}$ be the ring of integers, and $R=S[X_{ij}, Y_k]$, let $\phi_0: R_0^m\to R_0^n$ be given by the matrix (X_{ij}) and let $\beta_0: R_0^n\to R_0$ be given by the matrix $(Y_1,\ldots,Y_n)$ as in Theorem 2.2.

Then we know that $\mathbb{K}(\beta_0,\phi_0)$ is a resolution of Im (β_0)/Im (α_0) with $\alpha_0=\beta_0\phi_0: R_0^m\to R_0$. *If* R, ϕ and β are as in Theorem 2.1 then $\mathbb{K}(\beta,\phi)$ is a specialization of $\mathbb{K}(\beta_0,\phi_0)$, that is there is a unique homomorphism $\zeta: R_0\to R$ of rings, such that $\mathbb{K}(\beta,\phi)=\mathbb{K}(\beta_0,\phi_0)\otimes_{R_0}R$. If $r\in J(\beta_0,\phi_0)$ then the map $r:\mathbb{K}(\beta_0,\phi_0)\to\mathbb{K}(\beta_0,\phi_0)$, induced by multiplication with r, induces 0 on Im (β_0)/Im (α_0), and thus is homotopic to 0 by some homotopy s. But then $s\otimes_{R_0}R$ is a homotopy on $\mathbb{K}(\beta,\phi)$, which shows that multiplication by $\zeta(r)$ is homotopic to zero on $\mathbb{K}(\beta,\phi)$. Thus $\zeta(r)$ annihilates the homology of $\mathbb{K}(\beta,\phi)$, since $J(\beta,\phi)=R\zeta(J(\beta_0,\phi_0))$, part (1) is proven. As for part (3) we make use of Lemma 2.3, which shows that if the homology of $\mathbb{K}(\beta,\phi)$ is annihilated by an ideal of grade $\geq m$, then $\mathbb{K}(\beta,\phi)$ is exact, since $\mathbb{K}(\beta,\phi)$ has length m.

Thus if grade $J(\beta,\phi)\geq m$ then $\mathbb{K}(\beta,\phi)$ is exact and $pd(H_0)\leq m$ with $H_0=H_0(\mathbb{K}(\beta,\phi))$.

However $J(\beta,\phi)\subset$ ann (H_0), hence

$$\text{grade } J(\beta,\phi)\leq \text{grade (ann }(H_0))=\textit{grade }(H_0)\leq pd(H_0)\leq m.$$

So one concludes grade $J(\beta,\phi)=m$.

PROOF OF THEOREM 2.2. If R, ϕ and β are as in Theorem 2.2, then it is well known that grade $I_n(\phi)=m-n+1$ see [6]. Thus we may apply the following lemma (analogous to Lemma 5.5 of [3]):

LEMMA 2.4. In the set-up of Theorem 2.1, suppose that grade $I_n(\phi)=$ $=m-n+1$. Then some power of the ideal $J(\beta,\phi)$ annihilates the homology of $\mathbb{K}(\beta,\phi)$.

PROOF. We must give a proof without the help of Theorem 2.2. By [3] the rows of (**) are exact under the hypothesis of the lemma. By the spectral sequence of the double complex (**), the homology of $\mathbb{K}(\beta,\phi)$ is the same as the homology of the complex of cokernels of the map $\Lambda^q\phi$ for $g=1,\ldots,n$.

That is, if we let $C_{p-1} = \text{Coker}\ (\Lambda^p \phi)$ the maps in the two upper rows of the diagram:

$$\begin{array}{ccccccc}
\Lambda^n F \to \cdots \to & \Lambda^p F & \to \cdots \to & \Lambda^2 F & \longrightarrow & F \\
\downarrow \Lambda^n\phi & & \downarrow \Lambda^p\phi & & \downarrow \Lambda^2\phi & & \downarrow \phi \\
\Lambda^n G \to \cdots \to & \Lambda^p G & \to \cdots \to & \Lambda^2 G & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_{n-1} \to \cdots \to & C_{p-1} & \to \cdots \to & C_1 & \longrightarrow & C_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

induce maps $C_p \to C_{p-1}$, which make them into a complex $\mathbb{C}$, having the same homology as $\mathbb{K}(\beta, \phi)$. By Proposition 1.5 of [3], the C_p are themselves annihilated by $I_n(\phi)$, so the same goes for the homology of $\mathbb{K}(\beta, \phi)$. Thus we need only to show that some power of Im (α) kills $H(\mathbb{K}(\beta, \phi))$. To this end, let $\alpha = (\alpha_1, \dots, \alpha_m) \in R^{m*} = F^*$ and let T be a localization of R at the multiplicatively closed set generated by one of the α_i's, we must show that $T \otimes \mathbb{K}(\beta, \phi)$ is exact. Let $x_1, \dots, x_m$ be a basis for F and $y_1, \dots, y_n$ for G and $x_1^*, \dots, x_m^*$ a dual basis for F^* and $y_1^*, \dots, y_n^*$ for G^*. We have $\alpha = \Sigma\ \alpha_i x_i^*$ and $\beta = \Sigma\ \beta_j y_j^*$. Now $(\alpha_1, \dots, \alpha_m) T = T$ hence there exist $u_1, \dots, u_m \in T$ such that $\Sigma\ u_i \alpha_i = 1$. Take u equal to $\Sigma\ u_i x_i \in T \otimes F \subset \Lambda(T \otimes F)$ then $1 = \Sigma\ u_i \alpha_i = \Sigma\ u_i \phi_{ij} \beta_j = \Sigma\ v_j \beta_j$ with $\phi(u) = v$. Moreover $\phi^*(\beta) = \alpha$ and $u(\alpha) = 1$, so $v(\beta) = 1$. For $x \in \Lambda F$ one has by Corollary 1.2 of [3], since deg $u = 1$:

