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ON COMPLETE CONDITIONS IN ENUMERATIVE GEOMETRY

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§0 Introduction.

In enumerative geometry one deals with geometrical figures on which one imposes conditions. If the set of geometrical figures is a variety then one says that a condition is r-fold if the variety of figures which satisfy this condition has codimension r. In particular if one imposes an n-fold condition on geometrical figures of dimension n then one expects a finite number of solutions and one seeks to compute this number. Moreover varying this condition continuously, the "principle of continuity" also called the "principle of conservation of number", says that this number counted with appropriate multiplicities stays constant.

The first example of the above is Bézout's theorem [2]. Geometers like Chasles [4], Halphen [12] gave an abundance of examples culminating in Schubert's book "Kalkül der abzählenden Geometrie" [31], see also Zeuthen and Pieri [45], [46].

Hilbert poses as his 15$^{\text{th}}$ problem the question how the principles used by the enumerative geometers could be justified and whether the numbers obtained by them were correct, see [18].

After the foundational work of Severi [34], Van der Waerden [41], Weil [44] and Grothendieck [11] algebraic geometry got a rigorous basis. Algebraic varieties were defined and the language of schemes was developed to take account of non-reduced structures. In analogy with the cohomology ring in algebraic topology [40], the Chow ring was developed and Schubert calculus could be justified in any characteristic by doing the calculations in the Chow ring. Intersection theory was strongly developed by Fulton, Kleiman and MacPherson [8], [21]. Although the basic notions are well defined, it was still not clear whether all the principles underlying enumerative geometry were justified. This was clearly stressed by Kleiman [18]. For instance it is not obvious that the number one computes is obtained by taking the condition figures in general position, nor whether the "principle of conservation of number" is valid. Kleiman justified these for the class of enumerative problems where an algebraic group acts transitively on the geometrical figures [17]. The paradigm of enumerative geometry where a naive approach fails, are the plane conics. They are parametrized by $\mathbb{P}^5$ and the condition for a conic to be tangent to a given conic defines a hypersurface of degree 6 in $\mathbb{P}^5$. Hence one concludes, like Steiner did, that there are 6^5 conics tangent to 5 given conics, by Bézout's theorem. But the double lines are tangent to any given conic, hence one always has infinitely many solutions for any choice of the 5 given conics. Remark that the group of projective transformations does not act transitively on the plane conics, in fact one has three orbits: non-singular conics, the union of two lines and double lines.

One can proceed in two different ways. One can try to compute residual or excess intersections, that is an intersection theory where the intersections are not proper and one seeks to compute those solutions which are outside a base locus of degenerate solutions. Classical geometers like Severi developed a dynamic intersection theory. A modern treatment is given by Fulton and MacPherson [8] by their method of deforming to the normal cone and by Vogel and Stückrad [36], [38]. The link between them is given by Van Gastel [9]; he also gives a historical account of excess intersection theory. Up to now this approach only works if the geometrical figures are lying in projective n-space. Classically one remedied the naive approach by considering the collection of so called "complete conics", that is to say by considering a conic together with its dual. Chasles obtained the correct number 3264 of conics tangent to 5 given conics in case the characteristic is not 2. There are a lot of examples of complete geometrical figures, see (2.1), but no general definition seems to be known.

The primary aim of this paper is not to compute the numbers of a specific enumerative problem but to see which properties of a condition imply that the numbers one computes make sense, that is to say they are obtained by taking a generic choice of the condition figures, and the intersection of two conditions have again this property.

In Section 1 we define the notion of a condition and construct the sum and the intersection of two conditions and the pull back under a morphism of a condition and pose the question whether they are again conditions. In Section 2 we consider the construction of complete conics. In Section 3 we define the class of proper conditions which satisfies the sum, intersection and pull back property. In Section 4 we introduce the notion of a flat condition and show that it satisfies the intersection and pull back property. In Section 5 we prove that a condition has a flattening. In Section 6 we define Cohen-Macaulay conditions and show that they satisfy the intersection and pull back property and the principle of conservation of number. In Section 7 we give examples of conditions and consider their properties. In Section 8 we sketch a Schubert calculus on singular varieties.

§1 r-fold conditions.

Let k be an algebraically closed field. All schemes considered will be of finite type over k.

(1.1) **Definition.** Let X be a scheme of finite type over k. An *r-fold condition* on X is a triple (X,Γ,Y), where Y is a scheme of finite type over k and Γ is a closed subscheme of $X\times Y$, such that all its irreducible components have codimension r.

(1.2) **Some terminology.** Let $f: V \to W$ be a morphism of schemes over k. Let w be a point of W and $k(w)$ the residue field at w and let Spec $k(w) \to W$ be the natural morphism. Then we define the fibre $f^{-1}(w)$ of f over the point w to be the scheme

$$f^{-1}(w) = V \times_W \operatorname{Spec} k(w)\,,$$

see [13] II.3.3.

Let (X,Γ,Y) be an r-fold condition on X. We have two projections $\phi: X\times Y \to X$ and $\psi: X\times Y \to Y$. If Γ is a subscheme of $X\times Y$ then we denote the restrictions of ϕ and ψ to Γ by ϕ_Γ and ψ_Γ respectively. If y is a point of Y then we denote the fibre $\psi_\Gamma^{-1}(y)$ of ψ_Γ by Γ_y. By abuse of notation we denote $(X \times Y)_y$ by X_y. Now Γ_y is a subscheme of X_y. Similarly, if x is a point of X then define Γ^x to be the fibre $\phi_\Gamma^{-1}(x)$ and denote $(X \times Y)^x$ by Y^x. In enumerative problems X and Y will parametrize geometrical figures in some scheme Z. For instance, take for Z projective n-space and for the geometrical figures subschemes of a given dimension and degree or with a given Hilbert polynomial and take for Γ a relation between these figures like incidence or tangency. We abstract from this and forget that X and Y are parameter schemes of certain geometrical figures in some Z. We call X the scheme of geometrical figures and Y the scheme of condition figures and Γ a condition on X imposed by Y. For a point y of Y we call Γ_y the specialization of the condition Γ at y.

(1.3) **Proposition.** Let $f : V \rightarrow W$ be a flat morphism of schemes of finite type over k, and assume that W is irreducible. Then the following conditions are equivalent:

(i) Every irreducible component of V has dimension $\dim W + n$.

(ii) For any point w of W every irreducible component of the fibre $f^{-1}(w)$ has dimension n.

(iii) There exists an open dense subset U of W such that for any closed point w of U, every irreducible component of the fibre $f^{-1}(w)$ has dimension n.

Proof. See [13] III Corollary 9.6. and [11] IV_2 6.9.1.

(1.4) **Proposition.** Let X and Y be schemes of finite type over k. Suppose X is equidimensional and Y is integral. If (X, Γ, Y) is an r-fold condition on X then there exists an open dense subset V of Y such that Γ_y is empty for all points y of V or ψ_Γ is flat above V and all irreducible components of Γ_y have codimension r in X_y for all points y of V.

