Contemporary Mathematics
Volume **90**, 1989

SERIES OF ISOLATED SINGULARITIES

Ruud Pellikaan

INTRODUCTION. We start with quoting two remarks of V.I. Arnol'd: "Although the series undoubtedly exist, it is not all clear what a series of singularities is" [A] page 153.

"However a general definition of series of singularities is not known. It is only clear that the series are associated with singularities of infinite multiplicity (for example $D \sim x^2y$, $T \sim xyz$), so that the hierarchy of series reflects the hierarchy of non-isolated singularities" [A] page 154.

We do not give a definition of series of singularities either, but start to look at functions with a one dimensional singular locus Σ. We characterize those functions which have constant transversal R-type at every point of $\Sigma \setminus \{0\}$. We show that hypersurfaces with prescribed transversal singularities exist. We look at $f + x^{k+1}$, where x is a coordinate function such that $\Sigma \cap V(x) = \{0\}$. For $k >> 0$, $f + x^{k+1}$ has an isolated singularity. We give a formula for the Milnor number μ_k of $f + x^{k+1}$ in algebraic terms. Yomdin [Y] and Lê [Lê] proved a formula which relates the Euler characteristic of f and $f + x^{k+1}$, in greater generality. We combine these two results to get a formula for the Euler characteristic of the Milnor fibre of f.

$\mathcal{O}$ denotes the local ring of germs of analytic functions f: $(\mathbb{C}^{n+1},0) \to \mathbb{C}$, and $\mathcal{M}$ denotes the maximal ideal of $\mathcal{O}$.

0271-4132/89 $1.00 + $.25 per page

Sometimes in proofs we take representatives of the germs on an open neighbourhood U of 0 in $\mathbb{C}^{n+1}$ and then $\mathcal{O}$ is the sheaf of analytic functions on U.

If local coordinates $z_0,z_1, ..., z_n$ are chosen of $(\mathbb{C}^{n+1},0)$ then $f_0,f_1, ..., f_n$ denote the partial derivatives of f with respect to $z_0,z_1, ..., z_n$. We let $J_f = (f_0,f_1, ..., f_n)\mathcal{O}$ be the Jacobi ideal of f.

For an $\mathcal{O}$-module M we define $\ell(M) = \dim_{\mathbb{C}}(M)$.

For basic definitions and results in commutative algebra we refer to Matsumura [Ma].

ACKNOWLEDGEMENT. This work was done at the University of Utrecht and was part of the author's thesis [P,1]. I want to thank Prof. Dr. D. Siersma for his ideas and stimulating support. This work was supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research ZWO. The Free University of Amsterdam supported the travel to the conference on singularities at Iowa City.

§1. THE TRANSVERSAL TYPE OF f ALONG A CURVE

Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus $(\Sigma,0)$. We analyse the transversal type of f along Σ, with respect to different equivalence relations for isolated singularities, i.e. with respect to R-, K- and μ-type. We shall give a necessary and sufficient condition in the case the transversal R-type of f does not vary along Σ.

DEFINITION 1.1. Let $f,g\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be two germs of analytic functions, then f and g are called R (= *right*) *equivalent* if there exists a germ $h\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ of a local analytic isomorphism such that $g = f \circ h$. They are called K (= *contact*) *equivalent* if

$(f^{-1}(0),0)$ and $(g^{-1}(0),0)$ are isomorphic germs of analytic spaces. If moreover f and g have an isolated singularity at 0, then they are called *μ-homotopic*, if there exists a family $\{f_t\}$, $t \in [0,1]$, of germs f_t: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ of analytic functions with an isolated singular point at 0, such that the Milnor number $\mu(f_t,0)$ is constant, and $f_0 = f$, $f_1 = g$, and F: $\mathbb{C}^{n+1} \times [0,1] \to \mathbb{C}$, defined by $F(z,y) = f_t(z)$ is continuous.

REMARK 1.2. These relations define equivalence relations and the corresponding classes we call *R-class*, *K-class* and *μ-class* resp. or *R*-, *K*- and *μ-type* resp. The following inclusions of classes of a given function exist: R-class $\subset$ K-class $\subset$ μ-class.

DEFINITION 1.3. Let f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a singular locus $(\Sigma,0)$, such that $(\Sigma,0)$ is a germ of a non-singular curve. We say that the transversal *R-type of f along Σ at 0 is constant*, if for every choice of local coordinates x, y_1, ..., y_n of $(\mathbb{C}^{n+1},0)$ such that $\Sigma = V(y_1, ..., y_n)$, the R-type of f_t: $(\mathbb{C}^n,0) \to (\mathbb{C},0)$, with $f_t(y) = f(t,y)$ is the same for all t sufficiently small.

In the same way one defines what it means that the K-type (or μ-type) of f along Σ at 0 is constant.

PROPOSITION 1.4. *Let* f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an analytic function with a one dimensional singular locus* $(\Sigma,0)$. *Let* $(\Sigma_1,0)$, ..., $(\Sigma_r,0)$ *be the branches of* $(\Sigma,0)$. *The transversal μ-type of f along* Σ_i *is constant at all points of a punctured neighbourhood of* 0 *in* Σ_i.

PROOF. See [Y] or [Lê] (1.3.1) and (1.3.2).

REMARK 1.5. As a consequence we can associate to every branch $(\Sigma_i,0)$ a well-defined μ-class of an isolated singularity, which is the transversal μ-type of f along $\Sigma_i \setminus \{0\}$. We shall give some examples to show that the situation with respect to transversal R- and K-type is quite different.

EXAMPLE 1.6. Let $f(x,y,z) = y^4 + xy^2z^2 + z^4$, then the singular locus Σ of f is $V(y,z)$. The transversal μ-type is $\tilde{E}_7$ with Milnor number 9. The transversal K-type and R-type vary, since the cross ratio of the four lines of $f_x^{-1}(0)$ varies along Σ.