$$x = u(\alpha)(x) = u \wedge (\alpha(x)) + (-1)^{1 + \deg \alpha} \alpha(u \wedge x) =$$

$$= u \wedge (\alpha(x)) + \alpha(u \wedge x).$$

Thus $\text{id}_{\Lambda(T \otimes F)} = u\alpha + \alpha u$, i.e. $\text{id}_{\Lambda(T \otimes F)}$ is homotopic to 0 on the Koszul complex $(\Lambda(T \otimes F), \alpha)$.

In the same way $\text{id}_{\Lambda(T \otimes F)} = v\beta + \beta v$, i.e. $\text{id}_{\Lambda(T \otimes F)}$ is homotopic to 0 on the Koszul complex $(\Lambda(T \otimes G), \beta)$. In the following diagram, the second and third row are the Koszul complex $(\Lambda(T \otimes F), \alpha)$ and $(\Lambda(T \otimes G), \beta)$ resp. which are both exact, since Im $(\alpha) T =$ Im $(\beta) T = T$.

The homotopy's of both complexes are compatible, since $\phi^*(\beta) = \alpha$ and $\phi(u) = v$, so they induce a homotopy of $\text{id}_{T \otimes \mathbb{C}}$ to 0, so that $T \otimes H(\mathbb{C}) = 0$, thus some power of α_i annihilates $H(\mathbb{K}(\beta, \phi))$. Since i was arbitrary, the lemma is proven.

To finish the proof of Theorem 2.2 it is enough to show the exactness of $\mathbb{K}(\beta, \phi)$. By Lemma 2.3 and 2.4 it suffices to show that in this case grad $J(\beta, \phi) \geq m$. We have indeed the following lemma (analogous to Lemma 5.6 of [3]).

$$
\begin{array}{ccccccccc}
0\to\Lambda^m(T\otimes F)\to\cdots\to & \Lambda^{n+1}(T\otimes F) & \to & 0 & & & & & \\
\| & \| & & \downarrow & & & & & \\
0\to\Lambda^m(T\otimes F)\to\cdots\to & \Lambda^{n+1}(T\otimes F) & \to & \Lambda^n(T\otimes F) & \to\cdots\to & T\otimes F & \to & T & \to 0\\
 & \downarrow & & \downarrow{\scriptstyle \Lambda^n\phi} & & \downarrow{\scriptstyle \phi} & & \| & \\
 & 0 & & \to\Lambda^n(T\otimes G) & \to\cdots\to & T\otimes G & \to & T & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & & & \\
 & 0 & & \to T\otimes C_{n-1} & \to\cdots\to & T\otimes C_0 & \to & 0 & \\
 & & & \downarrow & & \downarrow & & & \\
 & & & 0 & & 0 & & &
\end{array}
$$

LEMMA 2.5. Let β and ϕ be as in Theorem 2.2. Then grade $J(\beta,\phi)\geq m$.

PROOF. Suppose grade $J(\beta,\phi)<m$, then there is a prime ideal $\mathfrak{p}$ in R such that $J(\beta,\phi)\subset\mathfrak{p}$ and $\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})<m$.

Suppose $Y_j\notin\mathfrak{p}$ for some $j=1,\dots,n$.

Let $f_i'=\Sigma_{k\neq j}X_{ik}Y_kY_j^{-1}+X_{ij}$ and put $f_i=y_if_i'$. Then the following sequence $Y_1,\dots,Y_{j-1},\ Y_{j+1},\dots,Y_n,\ f_1,\dots,f_m$ is R_{Y_j}-regular. Now $R\to R_{Y_j}\to R_\mathfrak{p}$ are localizations, so it is also an $R_\mathfrak{p}$-regular sequence, $R_\mathfrak{p}$ is a local ring, hence $f_1,\dots,f_m$ is an $R_\mathfrak{p}$-regular sequence in $\mathfrak{p}R_\mathfrak{p}$, thus $\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})\geq m$ which gives a contradiction. Thus $Y_j\in\mathfrak{p}$ for all $j=1,\dots,n$ so $I_n(\phi)+(Y_1,\dots,Y_n)\subset\mathfrak{p}$.