Proof. This follows from [11] IV_2 6.9.1 and Proposition (1.3).

(1.5) **Remark.** Let V be the open dense subset of Y mentioned in Proposition (1.4) and suppose $\psi_\Gamma^{-1}(V)$ is not empty. Define similarly the corresponding open dense subset U of X in case x is integral, after interchanging the roles of X and Y and suppose $\phi_\Gamma^{-1}(U)$ is not empty. Let $a = \dim X$, $b = a - r$, $c = \dim Y$ and $d = c - r$. Then $b = \dim \Gamma_y$ for all y of V and $d = \dim \Gamma^x$ for all x of U. Thus

$$\dim \Gamma = a + d = c + b.$$

This is called: *Prinzip der Konstantenzählung*, see [43].

(1.6) **Definition.** Let (X, Γ, Y) and (X, Λ, Z) be r-fold, respectively s-fold, conditions on X. Let

$$\tau : Y \times X \times Z \rightarrow X \times Y \times Z$$

be the isomorphism which interchanges the factors X and Y. Define the *sum* $(X, \Gamma + \Lambda, Y \times Z)$ of (X, Γ, Y) and (X, Λ, Z) by

$$\Gamma + \Lambda = (\Gamma \times Z) \cup \tau(Y \times \Lambda)).$$

Define the *product* or *intersection* $(X, \Gamma \circ \Lambda, Y \times Z)$ of (X, Γ, Y) and (X, Λ, Z) by

$$\Gamma \circ \Lambda = (\Gamma \times Z) \cap \tau(Y \times \Lambda)).$$

Define by induction

$$1\Gamma = \Gamma \text{ and } (n+1)\Gamma = (n\Gamma) + \Gamma$$

$$\Gamma^1 = \Gamma \text{ and } \Gamma^{n+1} = \Gamma^n \circ \Gamma.$$

(1.7) **Proposition.** Let (X, Γ, Y) and (X, Λ, Z) be r-fold, respectively s-fold, conditions on X and let x,y and z be closed points of X,Y and Z respectively. Then

(i) $\qquad (\Gamma \circ \Lambda)_{(y,z)} = \Gamma_y \cap \Lambda_z$

(ii) $\qquad (\Gamma \circ \Lambda)^x \cong \Gamma^x \times \Lambda^x$

where the schemes are considered as subschemes of X and Y, after identifying X_y and X_z with X and Y^x with Y.

Proof. (i) We have that $(\Gamma \circ \Lambda) = (\Gamma \times Z) \cap \tau(Y \times \Lambda)$.
Hence

$$\begin{aligned}(\Gamma \circ \Lambda)_{(y,z)} &= (\Gamma \times Z) \cap \tau(Y \times \Lambda) \cap X \times \{(y,z)\} \\ &= \Gamma_y \times \{(y,z)\} \cap \Lambda_z \times \{(y,z)\} \\ &= (\Gamma_y \cap \Lambda_z) \times \{(y,z)\}.\end{aligned}$$

We get the desired equality after the above mentioned identification.

(ii) $$\begin{aligned}(\Gamma \circ \Lambda)^x &= (\Gamma \times Z) \cap \tau(Y \times \Lambda) \cap \{x\} \times Y \times Z \\ &\cong (\Gamma^x \times Z) \cap \tau(Y \times \Lambda^x) \\ &= \Gamma^x \times \Lambda^x.\end{aligned}$$

(1.8) **Remark.** The underlying sets of $(\Gamma+\Lambda)_{(y,z)}$ and $\Gamma_y \cup \Lambda_z$ are the same for all closed points y and z of Y and Z respectively, but it is not true in general that $(\Gamma+\Lambda)_{(y,z)} = \Gamma_y \cup \Lambda_z$ as schemes, see Example (7.1). The underlying sets of $(\Gamma+\Lambda)^x$ and $(\Gamma^x \times Z) \cup (Y \times \Lambda^x)$ are the same for all closed points of X.

(1.9) **Proposition.** If (X, Γ, Y) and (X, Λ, Z) are both r-fold conditions on X then $(X, \Gamma+\Lambda, Y \times Z)$ is an r-fold condition on X.

Proof. All irreducible components of Γ have codimension r in $X \times Y$, hence all irreducible components of $\Gamma \times Z$ have codimension r in $X \times Y \times Z$, the same holds for $\tau(Y \times \Lambda)$, since τ is an isomorphism. Therefore all irreducible components of $(\Gamma \times Z) \cup \tau(Y \times \Lambda))$ have codimension r in $X \times Y \times Z$. Thus $(X, \Gamma+\Lambda, Y \times Z)$ is an r-fold condition on X. This proves the proposition.

(1.10) **Remark.** If (X, Γ, Y) and (X, Λ, Z) are r-fold, respectively s-fold, conditions on X then it is not in general true that $(X, \Gamma \circ \Lambda, Y \times Z)$ is an $(r+s)$-fold condition. Take for example a subvariety V of X of codimension $r > 0$. Let $\Gamma = V \times Y$. Then (X, Γ, Y) and $(X, \Gamma \circ \Gamma, Y \times Y)$ are both r-fold conditions on X, but $(X, \Gamma \circ \Gamma, Y \times Y)$ is not an $(r+r)$-fold condition, see (1.15) and Propositions (3.5), (4.7) and (6.6).

(1.11) **Definition.** Let (X, Γ, Y) be an r-fold condition on X. Let $f : X' \to X$ be a morphism of schemes of finite type over k. Define

$$f^{-1}(\Gamma) = (f \times \mathrm{id}_Y)^{-1}(\Gamma).$$

Then we call $(X', f^{-1}(\Gamma), Y)$ the *pull back* of (X, Γ, Y) under f.

(1.12) **Definition.** Let Z be a closed subscheme of a scheme X of finite type over k. If V is an irreducible component of Z then $O_{V,Z}$ is an Artinian local ring. Define the multiplicity of Z at V to be the length $O_{V,Z}$

$$m(V,Z) = \text{length } O_{V,Z}.$$

Define the cycle $[Z]$ associated to Z by

$$[Z] = \Sigma\, m(V,Z)\, V \,,$$

where the summation runs over all irreducible components V of Z, see [8] I.1.5. If Z is a zero dimensional subscheme of X then define

$$\int Z = \Sigma\, m(P,Z) \,,$$

where the summation runs over all closed points P of Z. If (X, Γ, Y) is an n-fold condition on a scheme X of dimension n then Γ_y is called the *solutions* of the condition Γ on X at the point y of Y, in case Γ_y is a zero dimensional subscheme of X_y. If moreover Y is irreducible then $\int \Gamma_Y$ is called the *generic number of solutions.*

(1.13) **Remark.** In case X and Y are smooth varieties over k, and (X, Γ, Y) is an n-fold condition on X, and X has dimension n, we could associate to the points of Γ_y the intersection multiplicities, according to Severi [33], [34], [35] which was made rigorous by Van der Waerden [39], [42] and Weil [44]. They proved the principle of conservation of number with their assignment of multiplicities. The disadvantage of this way is that one has to look at the number of solutions which emerge from a special solution to the generic solutions. So one has saved the principle of conservation of number, but one has to know the generic solutions in order to compute the multiplicities, whereas the classical geometers used the principle to get the number of generic solutions by specializing, where it is easier to compute the number of solutions. Another approach is to assign multiplicities according to Serre's Tor formula [32], of the intersection $\Gamma_y = \Gamma \cap X_y$ in $X \times Y$ for every closed point y in Y. Then the number of solutions would be constant for varying y in Y in case Y is irreducible, but this will not justify the methods used in the enumerative geometers, since this was not their way of assigning multiplicities to solutions. So one can say that our assignment of multiplicities is the naive one.

