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Multiplicative structures on the minimal resolution of determinantal rings

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ABSTRACT

It is shown that the minimal resolution of a determinantal ring has the structure of an associative differential graded algebra.

INTRODUCTION

Let R be a commutative ring with a unit. For an R linear map $\phi : R^m \to R^n$, Buchsbaum and Rim [4] defined complexes $\mathbb{K}(\Lambda^q\phi)$ of length $m-n+1$, which are free resolutions of coker $(\Lambda^q\phi)$ in case grade $I_n(\phi)=m-n+1$, where $I_n(\phi)$ is the ideal generated by the $(n\times n)$-minors of a matrix of ϕ. Later Gover [6] was able to define a multiplicative structure on $\mathbb{K}(\Lambda^q\phi)$ in such a way that $\mathbb{K}(\Lambda^n\phi)$ becomes an associative differential graded algebra and $\mathbb{K}(\Lambda^q\phi)$ is a differential graded module over $\mathbb{K}(\Lambda^n\phi)$. Although the complexes have the right length $m-n+1$, they are not minimal, except in the cases $m=n$ or $m=n+1$. Srinivasan [8] showed in her thesis that the Eagon Northcott complex of ϕ [5] has a graded commutative and associative differential graded algebra structure in case R contains the rationals. This Eagon Northcott complex is a minimal resolution of coker $(\Lambda^n\phi)$ in case grade $I_n(\phi)=m-n+1$ and the entries of ϕ are in the maximal ideal of the local ring R.

It is always possible to find a free resolution of R/I which has the structure of a differential graded algebra, for every ideal I in R. Buchsbaum and Eisenbud [2] showed that the minimal resolution of R/I has the structure of a graded commutative differential graded algebra, but this needs not to be associative.

Avramov [1] defined an obstruction to the existence of an associative differential graded algebra on the minimal resolution of R/I. We will show that there is an associative differential graded algebra structure on the minimal resolution of coker $(\Lambda^n\phi)$, without assuming that R contains the rationals as Srinivasan did. On the other hand our multiplication is not graded commutative if $m \geq n+2$.

§ 1. THE EAGON NORTHCOTT COMPLEX

(1.1) DEFINITION. A *differential graded algebra* over R is a triple $(\mathbb{L}, d, *)$ where $\mathbb{L} = \bigoplus_{r=0}^{\infty} L_r$ is a direct sum of R-modules and $L_0 = R$ and $d: \mathbb{L} \to \mathbb{L}$ is a differential, that is to say d is R linear and $d^2=0$ and $d(L_{r+1}) \subseteq L_r$ for all r. Furthermore there is an R bilinear map $\mu: \mathbb{L} \oplus \mathbb{L} \to \mathbb{L}$ such that

$$d(a*b) = da*b + (-1)^r a*db$$

for all $a \in L_r$ and $b \in L_s$, where $\mu(a,b)$ is denoted by $a*b$. We say for short that $(\mathbb{L}, d, *)$ is a DG algebra. If moreover the product $*$ is associative then we say it is an associative DG algebra, ADG algebra for short. If $*$ is graded commutative, that is to say

$$a*b = (-1)^{rs} b*a \text{ and } c*c = 0$$

for all $a \in L_r$ and $b \in L_s$ and $c \in L_{2t+1}$; then we say it is a graded commutative DG algebra, CDG algebra for short.

(1.2) Let $\phi: F \to G$ be an R linear map between free finitely generated R-modules of rank m and n respectively with $m \geq n$.