Let $\bar{R}=(R/(Y_1,\dots,Y_n)R)=S[X_{ij}]$, we have $\operatorname{grade}_{\bar{R}}(I_n(\bar{\phi}))=m-n+1$, by [6]. So $\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})\geq\operatorname{grade}\ \mathfrak{p}\geq\operatorname{grade}\ I_n(\phi)+(Y_1,\dots,Y_n)R\geq m+1$. This is a contradiction. Thus grade $J(\beta,\phi)\geq m$. This finishes the proof of Lemma 2.5 and Theorem 2.2.

3. RESOLUTIONS OF QUOTIENTS OF IDEALS

Let I and J be ideals in a ring R. Suppose $J\subset I$ and:

a) J is generated by m elements and grade $(I/J)\geq m$

b) One of the following three conditions is satisfied

  (i) I is a perfect ideal of grade $m-1$;

  (ii) $pd(R/I)=2$;

  (iii) I is generated by an R-sequence of length $n\leq m$.

Then we give a finite projective resolution of the quotient I/J and prove that I/J is a perfect R-module of grade m. See Propositions 3.3, 3.4 and 3.8 for (i), (ii) and (iii) respectively.

We show that the ideal $J(g,\phi)$ as defined in § 2 is an invariant of the pair of ideals $J\subset I$ in case I is generated by an R-sequence.

Let R be a ring and $f=(f_1,\ldots,f_m)$ a sequence of elements in R. Then we denote by $H_p(f, R)$ the p^{th} Koszul homology of the sequence f. Suppose J is an ideal in R and $f_1,\ldots,f_m$ generate J. Suppose grade $J=n$ and $a_1,\ldots,a_n$ is an R-sequence in J. Let (a) be the ideal in R generated by $a_1,\ldots,a_n$. Then the following two isomorphisms are well known:

$$\operatorname{Ext}^n_R(R/J, R)\simeq \operatorname{Hom}_R(R/J, R/(a)), \text{ see [13]}$$

$$H_{m-n}(f, R)\simeq \operatorname{Hom}_R(R/J, R/(a)), \quad \text{see [2]}.$$

Both isomorphisms can be proved by induction on n. We include the following form of this fact, since we could not find an explicit isomorphism in the literature.

LEMMA 3.1. Let R be a ring and J an ideal in R generated by $f_1,\ldots,f_m$. Suppose grade $J=n$. Let $(\mathbb{G}, d)$ be a projective resolution of R-modules of R/J. Take $F=R^m$ and let $f: F\to R$ be the map with matrix $(f_1,\ldots,f_m)$. Let $\phi:(\Lambda F, f)\to(\mathbb{G}, d)$ be a map of complexes which extends the identity map $1: R/J\to R/J$. Then the map induced by ϕ^* gives the following isomorphism:

$$\operatorname{Ext}^n_R(R/J, R)\simeq H^n(\mathbb{G}^*, d^*)\xrightarrow{\phi^n} H^n(\Lambda F^*, f^*)\simeq H_{m-n}(f, R).$$

PROOF. The complex $(\mathbb{G}, d)$ is a projective resolution of R/J. There exists an R-sequence $a_1,\ldots,a_n$ in J, since grade $J=n$. Let $A=R^n$ and let the map $a: A\to R$ be defined by the matrix $(a_1,\ldots,a_n)$. Then $(\Lambda A, a)$ is a free resolution of $R/(a)$ and $(\Lambda A\otimes\mathbb{G}, a\otimes d)$ is a resolution of $R/(a)\otimes R/J\simeq R/J$.

Let $\pi:(\Lambda A\otimes\mathbb{G}, a\otimes d)\to(\mathbb{G}, d)$ be a map of complexes which extends the isomorphism $R/(a)\otimes R/J\to R/J$. Let $i:(\Lambda F, f)\to(\Lambda A\otimes\Lambda F, a\otimes f)$ be the map of complexes defined by $i(v)=1\otimes v$ where 1 is the unit in $R=\Lambda^0 A$. Then $\pi\ (1\otimes\phi)\ i$ and ϕ are both maps of complexes from $(\Lambda F, f)$ to $(\mathbb{G}, d)$ which extend the identity map $1: R/J\to R/J$. Hence both maps are homotopy equivalent. So we have a commutative diagram

$$\begin{array}{ccc}
H^n((\Lambda A\otimes\mathbb{G})^*) & \xrightarrow{(1\otimes\phi)^n} & H^n((\Lambda A\otimes\Lambda F)^*) \\
\pi^n \uparrow & & \downarrow i^n \\
H^n(\mathbb{G}^*) & \xrightarrow{\phi^n} & H^n(\Lambda F^*)
\end{array}$$