(1.14) **Proposition.** Let X and Y be varieties over k and let $n = \dim X$. Let (X, Γ, Y) be an n-fold condition X. If $\mathrm{char}(k) = 0$ and Γ is reduced then there exists an open dense subset V of Y such that all solutions of Γ_y have multiplicity 1 for all y in V.

Proof. Γ is reduced, hence there exists a closed subscheme Λ of Γ such that $\Gamma\backslash\Lambda$ is smooth and open dense in Γ. Let Z be the closure of $\psi_\Gamma(\Lambda)$ in Y. Then $Y\backslash Z$ is open dense in Y and there exists an open dense subset V of $Y\backslash Z$ such that all the fibres of ψ_Γ over V are smooth, by [11] IV_3.9.9.4, since char(k) = 0. Hence all the points of Γ_y have multiplicity 1 for all y in V. This proves the proposition.

(1.15) **Definition.** Let X be a scheme of finite type over k. Define

$$\mathbf{C}^r(X) = \{(X,\Gamma,Y) \mid (X,\Gamma,Y) \text{ is an } r\text{-fold condition on } X\}$$

$$\mathbf{C}(X) = \bigcup \{C^r(X) \mid 0 \le r \le n\}, \text{ where } n = \dim X.$$

If (X,Γ,Y) is an r-fold condition then define

$$V(\Gamma) = \{y \in Y \mid \text{all irreducible components of } \Gamma_y \text{ have codimension } r \text{ in } X_Y\}.$$

If $\mathbf{S}$ is a subset of $\mathbf{C}(X)$ then define $\mathbf{S}^r(X) = \mathbf{S} \cap \mathbf{C}^r(X)$.
In this paper we are concerned with finding subsets $\mathbf{S}$ of $\mathbf{C}(X)$ which have one or more of the following properties:

(i) **Sum property.**
If (X,Γ,Y), $(X,\Lambda,Z) \in \mathbf{S}^r$ then $(X,\Gamma+\Lambda,Y\times Z) \in \mathbf{S}^r$.

(ii) **Intersection property.**
If $(X,\Gamma,Y) \in \mathbf{S}^r$ and $(X,\Lambda,Z) \in \mathbf{S}^s$ then $(X,\Gamma\circ\Lambda,Y\times Z) \in \mathbf{S}^{r+s}$.

(iii) **Principle of conservation of number.**
If $(X,\Gamma,Y) \in \mathbf{S}^n$ and $n = \dim X$ then the number of solutions of Γ at y stays constant for all y in Y as long as it is finite, that is to say $\int \Gamma_y$ is the same for all y in $V(\Gamma)$.

If moreover $f : X' \to X$ is a morphism of schemes of finite type over k and $\mathbf{T}$ is a subset of $\mathbf{C}(X')$ then we are interested in the following property of $\mathbf{S}$ and $\mathbf{T}$:

(iv) **Pull back property.**
If $(X,\Gamma,Y) \in \mathbf{S}^r$ then $(X',f^{-1}(\Gamma),Y) \in \mathbf{T}^r$.

(1.16) **Remark.** $\mathbf{C}(X)$ has the sum property by Proposition (1.9) but not the intersection property, by (1.10) nor does it satisfy the principle of conservation of number, by (7.1).

§2 Complete conics.

(2.1) **Example.** A suggestion to an answer of the above questions is given by the paradigm of enumerative geometry: plane conics, see Kleiman [20] for a historical survey of Chasles's work and Casas and Xambó [3] for an account of Halphen's work. Assume char(k) is not 2. The plane conics are parametrized by $\mathbb{P}^5$ and the lines in $\mathbb{P}^2$ are parametrized by $\check{\mathbb{P}}^2$. Consider the 1-fold condition $(\mathbb{P}^5,\Lambda,\check{\mathbb{P}}^2)$, where Λ is the subscheme of $\mathbb{P}^5 \times \check{\mathbb{P}}^2$ defining the tangency between a conic and a line. So if l is a line then Λ_l is as a set the collection of all conics q which are tangent to 1. If q is a double line then q is tangent to every line. Let D be the subscheme of $\mathbb{P}^5$ parametrizing all double lines. Then Λ^4 contains the subscheme $D\times(\check{\mathbb{P}}^2)^4)$, which has dimension 10. But for a 4-fold condition $(\mathbb{P}^5, \Gamma, (\check{\mathbb{P}}^2)^4)$ irreducible components of Γ have dimension 9. Thus Λ^4 is not a 4-fold condition on $\mathbb{P}^5$. The problem is that the double lines are too often a solution of the imposed condition Λ, that is to say

$$\dim \Lambda_q = \dim \phi_\Lambda^{-1}(q) = \begin{cases} 1 & \text{in case } q \in \mathbb{P}^5 \backslash D \\ 2 & \text{in case } q \in D. \end{cases}$$

To remedy this one classically considers the variety of complete conics. Let q be a (3×3)-symmetric matrix and define $\hat{q}$ by $\hat{q}_{ij} = (-)^{i+j}$ determinant of the submatrix of q obtained by deleting the i^{th} row and the j^{th} column. Now if we consider q as a point of $\mathbb{P}^5$ then $\hat{q} = 0$ if and only if q is an element of D. We have a map

$$f : \mathbb{P}^5 \backslash D \rightarrow \mathbb{P}^5 ,$$

sending q to $\hat{q}$. Define

$$X = \text{closure of the graph of } f \text{ in } \mathbb{P}^5 \times \mathbb{P}^5 .$$

Geometrically X consists of 4 kinds of complete conics:

(i) non-singular conics

(ii) union of two different lines

(iii) a double line with two different points on it

(iv) a double line with a double point on it.