EXAMPLE 1.7. Let $f(x,y,z) = y^5 + (1 + x)y^2z^2 + z^5$, then the singular locus Σ of f is the x-axis. The transversal μ-type and K-type are constant along Σ ($x \neq -1$) and of type $T_{5,5,2}$. But the transversal R-type of f along Σ varies.

EXAMPLE 1.8. Let $f(x,y,z) = y^6 + xy^3z^3 + z^6$, then the singular locus of f is the x-axis. Take $\bar{x} = x - y$, then $\bar{f}(\bar{x},y,z) = y^6 + (\bar{x} + y)y^3z^3 + z^6$. The transversal K-type of f along Σ with respect to the coordinates x, y, z is quasi-homogeneous, so $\mu(f_x,0) = \tau(f_x,0)$ the Tjurina number. But for the transversal K-type of f along Σ with respect to the coordinates $\bar{x}$, y, z one has $\mu(\bar{f}_{\bar{x}},0) > \tau(\bar{f}_{\bar{x}},0)$. Hence the transversal K- (and R-) type of f depends on the chosen local coordinates.

DEFINITION 1.9. The *analytic stratum* of a germ of an analytic space $(X,0)$ in $(\mathbb{C}^m,0)$ is the set-germ A of points in $(\mathbb{C}^m,0)$ along which a representative X of $(X,0)$ is trivial:

$$A = \{a \in (\mathbb{C}^m,0) \mid (X,a) \cong (X,0)\} \subseteq (\mathbb{C}^m,0)$$

where $\cong$ means isomorphism of germs of analytic spaces. See also [G-H].

The following theorem is due to Ephraim [E].

THEOREM 1.10. *Let $(X,0)$ be a germ of an analytic space in $(\mathbb{C}^m,0)$.*

1) *The analytic stratum $(A,0)$ of $(X,0)$ is the germ of an analytic manifold in $(\mathbb{C}^m,0)$.*
2) *The germ $(X,0)$ is a product along A:*

$$(X,0) \cong (X_0,0) \times A,$$

for some germ of an analytic space $(X_0,0)$ *in* $(\mathbb{C}^m,0)$.

An analogous proposition holds for R-equivalence of functions.

PROPOSITION 1.11. *Let* f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an analytic function with a one dimensional singular locus* Σ, *such that* $(\Sigma,0)$ *is a non-singular curve.*

a) *Then the following statements are equivalent:*
   1) *There are local coordinates* x, y_1, ..., y_n *such that* $f(x,y) = f(0,y)$ *and* $\Sigma = V(y_1,...,y_n)$.
   2) *There are local coordinates* $\overline{x}$, $\overline{y}_1$, ..., $\overline{y}_n$ *such that* $f_{\overline{x}}(\overline{y}) = f(\overline{x},\overline{y})$ *and* $\Sigma = V(\overline{y}_1,...,\overline{y}_n)$ *and the R-type of* $f_{\overline{x}}$ *is constant for all* $\overline{x}$ *sufficiently small.*
   3) *The derivative* $(\partial f/\partial z_0)$ *is an element of* $(\partial f/\partial z_1 , ..., \partial f/\partial z_n)$ *for any choice* $z_0,z_1,...,z_n$ *of local coordinates of* $(\mathbb{C}^{n+1},0)$ *such that* $V(z_0)$ *intersects* Σ *transversally at* 0.

b) *Moreover, if either* a) 1), 2) *or* 3) *holds then the transversal R-type of* f *along* Σ *at* 0 *is constant.*

See for the proof [P,1] Proposition 9.12. From now on we denote $\partial f/\partial z_i$ by f_i.

PROPOSITION 1.12. *Let* f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an analytic function with a one dimensional singular locus* $(\Sigma,0)$. *Let* $z_0,z_1, ..., z_n$ *be local coordinates of* $(\mathbb{C}^{n+1},0)$, *such that* $\Sigma \cap V(z_0) = \{0\}$. *Then the following two statements are equivalent:*

1) $\dim_{\mathbb{C}}(\mathcal{O}/J_f + ((f_1, ..., f_n): f_0)) < \infty$.
2) *The transversal R-type of* f *along* Σ_i *is constant at all points of a punctured neighbourhood of* 0 *in* Σ_i, *for every branch* Σ_i *of* Σ.

PROOF. Let U be an open neighborhood of 0 in $\mathbb{C}^{n+1}$, such that Σ and f are defined on U and $\Sigma \setminus \{0\}$ is non-singular. This is possible, since Σ is one dimensional. Let $\mathcal{O}$ be the sheaf of analytic functions on U. Then $\dim_{\mathbb{C}}(\mathcal{O}/J_f + (f_1, ..., f_n): f_0)) < \infty$ is equivalent with $V(J_f + ((f_1, ..., f_n): f_0)) = \{0\}$, after possibly shrinking U. Further $V(J_f + ((f_1, ..., f_n): f_0)) = \Sigma \cap V((f_1, ..., f_n): f_0)$.

The hyperplane $H_t = V(z_0 - t)$ intersects Σ transversally in finitely many points, for all $t \neq 0$ sufficiently small, since $\Sigma \cap V(z_0) = \{0\}$. Let $p \in \Sigma_i \cap H_t$, with $t \neq 0$. Then $p \notin V(f_1, ..., f_n): f_0)$ if and only if $f_0 \in (f_1,...,f_n)\,\mathcal{O}_p$ if and only if the transversal R-type of f along Σ_i at p is constant, by Proposition 1.11. thus $\Sigma \cap V(f_1, ..., f_n): f_0) = \{0\}$ is equivalent with 2). Hence 1) and 2) are equivalent. This proves the proposition.