The map π^n is an isomorphism, since $\Lambda A\otimes\mathbb{G}$ and $\mathbb{G}$ are both projective resolutions of R/J. The map i^n is an isomorphism since $a_1,\ldots,a_n$ is an R-sequence in J. Thus to prove that ϕ^n is an isomorphism it is enough to prove it for $(1\otimes\phi)^n$ and this follows from a spectral sequence argument. Consider the double complex $E_0^{pq}=(\Lambda^p A\otimes\Lambda^q F)^*$. The E_1 term is $E_1^{pq}=H^p((\Lambda A\otimes\Lambda^q F)^*)$ which is zero unless $p=n$, since $a_1,\ldots,a_n$ is an R-sequence. Thus the spectral sequence degenerates at E_2 and $E_2^{n0}\simeq\ker\ (R/(a)\otimes\Lambda^0 F^*\to R/(a)\otimes\Lambda^1 F^*)\simeq$ $\simeq\operatorname{Hom}_R(R/J, R/(a))$. In the same way the double complex $\bar{E}_0^{pq}=(\Lambda^p A\otimes G_q)^*$

has a spectral sequence $\{\bar{E}_r\}$ which degenerates at $\bar{E}_2$ and with $\bar{E}_1$ terms $\bar{E}_1^{pq}=0$ unless $p=n$. Thus $\bar{E}_2^{n0}\simeq\ker\,(R/(a)\otimes G_0^*\to R/(a)\otimes G_1^*)\simeq\mathrm{Hom}_R\,(R/J, R/(a))$. So the morphism between E_2^{pq} and $\bar{E}_2^{pq}$ is induced by $(1\otimes\phi)^n: H^n((\Lambda A\otimes\otimes\mathbb{G})^*)\to H^n((\Lambda A\otimes\Lambda F)^*)$ and is an isomorphism. Moreover $\mathrm{Ext}_R^n(R/J, R)\simeq\simeq H^n(\mathbb{G}^*)$ and $H_{m-n}(f, R)\simeq H^n(\Lambda F^*)$. This proves the Lemma.

CONSTRUCTION 3.2. Let R be a ring. Let I and J be two ideals in R. Suppose I is a perfect ideal of grade n and J is generated by $f_1,\dots,f_{n+1}$ in I. Suppose grade $(I/J)\geq n+1$. Then grade $J=n$. There is a finite projective resolution $(\mathbb{G}, d)$ of R/I as R-module, of length n, since I is perfect of grade n. Let $F=R^{n+1}$ and let $f: F\to R$ be the map given by the components $f_1,\dots,f_{n+1}$. Since grade $(I/J)\geq n+1$, we know that the map $\mathrm{Ext}_R^n(R/I, R)\to\mathrm{Ext}_R^n(R/J, R)$ induced by the map $R/J\to R/I$, is an isomorphism. Let the map of complexes $\phi:(\Lambda F, f)\to(\mathbb{G}, d)$ be an extension of the quotient map $R/J\to R/I$. From Lemma 3.1 and the remarks above we obtain that the map $\phi^n: H^n(\mathbb{G}^*, d^*)\to\to H^n(\Lambda F^*, f^*)$ is an isomorphism. Consider the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & G_1^*\otimes\Lambda^{n+1}F & \xrightarrow{\psi_n} & \Lambda^n F & \xrightarrow{\phi_n} & G_n \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & G_{n-q}^*\otimes\Lambda^{n+1}F & \xrightarrow{\psi_{q+1}} & \Lambda^{q+1}F & \xrightarrow{\phi_{q+1}} & G_{q+1} \\
 & (-1)^{n-q}d_{n-q}^*\otimes 1 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & G_{n-q+1}^*\otimes\Lambda^{n+1}F & \xrightarrow{\psi_q} & \Lambda^q F & \xrightarrow{\phi_q} & G_q \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & & G_n^*\otimes\Lambda^{n+1}F & \xrightarrow{\psi_1} & F & \xrightarrow{\phi_1} & G_1
\end{array}\qquad (*)$$

Question 23/4/'02
Posed by D. van Straten
Why is it that $\phi_q\circ\psi_q=0$?

The right-hand column is the complex $(\mathbb{G}, d)$ with G_0 deleted. The column in the middle is the complex $(\Lambda F, f)$ with $\Lambda^0 F$ and $\Lambda^{n+1}F$ deleted. The left-hand column is, up to signs, the dual $(\mathbb{G}^*, d^*)$ of the complex $(\mathbb{G}, d)$ tensored with $\Lambda^{n+1}F$ and with $G_0^*\otimes\Lambda^{n+1}F$ deleted. The map between the middle and the right-hand column is given by the map of complexes ϕ, with ϕ_0 and ϕ_{n+1} deleted. The maps $\{\psi_q\}$ between the left-hand and middle column are given as follows. The dual of ϕ_q is the map $\phi_q^*: G_q^*\to\Lambda^q F^*$. Let $\psi_q: G_{n-q+1}^*\otimes\otimes\Lambda^{n+1}F\to\Lambda^q F$ be given by $\psi_q(\gamma\otimes\upsilon)=(\phi_{n-q+1}^*(\gamma))(\upsilon)$ for $\gamma\in G_{n-q+1}^*$ and

$\upsilon \in \Lambda^{n+1}F$. Remember from § 1 that $\phi^*_{n-q+1}(\gamma)$ is an element of $\Lambda^{n-q+1}F^*$ and acts on ΛF.