Let Λ be the subscheme of $X \times \mathbb{P}^2$ defining the tangency correspondence between complete conics and lines. In the first two cases we already know what tangency means. If $\hat{q}$ is a double line with two points p_1 and p_2 on it then the line 1 is tangent to $\hat{q}$ if and only if p_1 or p_2 belongs to 1. We now have that all the fibres of the map $\phi_\Lambda : \Lambda \rightarrow X$ have dimension 1. For a conic being tangent to a line is dual to going through a point and there are no degenerate conics with respect to this last condition, see (3.1). But the condition for a conic to be tangent to a given conic is self dual and one can complete the conics in the above way in order to get a sensible answer.

(2.2) **Remark.** In a lot of enumerative problems one has to complete the scheme of geometrical figures in order to get a sensible answer. One considers for instance complete quadrics, see (7.7), complete collineations, complete correlations, see Laksov [23], [24], [26] and the references given there, complete triangles [30], complete twisted cubics [1], [27], [28] and complete symmetric varieties [5], [6]. It seems that the name giving "completeness" in this context is rather ad hoc. Of course there is the notion of a complete scheme [13] II.4.10, but this is not the only property of complete geometrical figures, since $\mathbb{P}^5$ parametrizing the plane conics is a complete scheme. In our opinion completeness is a property of the condition (X, Γ, Y) rather than of the scheme of geometrical figures, and sometimes one has to replace the degenerate figures, see (3.1), by new figures to get a scheme X' and a condition (X', Γ', Y) such that Γ' is "complete", see (5.1). Halphen pointed out that no matter how one completes the non-degenerate conics, there is always a condition whose number of solutions makes no sense for the completed object, [12], [3] 14.8. In the sequel we will investigate proper, flat and Cohen-Macaulay conditions and whether the classical enumerative conditions have these properties. We will not propose a definition for a complete condition and leave it as a classical terminology, but we think that a flat condition and a flattening of a condition are most close to what should be a complete condition and a completion of a condition.

§3 Proper conditions.

(3.1) **Definition.** Let X be a scheme of finite type over k and (X,Γ,Y) an r-fold condition on X. Define the *degeneracy locus* $D(\Gamma)$ or D for short, of the condition by

$$D(\Gamma) = \{x \in X \mid \Gamma^x \text{ is empty or has not codimension } r \text{ in } Y^x\}.$$

We call the condition *non-degenerate* or *proper* if the degeneracy locus is empty, that is, for all x in X the subscheme Γ^x of Y^x is not empty and has codimension r.

(3.2) **Example.** Let X be a variety and G an algebraic group over k acting transitively on the closed points of X. Let Y be a scheme of finite type over k such that G also acts on Y. Let (X,Γ,Y) be an r-fold condition on X such that the action of G on the product $X \times Y$ leaves Γ invariant. Then (X,Γ,Y) is a proper condition, see [17] and see (7.6).

(3.3) **Example.** Let $C(r,d,n)$ be the Chow variety of effective cycles of dimension r and degree d in $\mathbb{P}^n$ and Γ the subscheme of $C(r,d,n) \times \mathbb{P}^n$ defined by

$$(c,p) \in \Gamma \text{ if and only if } p \text{ is in the support of } c\ ,$$

where c and p are closed points of $C(r,d,n)$ and $\mathbb{P}^n$ respectively. Then $(C(r,d,n),\Gamma,\mathbb{P}^n)$ is a proper $(n-r)$-fold condition.

(3.4) **Remark.** For the definition of a proper morphism and a complete variety we refer to [13] II.4.6 and II.4.10 respectively.

(3.5) **Proposition.** Let X,Y and Z be smooth varieties over k. Suppose Y and Z are complete. If (X,Γ,Y) and (X,Λ,Z) are r-fold, respectively s-fold, proper conditions on X, then $(X,\Gamma \circ \Lambda, Y \times Z)$ is an $(r+s)$-fold proper condition on X. If $r = s$ then $(X,\Gamma + \Lambda, Y \times Z)$ is a proper r-fold condition on X.

(3.6) **Remark.** Let X be a smooth variety over k. Define

$$\mathbf{P}^r(X) = \{(X,\Gamma,Y) \mid (X,\Gamma,Y) \text{ is a proper } r\text{-fold and } Y \text{ is } a \text{ smooth complete variety}\}$$

$$\mathbf{P}(X) = \bigcup \{P^r(X) \mid 0 \le r \le n\}, \quad \text{where } n = \dim X.$$

Then $\mathbf{P}(X)$ satisfies the sum and the intersection property, by Proposition (3.5) and since the product of two complete smooth varieties over k is again a complete smooth variety over k, by [11] IV_2 6.8.5.

Proof of (3.5). The irreducible components of $\Gamma \times Z$ and $\tau(Y \times \Lambda)$ have codimension r and s respectively in $X \times Y \times Z$. Hence all irreducible components of $(\Gamma \times Z) \cap \tau(Y \times \Lambda)$ have codimension at most $r+s$, by [32] since $X\times Y\times Z$ is smooth. Let Σ be an irreducible component of $\Gamma\circ\Lambda$. Then

$$\dim \Sigma \geq l+m+n-(r+s),$$

where $l=\dim Z$, $m=\dim Y$ and $n=\dim X$. Let x be a closed point of X. Then $(\Gamma\circ\Lambda)^x=\Gamma^x\times\Lambda^x$, by (1.7), and $\dim\Gamma^x=m-r$ and $\dim\Lambda^x=l-s$, since Γ and Λ are both proper conditions. Hence $(\Gamma\circ\Lambda)^x$ has dimension $l+m-(r+s)$ for all closed points x of X. Y and Z are complete, so $Y\times Z$ is complete. Σ is a closed subscheme of $X\times Y\times Z$, hence $\phi(\Sigma)$ is closed in X. Therefore

$$\dim\Sigma \leq l+m-(r+s)+\dim\phi(\Sigma) \leq l+m+n-(r+s).$$

Hence equality holds and $\phi(\Sigma)=X$, since X is irreducible and X and $\phi(\Sigma)$ have the same dimension. Furthermore $\dim\phi_\Sigma^{-1}(x)=l+m-(r+s)$ for all closed points x of X, so $(\Gamma\circ\Lambda)^x$ has codimension $r+s$ for all closed points x of X. Thus $(X,\Gamma\circ\Lambda,Y\times Z)$ is a proper $(r+s)$-fold condition on X.
In case $r=s$ we know already that $(X,\Gamma+\Lambda,Y\times Z)$ is an r-fold condition. The fibre $(\Gamma+\Lambda)^x$ has the same underlying set as $(\Gamma^x\times Z)\cup(Y\times\Lambda^x)$, see Remark (1.8), which has constant codimension r in $Y\times Z$ for all closed points x in X. Hence the condition is proper. This proves the proposition.

(3.7) **Proposition.** Let X and Y be smooth varieties over k. Suppose Y is complete. Let (X,Γ,Y) be a proper r-fold condition on X and Z a subvariety of X. Then $(Z,\Gamma\cap(Z\times Y),Y)$ is a proper r-fold condition on Z.