SG is the symmetric algebra of G, it is a graded algebra $SG = \bigoplus S_t G$, we will give the elements of $S_t G$ degree $2t$. We denote by DG the divided power algebra of G. DG is a graded algebra $DG = \bigoplus D_t G$, we will give the elements of $D_t G$ degree $2t$. $D_t G$ is a free R-module with basis $g_1^{(p_1)} \cdots g_n^{(p_n)}$, where $p_1 + \cdots + p_n = t$ and $g_1, \ldots, g_n$ is a basis of G. There is an isomorphism of graded algebras $D(G^*) \cong (SG)^*_{qr}$, the graded dual of SG. From now on we will denote $(SG)^*_{qr}$ by $(SG)^*$ and identify it with $D(G^*)$. ΛF is the exterior algebra of F. It is graded $\Lambda F = \bigoplus \Lambda^q F$ and the elements of $\Lambda^q F$ have degree q. The algebra's SG, DG and ΛF are graded commutative and associative and $\Lambda F^* \otimes SG$ is a bigraded algebra and one can view $\Lambda F \otimes (SG)^*$ as a module over $\Lambda F^* \otimes SG$, see Buchsbaum and Eisenbud [3] for more details and proofs. The map $\phi: F \to G$ is an element of Hom (F, G) which is isomorphic with $F^* \otimes G$. We identify $F^* \otimes G$ with $\Lambda^1 F^* \otimes S_1 G$ which is contained in $\Lambda F^* \otimes SG$. We denote this element by c_ϕ and it has bidegree $(1,2)$, hence $c_\phi^2 = 0$. So the action of c_ϕ on $\Lambda F \otimes (SG)^*$, which we will denote by d, is a map of a complex $(\Lambda F \otimes (SG)^*, d)$. If we identify $(S_t G)^*$ with $D_t(G^*)$ then we can write the action of d as follows. Let $f_1, \ldots, f_m$ be a basis of F and $\gamma_1, \ldots, \gamma_n$ a basis of G^*, dual to a basis $g_1, \ldots, g_n$ of G.

Define

$$D_i(\gamma_1^{(p_1)}\cdots\gamma_n^{(p_n)})=\begin{cases}0 & \text{if } p_i=0\\ \gamma_1^{(q_1)}\cdots\gamma_n^{(q_n)} & \text{if } p_i>0,\end{cases}$$

where

$$q_j=\begin{cases}p_j & \text{if } j\neq i\\ p_j-1 & \text{if } j=i\end{cases}$$

ΛF is a ΛF^* module. For $\gamma\in\Lambda G^*$ and $f\in\Lambda F$ we denote by $\gamma(f)$ the action of γ on f. The map d is now equal to the map

$$d:\Lambda^{n+p+1}F\otimes D_{p+1}(G^*)\to\Lambda^{n+p}F\otimes D_p(G^*)$$

$$a\otimes\alpha\longmapsto\sum_{i=1}^{n}\gamma_i(a)\otimes D_i(\alpha).$$

It is possible to augment this complex by the term

$$\Lambda^n\phi:\Lambda^nF\to\Lambda^nG.$$

The complex we get in this way is called $\mathbb{L}_1^n(\phi)$ by Buchsbaum and Eisenbud [3], we will denote it by $(\mathbf{A},d)$, that means $A_0=\Lambda^nG$, $A_{r+1}=\Lambda^{n+r}F\otimes D_r(G^*)$, for $r\geq 0$.

§ 2. A MULTIPLICATIVE STRUCTURE ON A

In the appendix of Northcott [7] a map $\sigma:A_r\to A_{r+1}$ is defined for every choice of an element $f_{i_1}\wedge\cdots\wedge f_{i_n}$ with $i_1<\cdots<i_n$ of $A_1=\Lambda^nF$, such that

$$d\sigma(x)=d(f_{i_1}\wedge\cdots\wedge f_{i_n})x-\sigma(dx),$$

where $d(f_{i_1}\wedge\cdots\wedge f_{i_n})$ is an element of Λ^nG, which is identified with R, and multiplication with x is the scalar multiplication. In other words there is a map $\mu:A_1\otimes A_r\to A_{r+1}$ such that $d(x*y)=dx*y-x*dy$, where we denote $\mu(x,y)$ by $x*y$, and where $A_0\otimes A_r\to A_r$ is the scalar multiplication with elements in $A_0=R$. It is possible to extend this map to a graded map $*:\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$ in such a way that $(\mathbf{A},d,*)$ becomes an associative DG algebra.