PROPOSITION 3.3. Let R be a ring. Suppose I is a perfect ideal of grade n in R. Let J be an ideal in R generated by $f_1, \ldots, f_{n+1}$ in I. Then the above diagram (*) is commutative. Let $\mathbb{K}$ be the associated complex of (*). If grade $(I/J) \geq n+1$ then $\mathbb{K}$ is a projective resolution of I/J and I/J is a perfect R-module of grade $n+1$ and $pd(R/J) = n+1$.

PROOF. The right-hand squares of (*) are commutative by the construction of ϕ. The commutativity of the left-hand squares of (*) follows from the commutativity of the diagram:

$$\begin{array}{ccccc}
G^*_{n-q} \otimes \Lambda^{n+1}F & \xrightarrow{\phi^*_{n-q}\otimes 1} & \Lambda^{n-q}F^* \otimes \Lambda^{n+1}F & \xrightarrow{m_{q+1}} & \Lambda^{q+1}F \\
\downarrow {\scriptstyle (-1)^{n-q}d^*_{n-q}\otimes 1} & & \downarrow {\scriptstyle (-1)^{n-q}f^*\otimes 1} & & \downarrow {\scriptstyle f} \\
G^*_{n-q+1} \otimes \Lambda^{n+1}F & \xrightarrow{\phi^*_{n-q+1}\otimes 1} & \Lambda^{n-q+1}F^* \otimes \Lambda^{n+1}F & \xrightarrow{m_q} & \Lambda^{q}F
\end{array}$$

The left-hand square is commutative, since it is the dual of a commutative square. The right-hand square is commutative, since f is an element of $F^* \subset \Lambda F^*$ and acts on ΛF, thus f^* acts on $(\Lambda F)^* \simeq \Lambda F^*$. But f acts also on ΛF^* by left multiplication. Both actions of f are the same up to a sign, see [3], (1.4).

Next we prove the exactness of the complex $\mathbb{K}$ associated with the diagram (*). First, extend the diagram (*), by adjoining $G_0^* \otimes \Lambda^{n+1}F$ in the left upper corner, $\Lambda^{n+1}F$ and $\Lambda^0 F$ in the middle column and G_0 in the right bottom corner, together with the appropriate maps ψ_{n+1} and ϕ_0. Call this diagram (**) and its associated complex $\bar{\mathbb{K}}$. We may suppose that $G_0 = R$ and thus ϕ_0 and ψ_{n+1} are isomorphisms. Hence $\mathbb{K}$ and $\bar{\mathbb{K}}$ have the same homology. The right hand column of the diagram (**) has zero homology except at the place G_0, where its homology is R/I, since $\mathbb{G}$ is a resolution of R/I. The middle column of (**) is the Koszul complex of a sequence of $n+1$ generators of the ideal J of grade n. Hence this row has only non-zero homology at the places $\Lambda^1 F$ and $\Lambda^0 F$, which are $H_1(\Lambda F, f)$ and R/J. The left-hand column has only non-zero homology at the place $G_n^* \otimes \Lambda^{n+1}F$. Since this column is the dual of $\mathbb{G}$, up to signs and tensoring with a trivial factor, and since $\mathbb{G}$ is a resolution of a perfect R-module of grade n i.e. $H^i(\mathbb{G}^*) \simeq \mathrm{Ext}^i_R(R/I, R) = 0$ for all $i < n$. The cohomology $H^n(\mathbb{G}^* \otimes \Lambda^{n+1}F)$ is isomorphic with $H_1(\Lambda F, f)$ by the isomorphism induced by ψ_1, by the remarks in the beginning of (3.2). What we have computed is the E^1 term of the filtered complex $\bar{\mathbb{K}}$:

$$E^1_{p,q} \simeq \begin{cases} R/I & \text{if } p=0 \text{ and } q=0 \\ R/J & \text{if } p=1 \text{ and } q=0 \\ H_1(\Lambda F, f) & \text{if } p=1 \text{ and } q=1 \\ H^n(\mathbb{G}^* \otimes \Lambda^{n+1}F) & \text{if } p=2 \text{ and } q=1 \\ 0 & \text{otherwise} \end{cases}$$

And $E^1_{2,1} \xrightarrow{d^1} E^1_{1,1}$ is an isomorphism. Hence the E^2 term has only zero's except at one place: $E^2_{1,0} \simeq \ker\, (R/J \to R/J) \simeq I/J$. Thus $\mathbb{K}$ is exact and is a projective resolution of I/J of length $n+1$.

From $n+1 \leq \text{grade}\,(I/J) \leq pd(I/J) \leq n+1$ one concludes that I/J is perfect of grade $n+1$.

From the exactness of the sequence $0 \to I/J \to R/J \to R/I \to 0$ one concludes that $pd\,(R/J)=n+1$, since $pd\,(R/I)=n$ and $pd\,(R/J)=n+1$.