Proof. All irreducible components of Γ have codimension r in $X\times Y$ and $Z\times Y$ has codimension s in $X\times Y$, if Z has codimension s in X. Further $X\times Y$ is smooth, hence all irreducible components of $\Gamma\cap(Z\times Y)$ have codimension at most $r+s$ in $X\times Y$, by [32]. Let Σ be such an irreducible component, then

$$\dim\Sigma\geq m+n-(r+s),$$

where $\dim X=n$ and $\dim Y=m$. Then $\dim\phi_\Sigma^{-1}(x)\leq m-r$, since $\dim\phi_\Gamma^{-1}(x)=m-r$ for all closed points in $\phi(\Sigma)$. Σ is closed in $X\times Y$ and Y is complete, hence $\phi(\Sigma)$ is closed in X and contained in Z. Hence

$$\dim\Sigma\leq m-r+\dim\phi(\Sigma)\leq m-r+n-s.$$

Hence equality holds. Z and $\phi(\Sigma)$ have the same dimension, Z is irreducible and $\phi(\Sigma)$ is a closed subvariety, hence $\phi(\Sigma)=Z$. Furthermore $\dim\phi_\Sigma^{-1}(x)=m-r$ for all closed points in Z. Thus $(Z,\Gamma\cap(Z\times Y),Y)$ is a proper r-fold condition on Z. This proves the proposition.

(3.8) **Definition.** A scheme is called *pure dimensional* if all its components have the same dimension and it has no embedded components. Let V and W be two subschemes of a smooth variety X of pure codimension r and s respectively. Then the intersection $V\cap W$ is called *proper* if it is empty or of pure codimension $r+s$ in X.

(3.9) **Corollary.** Under the same assumptions as in Proposition (3.7) we have that there exists an open dense subset U of Y such that $\Gamma_y \cap Z$ is a proper intersection for all closed points y in U.

Proof. This follows from the Propositions (1.4) and (3.7).

(3.10) **Corollary.** Let X and Y be smooth varieties over k. Suppose Y is complete and $n = \dim X$. Let (X, Γ, Y) be a proper n-fold condition on X and Z a subvariety of X, not equal to X. Then there exists an open dense subset U of Y such that the solutions of Γ at y are disjoint from Z, for all closed points y of U.

Proof. This follows from Corollary (3.9), since $\Gamma \cap (Z \times Y)$ is an n-fold condition on Z which has dimension less than n.

§4 Flat conditions.

(4.1) **Definition.** An r-fold condition (X, Γ, Y) is called *flat* if the map $\phi_\Gamma : \Gamma \to X$ is flat.

(4.2) **Remark.** The idea of considering flat conditions and the flattening of a condition stems from Piene and Schlessinger [28] where they consider flat specializations of non-degenerate twisted cubics and the Hilbert scheme to make a completion.

(4.3) **Example.** (3.2) is also an example of a flat condition, since the map ϕ_Γ is flat over an open dense subset U of X, by Proposition (1.4) after interchanging the rôles of X and Y, and the group G acts transitively on the closed points of X, [17].

(4.4) **Proposition.** Let X and Y be schemes of finite type over k. Suppose X and Y are irreducible. If (X, Γ, Y) is a flat r-fold condition then all irreducible components of Γ^x have codimension r in Y^x for all points x of X, in particular (X, Γ, Y) is a proper condition.

Proof. Γ and X are schemes of finite type over k. The morphism ϕ_Γ is flat and X is irreducible and all irreducible components of Γ have the same codimension r in $X \times Y$, hence all irreducible components of Γ^x have codimension r in Y^x for all points x in X, by Proposition (1.3). Hence (X, Γ, Y) is a proper condition. This proves the proposition.

(4.5) **Remark.** We give in (7.2) an example of a condition which is proper but not flat.

(4.6) **Remark.** Let Y be a projective scheme over k and X a scheme of finite type over k. If (X, Γ, Y) be a flat r-fold condition on X and X is irreducible then the Hilbert polynomial of Γ^x in Y^x is the same for all points x of X, by [11] III.2.2.1. Conversely, if (X, Γ, Y) is an r-fold condition on X and the Hilbert polynomial of Γ^x in Y^x is the same for all x of X then (X, Γ, Y) is a flat condition, by [14]. Let P be a polynomial in one variable with rational coefficients. Let $\mathrm{Hilb}^P(Y)$ be the Hilbert scheme of subschemes of Y with Hilbert polynomial P, see [10] exposé 221. The Hilbert scheme has the following universal property. Let $\Lambda(P)$ be the subscheme of $\mathrm{Hilb}^P(Y) \times Y$ such that Λ^h is the subscheme of Y^h corresponding to the point h of $\mathrm{Hilb}^P(Y)$. Then $\phi_{\Lambda(P)}$ is flat. Moreover, for every flat condition (X, Γ, Y) such that Γ^x has Hilbert polynomial P in Y^x for all points x of X, there exists a unique map $f : X \to \mathrm{Hilb}^P(Y)$ such that $\Gamma^x = \Lambda(P)^{f(x)}$ for all x of X.

(4.7) **Proposition.** Let X, Y and Z be schemes of finite type over k. Suppose X, Y and Z are irreducible. Let (X, Γ, Y) and (X, Λ, Z) be flat r-fold, respectively s-fold, conditions on X. Then

$(X, \Gamma \circ \Lambda, Y \times Z)$ is a flat $(r+s)$-fold condition on X.

(4.8) **Remark.** Let X be an irreducible scheme of finite type over k. Define

$$\mathbf{F}^r(X) = \{(X, \Gamma, Y) \mid (X, \Gamma, Y) \text{ is a flat } r\text{–fold condition on } X \text{ and } Y \text{ is irreducible}\}$$

$$\mathbf{F}(x) = \bigcup \{F^r(X) \mid 0 \leq r \leq n\}, \quad \text{where } n = \dim X.$$

Then $\mathbf{F}(X)$ satisfies the intersection property by Proposition (4.7).
Before we proof the proposition we need a lemma.