For any monomial $\alpha\in D_r(G^*)$ with $\alpha=\gamma_1^{(p_1)}\cdots\gamma_n^{(p_n)}$ let

$$m(\alpha):=\min\{i\mid p_i\neq 0\}\text{ and }M(\alpha):=\max\{i\mid p_i\neq 0\}$$

and $m(1)=n$ and $M(1)=1$.

For $x=\lambda\ g_1\wedge\cdots\wedge g_n\in A_0$ and $y\in A_r$ define

$$x*y=y*x=\lambda y.$$

For $a\in\Lambda^{n+r}F$ and $b\in\Lambda^{n+s}F$ define

$$\gamma_k(a,b)=((\gamma_{k-1}\wedge\cdots\wedge\gamma_1)(a))\wedge((\gamma_n\wedge\cdots\wedge\gamma_{k+1})(b))\in\Lambda^{n+r+s+1}F.$$

We will denote $\gamma_{k-1}\wedge\cdots\wedge\gamma_1(a)$ by $\gamma_{k-1}\cdots\gamma_1 a$.

Let $x=a\otimes\alpha\in\Lambda^{n+r}F\otimes D_r(G^*)=A_{r+1}$ and $y=b\otimes\beta\in\Lambda^{n+s}F\otimes D_s(G^*)=$ $=A_{s+1}$ where α and β are monomials. Define

$$x*y=\sum_{M(\beta)\le k\le m(\alpha)}(-1)^{(k+1)r}\gamma_k(a,b)\otimes\alpha\beta\gamma_k.$$

Extend $*$ bilinearly to $\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}$.

(2.1) PROPOSITION. The product $*$ on $\mathbf{A}$ is associative.

PROOF. Let $a\otimes\alpha\in\Lambda^{n+r}F\otimes D_r(G^*)$, $b\otimes\beta\in\Lambda^{n+s}F\otimes D_s(G^*)$ and $c\otimes\gamma\in$ $\in\Lambda^{n+t}F\otimes D_t(G^*)$ with α, β and γ monomials. Then

$$((a\otimes\alpha)*(b\otimes\beta))*(c\otimes\gamma)=\sum_{M(\beta)\le k\le m(\alpha)}(-1)^{(k+1)r}(\gamma_k(a,b)\otimes\alpha\beta\gamma_k)*(c\otimes\gamma)$$

$$=\sum_{M(\beta)\le k\le m(\alpha)}(-1)^{(k+1)r}\sum_{M(\gamma)\le l\le m(\alpha\beta\gamma_k)}(-1)^{(l+1)(r+s+1)}\gamma_{l-1}\cdots\gamma_1(\gamma_{k-1}\cdots\gamma_1 a\wedge$$

$$\wedge\gamma_n\cdots\gamma_{k+1}b)\wedge\gamma_n\cdots\gamma_{l+1}c\otimes\alpha\beta\gamma\gamma_k\gamma_l.$$

Let A be this last expression. Now

$$l\le m(\alpha\beta\gamma_k)\le m(\beta)\le M(\beta)\le k.$$

So $l\le k$. Hence

$$\gamma_{l-1}\cdots\gamma_1(\gamma_{k-1}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}b)=$$

$$=(-1)^{(l-1)(r+1)}\gamma_{k-1}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{l-1}\cdots\gamma_1 b.$$

Therefore

$$A=\sum_{\substack{M(\beta)\le k\le m(\alpha)\\ M(\gamma)\le l\le m(\alpha\beta\gamma_k)}}(-1)^{(k+1)r+(l+1)(r+s+1)+(l-1)(r+1)}\gamma_{k-1}\cdots\gamma_1 a\wedge$$

$$\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{l-1}\cdots\gamma_1 b\wedge\gamma_n\cdots\gamma_{l+1}c\otimes\alpha\beta\gamma\gamma_k\gamma_l.$$

This last expression we denote by B.