§2. CERTAIN NON-ISOLATED HYPERSURFACE SINGULARITIES EXIST

THEOREM 2.1. *Let* $(\Sigma,0)$ *be a germ of a reduced, equidimensional analytic space in* $(\mathbb{C}^{n+1},0)$, *with irreducible components* $\Sigma_1, ..., \Sigma_r$ *of dimension k. Let* $K_1, ..., K_r$ *be K-classes of isolated hypersurface singularities in* $(\mathbb{C}^n,0)$. *Then there exists a germ of an analytic function* $f: (\mathbb{C}^{n+k},0) \to (\mathbb{C},0)$ *with singular locus* Σ *and there exist open dense subsets* U_i *of* Σ_i *such that the transversal K-type of f is constant and equal to* K_i *at every point of* U_i.

REMARK 2.2 1) If $(\Sigma,0)$ is a curve then the open dense subset U_i of Σ_i is a punctured neighbourhood of 0 in Σ_i. 2) The theorem above was posed in [P,1] as a question and proved in the case $K_1 = ... = K_r = A_1$.

In order to prove this theorem we need Bertini's Theorem and a lemma which is an exercise in prime avoidance. Bertini's Theorem is well-known and is a consequence of Sard's Theorem. We omit the proof.

BERTINI'S THEOREM 2.3. *Let* $f, \varphi_1, ..., \varphi_p$ *be germs of analytic functions from* $(\mathbb{C}^m,0)$ to $(\mathbb{C},0)$ *and define* $f_\lambda = f + \Sigma\lambda_i\varphi_i$ *for* $\lambda = (\lambda_1, ..., \lambda_p) \in \mathbb{C}^p$. *Then the singular locus of* $f_\lambda^{-1}(0)$ *is contained in* $V(\varphi_1, ..., \varphi_p)$ *for all* $\lambda \in U$, where *U is a dense subset of* $\mathbb{C}^p$.

The proof of the following lemma is probably known but we could not find a proof.

LEMMA 2.4. *Let R be a Noetherian commutative regular ring and I a radical ideal in R such that the associated primes $p_1, \ldots, p_r$ of I have the same height n. Then there exist $g_{i1}, \ldots, g_{in} \in p_i$ such that*

(i) $(g_{i1}, \ldots, g_{in})R_{p_i} = I_{p_i}$

(ii) $g_{kj} \notin p_i$ *for all j and $i \neq k$*

PROOF. I is a radical ideal in a Noetherian ring R, hence I has a prime decomposition $I = p_1 \cap \ldots \cap p_r$, with p_i a prime ideal and $p_i \not\subset p_j$ for all $i \neq j$. All prime ideals p_i have the same height n. Hence by induction we can construct a sequence of elements $g_{i1}, \ldots, g_{in}$ in p_i such that $g_{i1} \in p_i \setminus (\bigcup_{j \neq i} p_j \cup p_i^2)$ and

$$g_{ik+1} \in p_i \setminus \Big[\bigcup_{j \neq i} p_j \cup p_i^2 \cup (R \cap (g_{i1}, \ldots, g_{ik})R_{p_i}) \Big]$$

for $i = 1, \ldots, r$ and $1 \leq k < n$, see [Ma] 1.B. Let k_i be the residue field of the local ring R_{p_i}. Then by construction $g_{i1}, \ldots, g_{in}$ are linear independent over k_i. The localization R_{p_i} of the regular ring R is regular again. Thus R_{p_i} is a regular local ring of Krull dimension = height $p_i = n$.

So $(g_{i1}, \ldots, g_{in})R_{p_i} = p_i R_{p_i} = I_{p_i}$. And $g_{k_j} \notin p_i$ for all j and $i \neq k$, by construction. This proves the lemma

PROOF OF THEOREM 2.1. 1) Remark that a function f: $(\mathbb{C}^n,0) \to (\mathbb{C},0)$ with an isolated singularity and Milnor number μ is $(\mu+2)$-determined, see [M].

2) Let I be the ideal, defining Σ, in the local ring $\mathcal{O}$ of germs of analytic functions on $(\mathbb{C}^{n+k},0)$. I is radical and all associated primes have the same height n, since by assumption Σ is reduced and equidimensional and of dimension k. From the proof of Lemma 2.4 we conclude that there exist functions $g_{i1}, \ldots, g_{in}$ in $\mathcal{O}$ for $i = 1, \ldots, r$ such that

i) $g_{i1}, \ldots, g_{in}$ are part of local coordinates at every point of an open dense subset V_i of Σ_i.

ii) g_{kj} is not identically zero on Σ_i for all j and all $k \neq i$. Let $U_i = V_i \setminus \cup\{V(g_{kj}) \mid 1 \leqslant k \neq i \leqslant r$ and $1 \leqslant j \leqslant n\}$ then U_i is an open dense subset of Σ_i and $g_{i1}, ..., g_{in}$ are part of local coordinates at every point of U_i.

3) Let f_i: $(\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a function with an isolated singularity at 0 and K-type equal to K_i. Let μ_i be the Milnor number of f_i, it depends only on K_i. Define $N = \max\{\mu_i + 2 \mid 1 \leqslant i \leqslant r\}$ and let

$$F_0 = \sum_{i=1}^{r} \prod_{k \neq i} g_{k1}^N f_i(g_{i1}, ..., g_{in}).$$

Let $\varphi_1, ..., \varphi_p$ be generators of the ideal I^N and let:

$$F_\lambda = F_0 + \Sigma\lambda_i\varphi_i.$$

Then by Bertini's theorem, there exists a λ such that F_λ: $(\mathbb{C}^{n+k},0) \to (\mathbb{C},0)$ has a singular locus contained in $V(I^N) = \Sigma$.