Proof of (4.7). Consider the following commutative diagrams of schemes of finite type over k

$$\begin{array}{ccccc} \Gamma \circ \Lambda & \rightarrow & Y \times \Lambda & \rightarrow & \Lambda \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma \times Z & \rightarrow & X \times Y \times Z & \rightarrow & X \times Z \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \rightarrow & X \times Y & \rightarrow & X \end{array} \qquad \begin{array}{ccccc} & \Gamma \circ \Lambda & \xrightarrow{g} & \Lambda & \\ f & \downarrow & & \downarrow & \phi_\Lambda \\ & \Gamma & \xrightarrow{\phi_\Gamma} & X & \end{array}$$

, where the morphisms in the left diagram are the obvious inclusions or projections, g is the composition of the two morphisms in the top row and f is the composition of the two morphisms in the left column. The four squares in the left diagram are fibred, hence the right diagram is fibred by diagram chasing.. Now ϕ_Λ is flat, hence f is flat by the base change property of flatness [13] III Proposition 9.2.b. Further ϕ_Γ and f are flat, so the composition $\phi_{\Gamma \circ \Lambda}$ of ϕ_Γ and f is flat, by transitivity [13] III Proposition 9.2.c. Let x be a closed point of X, then

$$(\Gamma \circ \Lambda)^x \cong \Gamma^x \times \Lambda^x,$$

by (1.7.i). The irreducible components of Γ^x have codimension r in Y and the irreducible components of Λ^x have codimension s in Z, by Proposition (4.4). Hence all irreducible components of $(\Gamma \circ \Lambda)^x$ have codimension $r+s$ in $Y \times Z$. Hence all irreducible components of $\Gamma \circ \Lambda$ have codimension $r+s$ in $X \times Y \times Z$, by Proposition (1.3). Thus $(X, \Gamma \circ \Lambda, Y \times Z)$ is a flat $(r+s)$-fold condition on X. This proves the proposition.

(4.9) **Proposition.** Let Y and X' be irreducible schemes of finite type over k and $f : X' \rightarrow X$ a morphism of schemes over k. If (X, Γ, Y) is a flat r-fold condition on X then $(X', f^{-1}(\Gamma), Y)$ is a flat r-fold condition on X'.

(4.10) **Remark.** In this case $\mathbf{F}(X)$ and $\mathbf{F}(X')$ have the pull back property.

Proof of (4.9). Let $\Lambda = f^{-1}(\Gamma)$. The morphism ϕ_Γ is flat. The following diagram is a fibered square

$$\begin{array}{ccccc} & \Lambda & \rightarrow & \Gamma & \\ \phi_\Lambda & \downarrow & & \downarrow & \phi_\Gamma \\ & X' & \rightarrow & X & \end{array}$$

Hence ϕ_Λ is a flat morphism, by the base change property [13] III Proposition 9.2.b. The fibre $\Lambda^{x'}$ is isomorphic to $\Gamma^{f(x')}$ for all x' in X'. All irreducible components of Γ^x have the same dimension for all x in X, by Proposition (4.4). Hence all irreducible components of $\Lambda^{x'}$ have the same dimension for all x' in X'. Furthermore the map ϕ_Λ is flat and X' is irreducible. Thus all irreducible components of $f^{-1}(\Gamma)$ have codimension r in $X' \times Y$, by Proposition (1.3). Thus $(X', f^{-1}(\Gamma), Y)$ is a flat r-fold condition on X'. This proves the proposition.

§5 Flattening of a condition.

(5.1) **Definition.** Let (X, Γ, Y) be an r-fold condition on X. A flat r-fold condition (X', Γ', Y) on X' together with a morphism $\pi : X' \to X$ is called a *flattening* of (X, Γ, Y) if there exists an open dense subset U of X such that

$$\pi : \pi^{-1}(U) \to U \text{ and}$$

$$\pi \times \mathrm{id}_Y : \Gamma' \cap (\pi^{-1}(U) \times Y) \to \Gamma \cap (U \times Y)$$

are isomorphisms.

A flattening (X', Γ, Y) with a morphism $\pi : X' \to X$ is called *universal* if for every flattening (X'', Γ'', Y) together with a morphism $\pi' : X'' \to X$ there exists a unique morphism $f : X'' \to X'$ such that $\pi' = \pi \circ f$ and Γ'' is the pull back of Γ' under f.

(5.2) **Remark.** It is clear that the universal flattening is unique up to canonical isomorphisms, if it exists.

(5.3) **Proposition.** Let X be a scheme of finite type over k. Let (X, Γ, Y) be an r-fold condition on X. Then there exists a flattening of (X, Γ, Y).

Proof. The existence of a flattening is a result of Raynaud and Gruson [29]. We give a Hilbert scheme proof of the existence under the assumption that X is integral and Y is projective. The existence proof is in the same spirit as the way one constructs a completion of geometrical figures, see (2.1) and (7.5). If no irreducible component of Γ dominates X then we take $X' = X$ and $\Gamma' = \varnothing$. Otherwise there exists an open dense subset U of X such that ϕ_Γ is flat over U and the irreducible components of Γ^x have codimension r in Y^x for all x in U, by Proposition (1.4). The Hilbert polynomial P of Γ^x in Y^x is constant for all x in U, by Remark (4.5). Let $H = \mathrm{Hilb}^P(Y)$. Then there exists a map $g : U \to H$, where $g(x)$ is the point in H corresponding to the subscheme Γ^x of Y^x for all x in U. Consider the graph Γ_g in $U \times H$ of the map g. Let X' be the closure of Γ_g in $X \times H$. Let $\pi : X' \to X$ be the restriction to X' of the projection $X \times H \to X$ and let $g' : X' \to H$ be the restriction to X' of the projection $X \times H \to H$. Then $\pi : \pi^{-1}(U) \to U$ is an isomorphism and g' and $g \circ \pi$ are the same on $\pi^{-1}(U)$. Let $\Lambda(P)$ be the subscheme of $H \times Y$ as defined in Remark (4.6). Let Γ' be the pull back of $\Lambda(P)$ via g'. Then $\phi_{\Gamma'}$ is flat, by Proposition (4.9) and using the fact that $\phi_{\Lambda(P)}$ is flat, by Remark (4.5). Further

$$\pi \times \mathrm{id}_y : \Gamma' \cap (\pi^{-1}(U) \times Y) \to \Gamma \cap (U \times Y)$$

is an isomorphism, since $\Gamma^x = \Lambda(P)^{g(x)}$ for all x in U, and g' and $g \circ \pi$ are the same on $\pi^{-1}(U)$, hence $\Gamma'^{x'} = \Gamma^{\pi(x')}$ for all x' in $\pi^{-1}(U)$. All irreducible components of $\Gamma'^{x'}$ have codimension r in $Y^{x'}$ for all points x' in $\pi^{-1}(U)$, since this holds for $\Gamma^{\pi(x')}$. Furthermore $\pi^{-1}(U)$ is an open dense subset of X' and $\phi_{\Gamma'}$ is flat. Hence all irreducible components of Γ' have codimension r in $X \times Y$, by Proposition (1.3). Thus (X', Γ', Y) is a flat r-fold condition on X', together with the map π it gives a completion of (X, Γ, Y). This proves the proposition.

(5.4) **Remark.** If (X, Γ_i, Y_i), $i = 1, \ldots, r$ is a finite collection of conditions on X then there exists a scheme X' and a morphism $\pi : X' \to X$ and conditions (X', Γ_i', Y_i), $i = 1, \ldots, r$ such that they are flattenings of the original conditions. One can do this inductively by first making a flattening $(X^{(1)}, \Gamma_1^{(1)}, Y_1)$ of the first condition and pulling back conditions Γ_i, for $i = 2, \ldots, r$ to $X^{(1)}$ and then making a flattening of the second condition on $X^{(1)}$, etc. Another way is doing it in one step by taking the closure of the map $g = g_1 \times \cdots g_r : U \to H_1 \times \cdots \times H_r$, where $H_i = \mathrm{Hilb}^{P_i}(Y_i)$ and $g_i : U \to H_i$ is defined by $g_i(x)$ is the point in H_i corresponding to the subscheme Γ_i^x in Y_i^x.