In the same way one has

$$(a\otimes\alpha)*((b\otimes\beta)*(c\otimes\gamma))=$$

$$\sum_{\substack{M(\beta\gamma\gamma_l)\le k\le m(\alpha)\\ M(\gamma)\le l\le m(\beta)}}(-1)^{(k+1)r+(l+1)s}\gamma_{k-1}\cdots\gamma_1\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{l-1}\cdots\gamma_1 b\wedge$$

$$\wedge\gamma_n\cdots\gamma_{l+1}c\otimes\alpha\beta\gamma\gamma_k\gamma_l.$$

This last expression we denote by C.

The signs in the expressions B and C are the same, since

$$(k+1)r+(l+1)(r+s+1)+(l-1)(r+1)\equiv(k+1)r+(l+1)s\ (\mathrm{mod}\ 2).$$

Also the set of pairs (k,l) in the summations of B and C are the same, since

$$M(\beta)\le k\le m(\alpha)\text{ and }M(\gamma)\le l\le m(\alpha\beta\gamma_k)$$

implies $l \leq k$, as we have already seen. Moreover

$$M(\beta\gamma\gamma_l) = \max \{M(\beta), M(\gamma), l\}.$$

Assume $M(\beta) \leq k$ and $M(\gamma) \leq l \leq k$. Then $M(\beta\gamma\gamma_l) \leq k$.

Further $l \leq m(\alpha\beta\gamma_k)$ implies $l \leq m(\beta)$. Thus if $M(\beta) \leq k \leq m(\alpha)$ and $M(\gamma) \leq \leq l \leq m(\alpha\beta\gamma_k)$ then $M(\beta\gamma\gamma_l) \leq k \leq m(\alpha)$ and $M(\gamma) \leq l \leq m(\alpha)$.

The converse is proved similarly. This proves the associativity of the product $*$.

(2.2) PROPOSITION. If $x \in A_r$ and $y \in A_s$ then

$$d(x * y) = dx * y + (-1)^r x * dy.$$

PROOF. There is nothing to prove if $x \in A_0$ or $y \in A_0$. If $r = 1$ or $s = 1$ then the proof is in Northcott [7] appendix. So we may suppose $r \geq 2$ and $s \geq 2$. Let $x = a \otimes \alpha \in \Lambda^{n+r}F \otimes D_r(G^*)$ and $y = b \otimes \beta \in \Lambda^{n+s}F \otimes D_s(G^*)$. Suppose α and β are monomials.

Let $M = M(\beta)$ and $m = m(\alpha)$. Then

$$d(x * y) = d\Big(\sum_{k=M}^{m} (-1)^{(k+1)r} \gamma_k(a, b) \otimes \alpha\beta\gamma_k\Big) =$$

$$= \sum_{k=M}^{m} (-1)^{(k+1)r} \sum_{i=1}^{n} \gamma_i \gamma_k(a, b) \otimes D_i(\alpha\beta\gamma_k)$$

$$= \sum_{k=M}^{m} \sum_{i=1}^{n} A(i, k),$$

where

$$A(i, k) = (-1)^{(k+1)r} (\gamma_i \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b +$$

$$+ (-1)^{r+1} \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} \gamma_i b) \otimes D_i(\alpha\beta\gamma_k).$$