PROPOSITION 3.4. Let R be a local ring and I an ideal in R such that $pd\,(R/I)=2$. Let J be an ideal in R generated by m elements. Suppose $J \subset I$ and grade $(I/J) \geq m$. Then I/J is a perfect R-module of grade m and $pd\,(R/J) \leq m$.

PROOF. I is an ideal in the ring R such that $pd\,((R/I)=2$. We have a free resolution of length 2

$$0 \to R^{n-1} \xrightarrow{\beta} R^n \xrightarrow{g} R \to R/I \to 0$$

since R is a local ring. The ideal J is generated by m elements $f_1, \ldots, f_m$ in I so $f_i = \Sigma\, \phi_{ij} g_j$. Let ϕ be the map with matrix (ϕ_{ij}). We have the following commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & R^{n-1} & & & & \\
 & & & & \downarrow{\scriptstyle\beta} & & & & \\
 & & R^m & \xrightarrow{\phi} & R^n & & & & \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & & & \\
0 & \longrightarrow & J & \longrightarrow & I & \longrightarrow & I/J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Hence we get a presentation

$$R^m \oplus R^{n-1} \xrightarrow{\psi} R^n \to I/J \to 0$$

where $\psi = \phi + \beta$.

Further $\text{rad}\,(J:I) = \text{rad}\, I_n(\psi)$, so grade $I_n(\psi) \geq m$, since grade $(I/J) \geq m$. The matrix ψ has size $(m+n-1) \times n$ and $m=(m+n-1)-n+1$. So I/J is a perfect R-module of grade m, by [6]. Consider the exact sequence $0 \to I/J \to$ $\to R/J \to R/I \to 0$. We have that $pd\,(R/I)=2$ and $pd\,(I/J)=m$, hence $pd\,((R/J) \leq$ $\leq \max\,\{2, m\}$, so $pd\,(R/J) \leq m$ in case $m \geq 2$. Now suppose $m=1$, then there exists an exact sequence $0 \to R^p \to R \to R \to R/J \to 0$. Looking at the ranks one concludes that $p=0$, so $pd\,(R/J)=1$. This proves the proposition.

Let I and J be ideals in the ring R and $J \subset I$. Suppose I is generated by $g_1, \ldots, g_n$ and let J be generated by $f_1, \ldots, f_m$. Then $f_i = \Sigma\, \phi_{ij} g_j$. Define the ideal $J(g, \phi) = J + I_n(\phi)$ as in (2.1). Take $I_n(\phi) = (0)$ in case $m < n$.

Let $V(I) = \{\mathfrak{p} \in \operatorname{spec} R \mid I \subset \mathfrak{p}\}$ for any ideal in R.

Let M be a finitely generated R-module. Denote by $F_0(M)$ the Fitting ideal of M, i.e. for any presentation:

$$R^p \xrightarrow{\psi} R^q \to M \to 0 \text{ with } p \geq q$$

one defines $F_0(M) = I_q(\psi)$. This definition does not depend on the chosen presentation of M. Furthermore $V(F_0(M)) = \operatorname{Supp} M$.

PROPOSITION 3.5. Let I be an ideal in R generated by an R-sequence $g_1, \ldots, g_n$ and suppose $f_i = \Sigma\, \phi_{ij} g_j$ and $J = (f_1, \ldots, f_m)R$. Then

$$J(g, \phi) = J + F_0(I/J).$$

PROOF. We generalize the proof of Mond [10], where

$$R = \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$$

and $g_j = x_j - y_j$ for $j = 1, \ldots, n$ and $f_i = h_i(x) - h_i(y)$ and $h_i(x) \in \mathbb{C}\{x_1, \ldots, x_n\}$ for $i = 1, \ldots, m$.

A presentation of I/J is given by

$$R^m \oplus \Lambda^2 R^n \xrightarrow{\psi} R^n \to I/J \to 0$$

with $\psi = \phi + \delta$ and ϕ the map with matrix (ϕ_{ij}) and $\delta : \Lambda^2 R^n \to R^n$ is part of the Koszul complex of the sequence $g_1, \ldots, g_n$, as we have seen in proving Theorem 2.2.

Now $J(g, \phi) \subset J + F_0(I/J)$, since $I_n(\phi) \subset I_n(\psi)$. Let β be a $(n \times n)$-submatrix of ψ. If β is a submatrix of ϕ then $\det \beta \in I_n(\phi)$. If β is a submatrix of δ then $\det \beta = 0$. Otherwise β contains columns of δ and ϕ, i.e. a column of the form $g_j e_i - g_i e_j$ and one of the form $\Sigma\, \phi_{kl} e_{kl}$, where $e_1, \ldots, e_n$ is the standard basis of R^n. Let β' be the matrix obtained from β by multiplying the i^{th} row by g_i. Then $\det \beta' = g_i \cdot \det \beta$ and β has a column of the form $g_i g_j e_i - g_i^2 e_j$. Let β'' be the matrix obtained from β' by replacing the latter mentioned column by $g_j e_i - g_i e_j$. Then $\det \beta'' = g_i \det \beta'$. Thus $\det \beta = \det \beta''$, since g_i is a non-zero divisor on R. After multiplying the l^{th} row of β'' by g_l and adding this to the i^{th} row for every $l \neq i$, we get a matrix with only zero's or elements $\Sigma\, \phi_{kl} g_l$ on the i^{th} row. Hence $\det \beta'' \in J$. Thus $\det \beta \in J$. Therefore $F_0(I/J) \subset J + I_n(\phi)$. This proves the proposition.