(5.5) **Proposition.** Let X and Y be smooth varieties over k and let $n = \dim X$. Suppose Y is complete. Let (X, Γ, Y) be an r-fold condition and (X', Γ', Y) together with a morphism $\pi : X' \to X$ a flattening such that $\pi : \pi^{-1}(U) \to U$ is an isomorphism for an open dense subset U of X. Then there exists an open dense subset V of Y such that all the solutions of Γ_y' lie in $\pi^{-1}(U)$.

Proof. This follows from Corollary (3.10) by taking $Z = X' \backslash \pi^{-1}(U)$, since the condition Γ' is proper, by Proposition (4.4).

(5.6) **Corollary.** The generic number of solutions of a flattening is independent of the chosen flattening.

§6 Cohen-Macaulay conditions.

(6.1) **Definition.** An r-fold condition (X, Γ, Y) is called *Cohen-Macaulay*, or *CM* for short, if the condition is proper and Γ is a Cohen-Macaulay scheme.

(6.2) **Proposition.** Let $f : V \to W$ be a morphism of irreducible schemes of finite type over k.

(i) If V is *CM* and W is smooth then the map f is flat if and only if the fibres of $f^{-1}(w)$ have constant dimension $\dim V - \dim W$ for all (closed) points w of W.

(ii) If the map is flat then V is *CM* if and only if W is *CM* and the fibres $f^{-1}(w)$ are *CM* for all (closed) points w of W.

Proof. See [26] 5.1 and 23.1 for (i) and the corollary of 23.3 for (ii).

(6.3) **Lemma.** Let X and Y be irreducible smooth schemes of finite type over k. Suppose X is smooth. Then the following are equivalent:

(i) (X, Γ, Y) is an r-fold *CM* condition.

(ii) (X, Γ, Y) is a flat r-fold condition and Γ^x is a *CM* scheme for all points x of X.

(iii) (X, Γ, Y) is a flat r-fold condition and Γ^x is a *CM* scheme for all closed points x of X.

Proof. (i) $\Rightarrow$ (ii). If (X, Γ, Y) is an r-fold *CM* condition then Γ is *CM* and the condition is proper hence the fibre Γ^x of ϕ_Γ at x has constant dimension $\dim \Gamma - \dim X$ for all closed points x of X and

X is smooth, thus the map ϕ_Γ is flat, by Proposition (6.2.i), and (X,Γ,Y) is a flat r-fold condition. Furthermore Γ^x is *CM* for all points x of X, by Proposition (6.2.ii).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). If (X,Γ,Y) is a flat r-fold condition then it is a proper r-fold condition, by Proposition (4.4) since X is irreducible. Furthermore X is *CM* and the fibres Γ^x of ϕ_Γ at x are *CM* for all closed points x of X by assumption, and the map ϕ_Γ is flat, hence Γ is *CM*, by Proposition (6.2.ii). Thus (X,Γ,Y) is an r-fold *CM* condition. This proves the lemma.

(6.4) **Remark.** In (7.4) we give an example of a flat condition which is not *CM*.

(6.5) **Proposition.** Let X and Y be smooth varieties and Γ a hypersurface in $X\times Y$. If (X,Γ,Y) is a proper 1-fold condition then it is *CM*.

Proof. This follows immediately, since a hypersurface in a smooth scheme is *CM*.

(6.6) **Proposition.** Let X and Y be irreducible smooth schemes of finite type over k. Suppose X is smooth. If (X,Γ,Y) and (X,Λ,Z) are r-fold, respectively s-fold, *CM* conditions, then $(X,\Gamma\circ\Lambda,Y\times Z)$ is an $(r+s)$-fold *CM* condition.

(6.7) **Remark.** Let X and Y be irreducible smooth schemes of finite type over k. Suppose X is smooth. Define

$$\mathbf{CM}^r(X)=\{(X,\Gamma,Y)\mid (X,\Gamma,Y) \text{ is an } r\text{-fold } CM \text{ condition on } X\}$$

$$\mathbf{CM}(X)=\bigcup\{CM^r(X)\mid 0\le r\le n\}, \text{ where } n=\dim X.$$

Then $\mathbf{CM}(X)$ satisfies the intersection property by Proposition (6.6). In general $\mathbf{CM}(X)$ does not satisfy the sum property, see example (7.3).

Proof of (6.6). (X,Γ,Y) and (X,Λ,Z) are flat r-fold respectively s-fold conditions, by Lemma (6.3). Hence $(X,\Gamma\circ\Lambda,Y\times Z)$ is a flat $(r+s)$-fold condition, by Proposition (4.7), and $\phi_{\Gamma\circ\Lambda}$ is a flat map. The fibers Γ^x and Λ^x of ϕ_Γ and ϕ_Λ respectively are *CM* for all closed points x of X. The fibre $(\Gamma\circ\Lambda)^x$ of $\phi_{(\Gamma\circ\Lambda)}$ is isomorphic with the product $\Gamma^x\times\Lambda^x$, by (1.7.ii) and is therefore also *CM*, for all closed points x of X. Thus $(X,\Gamma\circ\Lambda,Y\times Z)$ is an $(r+s)$-fold *CM* condition on X, by Lemma (6.3) since we assumed X to be irreducible and smooth. This proves the proposition.

(6.8) **Proposition.** Let X,Y and X' be irreducible schemes of finite type over k and $f:X'\to X$ a morphism. Suppose X and X' are smooth. If (X,Γ,Y) is an r-fold *CM* condition on X then $(X',f^{-1}(\Gamma),Y)$ is an r-fold *CM* condition on X'.

(6.9) **Remark.** In this case $\mathbf{CM}(X)$ and $\mathbf{CM}(X')$ have the pull back property.

Proof. If (X,Γ,Y) is a r-fold *CM* condition then this condition is flat and Γ^x is *CM* for all closed points of X, by Lemma (6.3). Hence $(X,f^{-1}(\Gamma),Y)$ is flat, by Proposition (4.10) and $f^{-1}(\Gamma)^{x'}=\Gamma^{f(x')}$ is *CM* for all closed points x' of X'. Thus $(X',f^{-1}(\Gamma),Y)$ is an r-fold *CM* condition, by Lemma (6.3) since X' is irreducible and smooth. This proves the proposition.

(6.10) **Proposition.** Let X be a complete variety over k of dimension n and Y an irreducible smooth variety over k. If (X,Γ,Y) is an n-fold *CM* condition then the principle of conservation of number holds.