4) At every point a of U_i and transversal slice (S,a) at (Σ_i,a), the functions $g_{i1}, ..., g_{in}$ restricted to S are local coordinates of (S,a) by 2). Let τ: $(\mathbb{C}^n,0) \to (\mathbb{C}^{n+k},a)$ be the embedding with image (S,a) such that $\tau^*(g_{ij}) = y_j$ and $y_1, ..., y_n$ the standard coordinates of $\mathbb{C}^n$. Consider $F_\lambda o\tau$: $(\mathbb{C}^n,0) \to (\mathbb{C},0)$ and let

$$G = \sum_{j \neq i} \prod_{k \neq j} g_{k1}^N f_j(g_{j1}, ..., g_{jn}) + \Sigma\lambda_i\varphi_i \quad \text{and}$$

$$u = \prod_{k \neq i} g_{k1}^N.$$

Then $F_\lambda o\tau = (uo\tau)\cdot f_i + Go\tau$. Of course, u and G depend on i and τ depends on i and a. Now $Go\tau \in (y_1, ..., y_n)^N$ and $N \geqslant \mu_i + 2$ hence $F_\lambda o\tau$ is R-equivalent with $(uo\tau)f_i$ by 1). Further, $uo\tau$ is a unit, since g_{k1} is not zero at a for all $k \neq i$, by definition of u_i. Hence $F_\lambda o\tau$ is K-equivalent with f_i.

5) F_λ is singular at all points of Σ, hence the singular locus of F_λ is equal to Σ, by 3).

This proves the theorem.

REMARK 2.10. It is still an open question whether we can replace the K-classes by R-classes.

§3. SERIES OF ISOLATED SINGULARITIES

Let $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ define a germ of an analytic function with a one dimensional singular locus Σ. Let $z_0, z_1, ..., z_n$ be local coordinates of $(\mathbb{C}^{n+1},0)$ such that $\Sigma \cap V(z_0) = \{0\}$. In this section we show that $f + z_0^k$ has an isolated singularity at 0 for $k >> 0$. We shall give a formula for its Milnor number in case the transversal R-type of f at every point of $\Sigma_i \setminus \{0\}$ is constant for every branch Σ_i of Σ.

PROPOSITION 3.1. *Let* $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an analytic function with singular locus* Σ *of dimension* d. *Suppose* $z_0, ...,$ z_n *are local coordinates such that* $\Sigma \cap V(z_0, ..., z_{d-1}) = \{0\}$ *as a germ at* 0. *Then* $V(f_d, ..., f_n)$ *is a complete intersection of dimension* d.

PROOF. Define $f_\lambda = f + \Sigma_{i=0}^{d-1}\lambda_i z_i^2$. Then by Bertini's theorem there exists a λ such that $f_\lambda^{-1}(0)$ has a singular locus Σ_λ contained in $V(z_0^2, ..., z_{d-1}^2)$. So $\Sigma_\lambda = \Sigma_\lambda \cap V(z_0,...,z_{d-1})$. Hence

$$\Sigma_\lambda = V(f_0 + 2\lambda_0 z_0, ..., f_{d-1} + 2\lambda_{d-1}z_{d-1}, f_d, ..., f_n) \cap V(z_0, ..., z_{d-1})$$

$$= V(f_0, ..., f_n) \cap V(z_0, ..., z_{d-1}) = \Sigma \cap V(z_0, ..., z_{d-1})$$

which is equal to $\{0\}$ by assumption. Thus

$$f_0 + 2\lambda_0 z_0, ..., f_{d-1} + 2\lambda_{d-1}z_{d-1}, f_d, ..., f_n$$

is an $\mathcal{O}$-regular sequence and a fortiori $f_d, ..., f_n$ is an $\mathcal{O}$-regular sequence. Hence $V(f_d, ..., f_n)$ is a complete intersection of dimension d.

REMARK 3.2. There is a tedious proof of this proposition in [P,1], in the case $d = 1$.

PROPOSITION 3.3. *Let* $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an* *analytic function with a one dimensional singular locus* Σ *and* z_0 *a coordinate such that* $\Sigma \cup V(z_0) = \{0\}$. *Then* $f + z_0^k$ *defines an isolated singularity for* $k >> 0$.

PROOF. The proof is the same as in [Lê] and is due to B. Teissier. A stronger condition is required there: $V(z_0)$ is assumed to be an admissible hyperplane, i.e. $f^{-1}(0) \cap V(z_0)$ has an isolated singularity.

Let $\Gamma = V(f_1, ..., f_n)$, then Γ is one dimensional, by Proposition 3.1, since $\Sigma \cap V(z_0) = \{0\}$ and Σ is one dimensional. Let $p_1, ..., p_s$ be the minimal prime ideals in $\mathcal{O}$ lying over $(f_1, ..., f_n)\mathcal{O}$. Let $\Gamma_i = V(p_i)$, then $\Gamma_1, ..., \Gamma_s$ are the branches of Γ. There is for every i at most one k such that $f_0 + kz_0^{k-1} \in p_i$. Otherwise, there are k and l with $k < l$ such that $f_0 + kz_0^{k-1}$, $f_0 + z_0^{l-1} \in p_i$. Hence $z_0^{k-1}(k + lz_0^{l-k}) \in p_i$. So $z_0^{k-1} \in p_i$, since $(k + lz_0^{l-k})$ is a unit in $\mathcal{O}$. But then also $f_0 \in p_i$ and we have that $(z_0^k, f_0, f_1, ..., f_n) \subseteq p_i$. Thus $\Gamma_i \subseteq \Sigma \cap V(z_0) = \{0\}$ and this is a contradiction. Therefore there exists a k_0 such that for all $k \geqslant k_0$ and all i we have that $f_0 + kz_0^{k-1} \notin p_i$ so $f_0 + kz_0^{k-1} \notin p_1 \cup ... \cup p_s = \cup \operatorname{Ass}(\mathcal{O}_\Gamma)$. Hence $f_0 + kz_0^{k-1}$ is not a zero-divisor of $\mathcal{O}_\Gamma$. Γ is one dimensional hence

$$V(f_0 + kz_0^{k-1}, f_1, ..., f_n) = \{0\}$$

and $f + z_0^k$ has an isolated singularity for all $k \geqslant k_0$.