Since $D_k(\alpha\beta\gamma_k) = \alpha\beta$, the last summation is equal to

$$\sum_{k=M}^{m} \sum_{i=1, i \neq k}^{n} A(i, k) + \sum_{k=M}^{m} (-1)^{(k+1)r} (\gamma_k \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b +$$

$$+ (-1)^{r+1} \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} \gamma_k b) \otimes \alpha\beta$$

$$= \sum_{k=M}^{m} \sum_{i=1, i \neq k}^{n} A(i, k) + (-1)^{(m+1)r} (\gamma_m \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{m+1} b) \otimes \alpha\beta +$$

$$+ \sum_{k=M+1}^{m} \{(-1)^{kr} (\gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_k b) \otimes \alpha\beta +$$

$$+ (-1)^{(k+1)r+r+1} (\gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_k b) \otimes \alpha\beta\} +$$

$$+ (-1)^{(M+1)r+r+1} (\gamma_{M-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_M b) \otimes \alpha\beta$$

$$= \sum_{k=M}^{m} \sum_{i=1, i\neq k}^{n} A(i,k) + (-1)^{(m+1)r}(\gamma_m \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{m+1} b) \otimes \alpha\beta +$$

$$+ (-1)^{(M+1)r+r+1}(\gamma_{M-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_M b) \otimes \alpha\beta$$

$$= A_1 + A_2 + B_1 + B_2,$$

where

$$A_1 = \sum_{k=M}^{m} \sum_{i=1, i\neq k}^{n} (-1)^{(k+1)r}(\gamma_i \gamma_k \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b) \otimes D_i(\alpha\beta\gamma_k)$$

$$A_2 = (-1)^{(m+1)r}(\gamma_m \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{m+1} b) \otimes \alpha\beta$$

$$B_1 = \sum_{k=M}^{m} \sum_{i=1, i\neq k}^{n} (-1)^{(k+1)r+r+1}(\gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} \gamma_i b) \otimes D_i(\alpha\beta\gamma_k)$$

$$B_2 = (-1)^{(M+1)r+r+1}(\gamma_{M-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_M b) \otimes \alpha\beta.$$

If $i<k$ then $\gamma_i \gamma_{k-1} \cdots \gamma_1 a = 0$. If $i \geq k$ and $M \leq k \leq m$ then $D_i(\alpha\beta\gamma_k) = D_i(\alpha)\beta\gamma_k$. In the same way, if $i>k$ then $\gamma_n \cdots \gamma_{k+1} \gamma_i b = 0$. If $i \leq k$ and $M \leq k \leq m$ then $D_i(\alpha\beta\gamma_k) = \alpha D_i(\beta)\gamma_k$. Hence

$$A_1 = \sum_{k=M}^{m} \sum_{i\geq k} (-1)^{(k+1)r}(\gamma_i \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b) \otimes D_i(\alpha)\beta\gamma_k$$

and

$$B_1 = \sum_{k=M}^{m} \sum_{i\leq k} (-1)^{(k+1)r+r+1}(\gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} \gamma_i b) \otimes \alpha D_i(\beta)\gamma_k.$$

Now

$$d(a\otimes\alpha) * (b\otimes\beta) = \sum_{i=1}^{n} (\gamma_i a \otimes D_i(\alpha)) * (b\otimes\beta) =$$

$$= \sum_{i=1}^{n} \sum_{M\leq k\leq m(D_i(\alpha))} (-1)^{(k+1)(r-1)} \gamma_k \gamma_i (a,b) \otimes D_i(\alpha)\beta\gamma_k.$$

Call this last expression C.

If $i \neq m$ then $D_i(\alpha) = 0$ or $m(D_i(\alpha)) = m$. If $k>i$ then

$$\gamma_k \gamma_i (a,b) = \gamma_{k-1} \cdots \gamma_1 \gamma_i a \wedge \gamma_n \cdots \gamma_{k+1} b = 0.$$

If $k<m$ then $D_k(\alpha) = 0$ hence

$$C = \sum_{M\leq k\leq m(D_i(\alpha))} \sum_{i\geq k} (-1)^{(k+1)(r-1)}(\gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b) \otimes D_i(\alpha)\beta\gamma_k$$

$$= \sum_{M}^{m} \sum_{i\geq k} (-1)^{(k+1)(r-1)+(k-1)}(\gamma_i \gamma_{k-1} \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b) \otimes D_i(\alpha)\beta\gamma_k$$

$$+ (-1)^{(m+1)(r-1)+(m-1)}(\gamma_m \cdots \gamma_1 a \wedge \gamma_n \cdots \gamma_{k+1} b) \otimes D_m(\alpha)\beta\gamma_m$$

$$= A_1 + A_2,$$

since $(k+1)(r-1) + (k-1) \equiv (k+1)r \pmod 2$ and $D_m(\alpha)\beta\gamma_m = \alpha\beta$.