REMARK 3.6. The assumption that I is generated by an R-sequence is essential in Proposition 3.5 as one sees in the following example. Take $R = k[x, y, z]$ with k a field containing $\frac{1}{2}$. Let

$$I = (yz, zx, xy)R \text{ and } J = (x(y^2 + z^2), y(x^2 + z^2), z(x^2 + y^2))R.$$

Define the matrices ϕ and ψ by

$$\phi=\begin{pmatrix}0 & z & y\\ z & 0 & x\\ y & x & 0\end{pmatrix} \text{ and } \psi=\begin{pmatrix}x & z-y & y\\ z & 0 & x\\ y & x & 0\end{pmatrix}.$$

Then $J(g,\phi)=(xyz)R+J\subset I$ and $\det(\psi)\equiv -x^3 \pmod I$.

So $J(g,\phi)\neq J(g,\psi)$.

COROLLARY 3.7. Let R be a ring containing two ideals I and J. Suppose I contains J and is generated by an R-sequence. Then $V(J(g,\phi))=V(J:I)$.

PROOF. Since $J(g,\phi)=J+F_0(I/J)$ and $\operatorname{Supp}(I/J)=V(F_0(I/J))$ is contained in $V(J)$ we have:

$$V(J(g,\phi))=V(J)\cap V(F_0(I/J))=\operatorname{Supp}(I/J)=V(J:I).$$

PROPOSITION 3.8. Suppose I is an ideal in the ring R generated by an R-sequence. Let J be an ideal in R generated by m elements. Suppose $J\subset I$ and grade $(I/J)\geq m\geq n$. Then I/J is a perfect R-module of grade m and $pd\,(R/J)\leq m$.

PROOF. The ideal I is generated by an R-sequence $g_1,\dots,g_n$ and J is generated by m elements $f_1,\dots,f_m$ in I. We can write $f_i=\Sigma\, \phi_{ij}g_j$. Proposition 3.5 implies $\operatorname{rad}(J:I)=\operatorname{rad} J(g,\phi)$. By assumption grade $(I/J)\geq m$, hence grade $J(g,\phi)\geq m$. Now $m\geq n$, so we can apply Theorem 2.1 and we get a finite free resolution of I/J of length m. Hence $m\leq$ grade $(I/J)\leq pd\,(I/J)\leq m$. So I/J is a perfect R-module of grade m. From the exact sequence $0\to I/J\to R/J\to R/I$ and $pd\,(R/I)=n\leq m=pd\,(I/J)$ we derive $pd\,(R/J)\leq m$. This proves the proposition.

4. A LENGTH FORMULA

The proof of the following proposition I owe to C. Huneke, see [17].

PROPOSITION 4.1. Let $(R,\mathfrak{m})$ be a local CM ring of dimension m. Let I be an ideal in R generated by the R-sequence $g_1,\dots,g_n$. Let J be an ideal in R generated by $f_1,\dots,f_m$ with $f_i=\Sigma\,\phi_{ij}g_j$. Let ϕ be the $(m\times n)$-matrix with entries ϕ_{ij}.

Then $\ell(I/J)=\ell(R/J(g,\phi))$.

PROOF. Since I/J and $R/J(g,\phi)$ have the same support by (3.7), we may assume that they are both of finite length. We may assume that R is a complete local CM ring, since taking the $\mathfrak{m}$-adic completion $\hat{R}$ of R does not change the length of finitely generated R-modules and $g_1,\dots,g_n$ is again an $\hat{R}$-sequence. Cohen's structure theorem [4] implies that R is a quotient of a regular complete local ring S, which is unramified in case of unequal characteristic. Take

$T = S[[X, Y]]$ with $Y = (Y_1, \ldots, Y_n)$ and X an $(m \times n)$-matrix with entries X_{ij}, $1 \le i \le m$, $1 \le j \le n$, such that the Y_j's and X_{ij}'s are algebraic independant over S. Let $T \to R[[X, Y]]$ be the quotient map corresponding with the quotient $S \to R$. Let $R[[X, Y]] \to R$ be the specialisation map obtained by sending X_{ij} to ϕ_{ij} and Y_i to g_j. Let $p: T \to R$ be the composition of these two maps. Define $\tilde{I} = (Y_1, \ldots, Y_n)T$ and $F_i = \Sigma X_{ij} Y_j$. Let $\tilde{J} = (F_1, \ldots, F_m)T$. Then $J(Y, X)$ is a perfect ideal of grade m by [5], since it is an ideal of a variety of complexes, and $\tilde{I}/\tilde{J}$ is a perfect module of grade m by Theorem 2.2.