NOTATION 3.4. Let $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ. Let $J = J_f = (f_0, f_1, ..., f_n)\mathcal{O}$ be the Jacobi ideal of f in $\mathcal{O}$. Let J_k be the Jacobi ideal of $f + (1/k+1)z_0^{k+1}$ and let μ_k be its Milnor number, i.e., $\mu_k = \ell(\mathcal{O}/J_k)$.

Let $J = q_0 \cap q_1 \cap ... \cap q_r$ be a primary decomposition of J, where q_0 is a $\mathcal{M}$-primary component of J, if J has one, and take $q_0 = \mathcal{O}$ otherwise. The primary components $q_1, ..., q_r$ are uniquely defined for the given J. Let $I = q_1 \cap ... \cap q_r$. The singular locus Σ of f is one dimensional. So $\mathcal{O}/I$ has krull dimensional one and has no embedded component and $\Sigma = V(I)$. We give Σ the analytic structure defined by I, i.e., $\mathcal{O}_\Sigma = \mathcal{O}/I$. Let $z_0, z_1, ..., z_n$ be local coordinates of $\mathbb{C}^{n+1}$ at 0. Take $x = z_0$ and suppose $\Sigma \cap V(x) = \{0\}$. This implies that x is not a zero-divisor on $\mathcal{O}_\Sigma$.

In such a case is $e_x(\Sigma)$, the multiplicity of Σ with respect to x at 0, equals $\ell(\mathcal{O}_\Sigma/(x))$ and $ke_x(\Sigma) = \ell(\mathcal{O}_\Sigma/(x^k))$.

In this situation we have the following theorem.

THEOREM 3.5. *Let* f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ *be a germ of an analytic function with a one dimensional singular locus* Σ. *Let* $x,z_1, ..., z_n$ *be local coordinates such that* $\Sigma \cap V(x) = \{0\}$. *Suppose the transversal R-type of* f *is constant at every point of a punctured neighbourhood of* 0 *in* Σ_i, *for every branch* Σ_i *of* Σ. *Then, the Milnor number* μ_k *of* $f + (1/k+1)z_0^{k+1}$ *for* $k >> 0$ *is equal to*

$$\mu_k = \ell(I/J) + ke_x(\Sigma) + \ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)).$$

For the proof of this theorem we need several lemma's.

LEMMA 3.6. $\mu_k = \ell(I/I \cap J_k) + ke_x(\Sigma)$ *for* $k >> 0$.

PROOF. The following sequence is exact:

$$0 \to (I + J_k)/J_k \to \mathcal{O}/J_k \to \mathcal{O}/(I + J_k) \to 0,$$

and consists for $k >> 0$ of $\mathcal{O}$-modulus of finite length, by Proposition 3.3. Now $I/I \cap J_k \cong (I + J_k)/J_k$ and $\mu_k = \ell(\mathcal{O}/J_k)$. Furthermore $I + J_k = I + x^k)$, thus $\ell(\mathcal{O}/I + J_k) = ke_x(\Sigma)$ by 3.4.

LEMMA 3.7. *Suppose* $\ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)) < \infty$ *then* $I \cap J_k = f_0.I + (f_1,...,f_n)$.

PROOF. Let $J' := f_0.I + (f_1, ..., f_n)\mathcal{O}$. Suppose $a \in I \cap J_k$ then $a = \Sigma a_i f_i + a_0 x^k \in I$ for some $a_0,a_1, ..., a_n \in \mathcal{O}$. Now $J \subseteq I$, hence $a_0 x^k \in I$, so $a_0 \in I$, since x is not a zero- divisor of $\mathcal{O}/I$.

Since $J = I \cap q_0$ with q_0 an $\mathcal{M}$-primary ideal or equal to $\mathcal{O}$. We have that $\ell(I/J) < \infty$. So we can find $n_1 \in \mathbf{N}$ such that $x^{n_1}I \subseteq J$. So, for all $k \geq n_1$, $a_0 x^k \in (f_0,f_1, ..., f_n)\mathcal{O}$. We assumed

$$\ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)) < \infty,$$

so there exists $n \in \mathbf{N}$ such that

$$x^{n_2} \in I + ((f_1, ..., f_n): f_0),$$

hence $x^{n_2}f_0 \in f_0.I + (f_1, ..., f_n)\mathcal{O}$. Therefore $a_0x^k \in J'$ for $k \geq n_1 + n_2$. Thus $a \in J'$ and we have proved $I \cap J_k \subseteq J'$ for $k \geq n_1 + n_2$. There exists $n_3 \geq N$ such that $x^{n_3}I \subseteq J'$, since $(I/I \cap J_k) < \infty$ for $k >> 0$, by Lemma 3.6, and $I \cap J_k \subset J' \subset I$ for $k \geq n_1 + n_2$. But then

$$J' \subseteq (f_0 + x^k)I + (f_1, ..., f_n)\mathcal{O} + x^k.I \subseteq I \cap J_k + x^k.I \subseteq I \cap J_k + MJ'$$

for $k \geq n_3 + 1$. Hence $J' \subseteq I \cap J_k$ for $k \geq n_3 + 1.$, by Nakayama's lemma. Thus $J' = I \cap J_k$ for all $k >> 0$.