Similarly one proves

$$(-1)^r(a\otimes\alpha)*d(b\otimes\beta)=B_1+B_2.$$

This proves the proposition.

(2.3) THEOREM. $(\mathbf{A}, d, *)$ is an *ADG* algebra.

PROOF. This is a consequence of Propositions (2.1) and (2.2).

(2.4) LEMMA. If $m=n$ or $m=n+1$ then $(\mathbf{A}, d, *)$ is also graded commutative.

PROOF. If $m=n$ then $\mathbf{A}$ has length one, so there is nothing to prove. Suppose $m=n+1$. Let $a, b\in A_1=\Lambda^n F$. Then $(\gamma_{k-2}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}b)\in\Lambda^{n+2}F\cong \cong\Lambda^{n+2}R^{n+1}=0$. Hence

$$0=\gamma_{k-1}(\gamma_{k-2}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}b)=$$
$$=\gamma_{k-1}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}b+\gamma_{k-2}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{k-1}b.$$

With induction one proves that

$$\gamma_k(a,b)=\gamma_{k-1}\cdots\gamma_1 a\wedge\gamma_n\cdots\gamma_{k+1}b=(-1)^{k-1}a\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{k-1}\cdots\gamma_1 b.$$

In the same way

$$\gamma_k(b,a)=(-1)^{n-k}\gamma_n\cdots\gamma_{k+1}\gamma_{k-1}\cdots\gamma_1 b\wedge a.$$

Thus

$$\gamma_k(a,b)=(-1)^{k-1}a\wedge\gamma_n\cdots\gamma_{k+1}\gamma_{k-1}\cdots\gamma_1 b$$
$$=(-1)^{k-1+n}\gamma_n\cdots\gamma_{k+1}\gamma_{k-1}\cdots\gamma_1 b\wedge a=-\gamma_k(b,a).$$

Therefore

$$a*b=\sum_{k=1}^{n}\gamma_k(a,b)\otimes\gamma_k=-\sum_{k=1}^{n}\gamma_k(b,a)\otimes\gamma_k=-b*a.$$

Hence $(\mathbf{A}, *)$ is graded commutative.

REFERENCES

1. Avramov, L.L. – Obstructions to the existence of multiplicative structures on minimal free resolutions, Amer. J. Math. **103**, 1-31 (1981).
2. Buchsbaum, D.A. and D. Eisenbud – Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. **99**, 447-485 (1977).
3. Buchsbaum, D.A. and D. Eisenbud – Generic free resolutions and a family of generically perfect ideals, Adv. Math. **18**, 245-301 (1975).
4. Buchsbaum, D.A. and D. Rim – A generalized Koszul complex II. Depth and multiplicity Trans. Amer. Math. Soc. **111**, 197-224 (1964).
5. Eagon, J.A. and D.G. Northcott – Ideals defined by matrices and a certain complex associated with them, Proc. Royal Soc. **A 269**, 188-204 (1962).

6. Gover, E.H. – Multiplicative structure of generalized Koszul complexes, Trans. Amer. Math. Soc. **185**, 287-307 (1973).
7. Northcott, D.G. – Finite free resolutions, Cambridge Tracts in Mathematics and Mathematical Physics **71**, Cambridge University Press, 1976.
8. Srinivasan, H. – Multiplicative structures on some canonical resolutions, thesis Brandeis University, 1986.