Now $g_1, \ldots, g_n$ is an R-sequence and $Y_1, \ldots, Y_n$ is a T-sequence hence $H_0(\mathbb{K}(g, \phi)) \simeq I/J$ and $H_0(\mathbb{K}(Y, X)) \simeq \tilde{I}/\tilde{J}$ by (2.1). The functor $R \otimes_T$- is right exact hence

$$R \otimes_T \tilde{I}/\tilde{J} \simeq H_0(R \otimes_T \mathbb{K}(Y, X)) \simeq H_0(\mathbb{K}(g, \phi)) \simeq I/J$$

Further $R \otimes_T T/J(Y, X) \simeq R/J(g, \phi)$.

The T-modules R and I/J are CM and have a proper intersection, hence $\mathscr{X}(R, \tilde{I}/\tilde{J}) = \ell(R \otimes_T \tilde{I}/\tilde{J}) = \ell(I/J)$ by [14], where

$$\mathscr{X}(M, N) = \Sigma\, (-1)^i \ell(\mathrm{Tor}_i^R(M, N)).$$

In the same way $\mathscr{X}(R, T/J(Y, X)) = \ell(R \otimes_T T/J(Y, X)) = \ell(R/J(g, \phi))$. Consider the T-linear map $\alpha: T/J(Y, X) \to \tilde{I}/\tilde{J}$ defined by multiplication by Y_n. This map is well defined, since $J(Y, X) \subseteq (\tilde{J} : \tilde{I})$ by Theorem 2.1 and a fortiori $Y_n \cdot J(Y, X) \subseteq \tilde{J}$.

Denote the kernel of α by K and the cokernel by C. Let $\mathfrak{p}$ be any prime ideal in T containing $J(X, Y)$. If $Y_n \notin \mathfrak{p}$ then $\alpha \otimes T_{\mathfrak{p}}$ is an isomorphism, hence $K_{\mathfrak{p}} = C_{\mathfrak{p}} = 0$. If $Y_n \in \mathfrak{p}$ and $Y_j \notin \mathfrak{p}$ for some $j < n$ then $Y_n, F_1, \ldots, F_m$ is a $T_{\mathfrak{p}}$-sequence and $I_n(X) \subseteq (F_1, \ldots, F_m)T_{\mathfrak{p}}$, see the proof of Lemma 2.5. Hence the map $\alpha \otimes T_{\mathfrak{p}}$ is isomorphic with multiplication by Y_n on $T_{\mathfrak{p}}/(F_1, \ldots, F_m)T_{\mathfrak{p}}$, which is injective. So $K_{\mathfrak{p}} = 0$. Thus supp $K \subseteq V(J(X, Y) + (Y_1, \ldots, Y_n))$ which has codimension $m+1$ in Spec T.

Further $C \simeq \tilde{I}/(Y_n) + \tilde{J}$ which has support $V((Y_n) + J(Y', X'))$, with $Y' = (Y_1, \ldots, Y_{n-1})$ and X' is the $m \times (n-1)$-matrix with entries X_{ij}; $1 \le i \le m$, $1 \le j \le n-1$. Hence codim Supp $C = m+1$. T is a regular unramified local ring, moreover R and C are T-modules which have an improper intersection, so $\mathscr{X}(R, C) = 0$ by [14]. Analogously $\mathscr{X}(R, K) = 0$. The sequence $0 \to K \to T/J(X, Y) \to \tilde{I}/\tilde{J} \to C \to 0$ is exact. Thus $\mathscr{X}(R, T/J(Y, X)) = \mathscr{X}(R, \tilde{I}/\tilde{J})$ and we have proved the proposition.

5. APPLICATIONS

Theorem 2.1 and the Propositions 3.3 and 3.8 are used to give a connection between algebraic and topological invariants of germs of analytic functions $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with a one dimensional singular locus Σ.

Let J_f be the Jacobi ideal of f, i.e.

$$J_f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)\mathcal{O}$$

where $\partial f/\partial z_i$ is a partial derivative of f and $\mathcal{O}$ is the local ring of germs of analytic functions on $(\mathbb{C}^{n+1}, 0)$. Let I be the radical ideal of J_f. If $\dim_{\mathbb{C}}(I/J_f)$ is finite then this number is equal to the number of so called A_1 and D_∞ singularities in a generic deformation of f. This answers a conjecture of Siersma, see [11]. These A_1 and D_∞ singularities are used by Siersma [15] to give the homotopy type of the Milnor fibre of f if moreover Σ is a complete intersection.

By means of the Gauss-Manin connection of f, Van Straten [16] was able to compute the Betti numbers of the Milnor fibre of f in case Σ is a complete intersection, using the explicit isomorphism given by Lemma 3.1. In the article of Mond [10] a quotient of two ideals appears, which has as support the double point locus of f. This quotient is a perfect module by Theorem 2.1. One can also use a result of Artin and Nagata [1] and corrected by Huneke [8] to conclude that the quotient I/J is a Cohen-Macaulay module in the cases treated in section 3.

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