PROOF OF 3.5. The transversality assumption implies by Proposition 1.12 $\ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)) < \infty$. So we can apply Lemmas 3.6 and 3.7.

$$I \cap J_k = f_0.I + (f_1, ..., f_n)\mathcal{O} = J'$$

The module J/J' is generated by f_0 and one easily shows that the annihilator of J/J' is $I + ((f_1, ..., f_n): f_0)$. So $J/I \cap J_k = J/J' \cong \mathcal{O}/I + ((f_1, ..., f_n): f_0)$. The following sequence is exact

$$0 \to J/I \cap J_k \to I/I \cap J_k \to I/J \to 0$$

Combining this with Lemma 3.6 we get the desired result:

$$\mu_k = \ell(I/J) + ke_x(\Sigma) + \ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)) \quad \text{for} \quad k >> 0.$$

§4. HYPERSURFACES WITH A COMPLETE INTERSECTION CURVE AS SINGULAR LOCUS

Let $f: (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional locus Σ, which is a complete intersection. Suppose

f has A_1 singularities transversally to Σ. In this situation we give a more detailed formula for the Milnor number of $f + x^k$ and a formula for the Euler characteristic of the Milnor fibre of f.

NOTATION 4.1. It follows from [P,1] that the ideal I, defining Σ, and f have the following properties. Since f has transversal A_1-singularities $I = \mathrm{rad}(J_f)$.

The ideal I is generated by an $\mathcal{O}$-sequence $g_1, \ldots, g_n$, since Σ is complete intersection. Moreover, $f \in I^2$, hence $f = \Sigma\, h_{kl}g_kg_l$ for some $h_{kl} \in \mathcal{O}$ with $h_{kl} = h_{lk}$.

Define $\delta_f = \ell(\mathcal{O}_\Sigma/\det(h_{kl}))$, then δ_f is finite.

THEOREM 4.2. *Let f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ, which is a complete intersection. Suppose f has A_1 singularities transversally to Σ. Let $x,z_1, \ldots, z_n$ be local coordinates such that $\Sigma \cap V(x) = \{0\}$. Then*

$$\ell(\mathcal{O}/I + ((f_1, \ldots, f_n)\colon f_0)) = \delta_f + \mu(\Sigma,0) + e_x(\Sigma) - 1$$

and the Milnor number μ_k of $f + (1/k+1)x^{k+1}$ is equal to

$$\ell(I/J) + \delta_f + \mu(\Sigma,0) + (k+1)e_x(\Sigma) - 1, \quad k >> 0$$

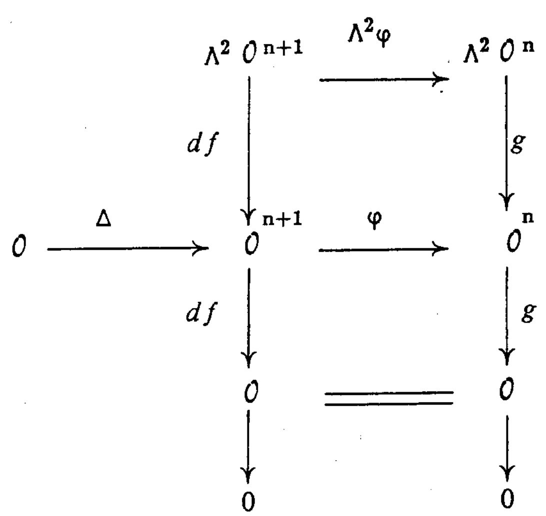
PROOF. The second formula follows from the first one, by Theorem 3.5, since the transversal R-type is A_1 on every branch of Σ. We can write $f = \Sigma h_{kl}g_kg_l$ with $h_{kl} = h_{lk}$, so $f_i = \Sigma\varphi_{ik}g_k$ for $i = 0,1, \ldots, n$, where

$$\varphi_{ik} \equiv 2\Sigma h_{kl} \frac{\partial g_l}{\partial z_i} \pmod{I}.$$

Let φ be the $((n+1) \times n)$-matrix with entries φ_{ik} and φ^i the $(n \times n)$-matrix obtained from φ by deleting the $(i+1)$th row, for $i = 0,1, \ldots, n$. Let $\Delta_i = (-i)^i\det \varphi^i$. We need the following lemma.

LEMMA 4.3. $((f_1, \ldots, f_n)\colon f_0) = (\Delta_0, f_1, \ldots, f_n)\mathcal{O}$.

PROOF. Consider the commutative diagram:



where the columns are the beginnings of the Koszul complexes of the sequence $df = (f_0, f_1, \ldots, f_n)$ resp. $g = (g_1, \ldots, g_n)$. The map Δ: $\mathcal{O} \to \mathcal{O}^{n+1}$ has components $\Delta_0, \ldots, \Delta_n$, with $\Delta_i = (-1)^i \det(\varphi^i)$. The complex $\mathbf{K}(g,\varphi)$ constructed in [P,2] has components $K_0(g,\varphi) = \mathcal{O}^n$, $K_1(g,\varphi) = \mathcal{O}^{n+1} \oplus \Lambda^2 \mathcal{O}^n$ and $K_3(g,\varphi) = \mathcal{O} \oplus \Lambda^2 \mathcal{O}^{n+1} \oplus \Lambda^3 \mathcal{O}^n$. The support of I/J is $\{0\}$, hence of codimension $n + 1$, so grade $(I/J) = n + 1$ and the complex $\mathbf{K}(g,\varphi)$ is exact, by [P,2]. $\Sigma \Delta_i f_i = g\varphi\Delta = 0$, so $\Delta_0 \in ((f_1, \ldots, f_n): f_0)$. Diagram chasing and using the fact that the right column is exact, since $g_1, \ldots, g_n$ is a $\mathcal{O}$-regular sequence, one obtains $((f_1, \ldots, f_n): f_0) \subseteq (\Delta_0, f_1, \ldots, f_n)\mathcal{O}$. So they are equal. This proves Lemma 4.3.

We continue the proof of Theorem 4.2.

$I + ((f_1, \ldots, f_n): f_0) = I + (\Delta_0)\mathcal{O}$, by Lemma 4.3, since $(f_1, \ldots, f_n)\mathcal{O} \subseteq I$. The derivative dg is an $((n+1) \times n)$-matrix. Let $(dg)^0$ be the $(n \times n)$-matrix obtained from dg by deleting the first row. Let h be the matrix with entries h_{kl}. Then $\varphi^0 \equiv 2h(dg)^0 \pmod{I}$

$$\Delta_0 = \det(\varphi^0) \equiv 2^n \det(h) \cdot \det(dg)^0 \pmod{I}.$$

Thus $\ell(\mathcal{O}/I + (\Delta_0)) = \ell(\mathcal{O}/I + (\det(h))) + \ell(\mathcal{O}/I + (\det(dg)^0))$, by [K], V, §2, exercise 2c. The first term at the right hand side is by definition δ_f. The map $\tilde{g} = (x, g_1, \ldots, g_n)$: $(\mathbb{C}^{n+1},0) \to (\mathbb{C}^{n+1},0)$ defines a zero-dimensional complete intersection with Milnor number $\mu(\tilde{g})$ and $\mu(\tilde{g}) = e_x(\Sigma) - 1$. Further $\ell(\mathcal{O}/I + (\det(dg)^0)) = \mu(\Sigma,0) + \mu(\tilde{g})$, see [Lo] (5.11). Hence $\ell(\mathcal{O}/I + ((f_1, ..., f_n): f_0)) = \delta_f + \mu(\Sigma,0) + e_x(\Sigma) - 1$. This proves the theorem.

PROPOSITION 4.4. *Let f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ. Suppose f as A_1 singularities transversally to Σ. Let x be a coordinate such that $f^{-1}(0) \cap V(x)$ has an isolated singularity at 0. Then*

$$\chi(F) = \chi(F_k) + (-1)^{n+1}(k+1)e_x(\Sigma) \quad \text{for } k >> 0,$$

where F and F_k are the Milnor fibres of f resp. $f + (1/k+1)x^{k+1}$ and χ denotes the Euler characteristic.

PROOF. See [Y] or [Lê] (2.2.2). A general theorem is proved by Yomdin and Lê with arbitrary transversal singularities.

PROPOSITION 4.5. *Let f: $(\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of an analytic function with a one dimensional singular locus Σ, which is a complete intersection. Suppose f has A_1 singularities transversally to Σ. Then*

$$\chi(F) = 1 + (-1)^n[\ell(I/J) + \delta_f + \mu(\Sigma,0) - 1]$$

PROOF. The Milnor fibre of an isolated hypersurface singularity is a bouquet of μ n-spheres, where μ is the Milnor number. So $\chi(F_k) = 1 + (-1)^n\mu_k$. If we choose a general coordinate function x then $f^{-1}(0) \cap V(x)$ has an isolated singularity. Combining the formulas for μ_k in Theorem 4.2 and for $\chi(F)$ in Proposition 4.4 with the above remark, we get the desired result.

REMARK 4.6. a) In the situation of Proposition 4.5 there is (f_t,Σ_t) a deformation of the pair (f,Σ) such that $f_t^{-1}(0)$ has Σ_t as singular locus and Σ_t is a nonsingular curve, i.e. the Milnor fibre of Σ, moreover f_t has only critical points of type A_1 outside Σ_t and only so called A_∞ and D_∞ singularities on Σ_t, for all $t \neq 0$.

Furthermore

$$\delta_f = \#\{D_\infty \text{ points of } f_t\}$$
$$\ell(I/J) = \#\{D_\infty \text{ points of } f_t\} + \#\{A_1 \text{ points of } f_t\},$$

see [P,1]. Siersma [S,2] computed the homotopy type of the Milnor fibre F of f and has the following result:

(i) F has the homotopy type of a bouquet of b_n n-spheres where $b_n = \ell(I/J) + \delta_f + \mu(\Sigma,0) - 1$, in case $\delta_f > 0$.

(ii) F has the homotopy type of a bouquet of one $(n-1)$-sphere and b_n n-spheres, where $b_n = \ell(I/J) + \mu(\Sigma,0)$, in case $\delta_f = 0$.

b) Van Straten [St.1] was able to compute the Betti numbers of F in the situation of Proposition 4.5, by computing the Gauss-Manin system of f.

c) De Jong [J] computed the homotopy type of F in case Σ is a line and transversal singularities of type A_2, A_3, D_4, E_6, E_7 or E_8.

d) Vannier [V] proved a theorem in the spirit of Lê-Ramanujam (μ constant implies topological triviality) for analytic functions with a one dimensional singular locus.

§5. CONCLUDING REMARKS AND QUESTIONS

5.1. Although we give a formula for the Milnor number of $f + x^{k+1}$, what we need is a formula for $f + \varphi$, with φ an element of $\mathcal{O}$ of high enough order and $\Sigma \cap V(\varphi) = \{0\}$, $\ell(\mathcal{O}_\Sigma/(\varphi)) >> 0$ and φ is not "degenerate".

For instance the series $W^{\#}_{1,k}$ is obtained from the non-isolated singularity $f = (x^2 + y^3)^2$ as follows. Let $\nu(\varphi) = \ell(\mathcal{O}_\Sigma/(\varphi))$. In this case is $\Sigma = V(x^2 + y^3)$.

$$W^{\#}_{1,2q-1} : (x^2 + y^3)^2 + axy^{4+q}, \quad \mu(W^{\#}_{1,2q-1}) = 14 + 2q = 3 + \nu(axy^{4+q}).$$

$$W^{\#}_{1,2q} : (x^2 + y^3)^2 + ax^2y^{3+q}, \quad \mu(W^{\#}_{1,2q}) = 15 + 2q = 3 + \nu(ax^2y^{3+q}).$$

Where $a = a_0 + a_1 y$ and $a_0, a_1 \in \mathbb{C}$, $a_0 \neq 0$. See [A] page 163.

We expect a formula of the form:

$$\mu(f + \varphi) = c(f) + \nu(\varphi),$$

where $c(f)$ is a constant only depending on f and which must be equal to $(-1)^n(\chi(F) - 1)$, by Proposition 4.4.

If φ is a kth-power of a coordinate function and $f = (x^2 + y^3)^2$, we can get only the Milnor numbers $3 + 2k$ or $3 + 3k$ depending whether the coordinate hyperplane intersects $V(x)$ transversally or not.

We show in some examples how φ can be "degenerate". Let $f = y^2 \in \mathbb{C}\{x,y\}$ then $\Sigma = V(y)$. If $\varphi = 2x^k y + x^{2k}$ then φ has arbitrary high order and $\nu(\varphi) = 2k$, but $f + \varphi = (y + x^k)^2$ has a non-isolated singularity. One can show that $f + t\varphi$ has an isolated singularity for all but finitely many t if $\Sigma \cap V(\varphi) = \{0\}$. If $\varphi = x^k y + x^\ell$ then

$$\mu(f + \varphi) = \begin{cases} 2k-1 & \text{if } 2k \leq \ell \\ \ell-1 & \text{if } 2k > \ell, \end{cases}$$

whereas $\nu(\varphi) = \ell$ is independent of k.

5.2. Van Straten [St,2] takes another point of view in his thesis. Series of normal surface singularities are obtained from a non-isolated singularity, which is a weakly normal surface. The resolution graphs of the series have the same center but different chains of $\mathbf{P}^1$'s with self intersection -2.

5.3. Examples suggest that in the case f has a one dimensional singular locus Σ with transversal A_1 singularities. One obtains a series of singularities $\chi_{i_1,\dots,i_r}$ with r indices, where r is the number of branches of Σ. Adjacencies $\chi_{i_1,\dots,i_r} \leftarrow \chi_{j_1,\dots,j_r}$ should exist if $i_k \leq j_k$ for all $1 \leq k \leq r$.

5.4. See also Wall's [W] remarks concerning series.

REFERENCES

[A] V. I. Arnol'd: Singularity theory, selected papers, Lecture Note Ser. **53**, Cambridge Univ. Press. Cambridge, 1981.

[E] R. Ephraim: Isosingular loci and the cartesian product-structure of complex-analytic singularities, Trans. Am. Math. Soc. **241** (1978), 357-371.

[G-H] T. Gaffney and H. Hauser: Characterizing singularities of varieties and of mappings, Invent. Math. **81** (1985), 427-447.

[J] T. de Jong: Some classes of line singularities, to appear in Math. Zeitschrift (1988).

[K] E. Kunz: Introduction to commutative algebra and algebraic geometry, Birkhäuser, Boston etc., 1985.

[Lê] Lê D. T.: Ensembles analytiques complexes avec lieu singulier de dimension un (d'après I. N. Iomdin), Publ. Math. de l'Un. Paris VII, ed. Lê D. T., (1980), 87-94.

[Lo] E. Looijenga: Isolated singular points on complete intersections, Lecture Note Ser. **77**, Cambridge Univ. Press. Cambridge, 1984.

[M] J. Mather: Stability of C^∞-mappings III, Finitely determined map germs, IHES Publ. Math. **35** (1968), 127-156.

[Ma] H. Matsumura: Commutative algebra, Math. Lecture Note Ser. **56**, Benjamin, New York, 1970.

[P,1] R. Pellikaan: Hypersurface singularities and resultions of Jacobi modules, Thesis, Rijksuniversiteit Utrecht, 1985.

[P,2] R. Pellikaan: Projective resolutions of the quotient of two ideals, Indagationes Mathematicae **91** (1988), 65-84.

[S,1] D. Siersma: Isolated line singularities, in Proc. Symp. Pure Math. **40**, Singularities at Arcata 1981, ed. P. Orlik, part 2, 485-496.

[S,2] D. Siersma: Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1, Topology and its Applications, **27** (1987), 51-73.

[St,1] D. van Straten: On the Betti numbers of the Milnor fibre of a certain class of hypersurface singularities, in Proceedings of Lambrecht Conf., 1985, Lecture Notes in Math. 1273, Springer (1988).

[St,2] D. van Straten: Weakly normal surface singularities and their improvements, Thesis, Rijksuniversiteit Leiden, 1986.

[V] J.-P. Vannier: Familles à un paramètre de fonctions analytiques à lieu singulier de dimension un, C.R. Acad. Sc. Paris **303** (1986), 367-370.

[W] C. T. C. Wall: Notes on the classification of singularities, Proc. London Math. Soc. **48** (1984), 461-573.

[Y] I. N. Yomdin: Variétés complexes avec singularités de dimension un, Siberian Math. J. **15** (1974), 1061-1082.

DEPARTMENT OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF TECHNOLOGY
P.O. BOX 513
5600 MB EINDHOVEN
THE NETHERLANDS