Proof. Γ is a closed subscheme of $X \times Y$ and X is complete, hence the map ψ is closed and ψ_Γ is a proper map. We define $V(\Gamma) = \{y \in Y \mid \Gamma_y \text{ is zero dimensional}\}$ in (1.15). Let $U = \psi_\Gamma^{-1}(V(\Gamma))$. Then $\psi_\Gamma : U \to V(\Gamma)$ is a finite map. Moreover U is CM since it is an open subscheme of Γ which is CM by assumption. The sheaf $\psi_{\Gamma *}(O_U)$ is a coherent on $V(\Gamma)$ and depth $\psi_{\Gamma *}(O_U)_p = \dim_p Y$ for all points p of Y. Furthermore Y is smooth, hence

$$\operatorname{depth} M_p + \operatorname{projdim} M_p = \dim_p Y \,,$$

for all points p of Y, by [32]. Thus $\psi_{\Gamma *}(O_U)$ is a locally free sheaf on $V(\Gamma)$ of constant rank, say N, since Y is irreducible. Thus for every y in $V(\Gamma)$ one has that

$$N = \Sigma \operatorname{length} O_{p,U} \,,$$

where p runs through all the points in the fibre $\psi_\Gamma^{-1}(y)$, which is equal to $\int \Gamma_y$. Hence the number of solutions $\int \Gamma_y$ is constant and equal to N for all points y of $V(\Gamma)$. Thus the principle of conservation of number holds. This proves the proposition.

§7 Examples.

(7.1) **Example.** Let $(\mathbb{P}^2, \Gamma, \check{\mathbb{P}}^2)$ be the condition defining the incidence between points and lines in $\mathbb{P}^2$, where the lines of $\mathbb{P}^2$ are parametrized by $\check{\mathbb{P}}^2$. Γ is defined by the bihomogeneous ideal

$$(l_0 x_0 + l_1 x_1 + l_2 x_2) \quad \text{in} \quad k[x_0, x_1, x_2, l_0, l_1, l_2] \,,$$

the bihomogeneous coordinate ring of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$. The condition Γ^2 is defined by the ideal (lx, mx) in the trihomogeneous coordinate ring $k[x,l,m]$ of $\mathbb{P}^2 \times (\check{\mathbb{P}}^2)^2$, where we denote (x_0, x_1, x_2) by x and $l_0 x_0 + l_1 x_1 + l_2 x_2$ by lx. The condition $\Gamma^2 + \Gamma^2$ is defined by the ideal

$$(lx, mx) \cap (l'x, m'x) = ((lx)\,(l'x),\, (lx)\,(m'x),\, (mx)\,(l'x),\, (mx)\,(m'x)) \,,$$

in the 5-homogeneous coordinate ring $k[x,l,m,l',m']$ of $\mathbb{P}^2(\check{\mathbb{P}}^2)^4$. Let l and m be two different lines in $\mathbb{P}^2$ which intersect at P and let $y = (l,m)$ then the scheme $(\Gamma^2 + \Gamma^2)_{(y,y)}$ is supported at P and has multiplicity 3, whereas $\Gamma_y^2 \cup \Gamma_y^2$ is equal to Γ_y^2 and is supported at P with multiplicity 1. Thus

$$(\Gamma^2 + \Gamma^2)_{(y,y)} \neq \Gamma_y^2 \cup \Gamma_y^2 \,,$$

and the principle of conservation does not hold for $\Gamma^2 + \Gamma^2$.

(7.2) **Example.** Let Γ be the closed subscheme of $\mathbb{P} \times \mathbb{P}^3$ defined by the bihomogeneous ideal

$$(y_0, y_1) \cap (y_0, y_2) \cap (x_0 y_0 + x_1 y_1 + x_2 y_3 \,,\, y_1 + y_2)$$

in the bihomogeneous coordinate ring $k[x_0, x_1, x_2, y_0, y_1, y_2, y_3]$ of $\mathbb{P}^2 \times \mathbb{P}^3$. Then $(\mathbb{P}^2, \Gamma, \mathbb{P}^3)$ is a proper 2-fold condition, since Γ^x consists of three lines in $\mathbb{P}^3$ for all closed points x in $\mathbb{P}^2$. If $x = (0{:}\,1{:}\,0)$ then Γ^x is defined by the ideal $(y_0, y_1) \cap (y_0, y_2) \cap (y_1, y_2)$, so it is the union of three lines going through $(0{:}\,0{:}\,0{:}\,1)$ and not lying in a plane, it has Hilbert polynomial $3t+1$. If $x = (1{:}\,0{:}\,0)$ then Γ^x is defined by the ideal $(y_0, y_1) \cap (y_0, y_2) \cap (y_0, y_1 + y_2)$, so it is the union of three lines in a plane going through one point and it has Hilbert polynomial $3t$. So the Hilbert polynomial of Γ^x is not constant, hence the map ϕ_Γ is not flat, by Remark (4.6), and the condition is not flat.

(7.3) **Example.** Let $G(2.4)$ be the Grassmann variety of planes in $\mathbb{P}^4$ and $(G(2,4),\Gamma,\mathbb{P}^4)$ the condition such that $(v,p) \in \Gamma$ if and only if p is a point of v, for all v and p closed points of $G(2,4)$ and $\mathbb{P}^4$ respectively. Then Γ is a 2-fold *CM* condition, see (7.6) and $\Gamma \circ \Gamma$ is a 4-fold *CM* condition, by Proposition (6.6). We have the following exact sequence of local rings

$$0 \to O_{p,\Gamma+\Gamma} \to O_{p,\Gamma\times\mathbb{P}^4} \oplus O_{p,\mathbb{P}^4\times\Gamma} \to O_{p,\Gamma\circ\Gamma} \to 0\ ,$$

for every closed point p of $\Gamma \circ \Gamma$. The middle term is a direct sum of two *CM* local rings of dimension 12, hence it has depth 12. The third term is a *CM* local ring of dimension 10, so of depth 10. Thus the first term has depth 11, whereas it has dimension 12. Therefore $\Gamma + \Gamma$ is not *CM*.

(7.4) **Example. Twisted cubics.** Let $P = 3t + 1$. Then $\text{Hilb}^P(\mathbb{P}^3)$ consists of two smooth irreducible components H and H' of dimensions 12 and 15 respectively and the intersection is smooth of dimension 11. A point of $H_0 = H \backslash H'$ corresponds to a non-degenerate twisted cubic, a point of $H' \backslash H$ corresponds to a plane cubic curve with a point outside the plane and a point of $H \cap H'$ corresponds to a singular plane cubic curve with an embedded point at a singular point. See Piene and Schlessinger [28]. Let Λ be the pull back of $\Lambda(P)$ under the inclusion of H in $\text{Hilb}^P(\mathbb{P}^3)$. Then $(H, \Lambda, \mathbb{P}^3)$ is a flat 2-fold condition, since the Hilbert polynomial of Γ^x is constant $3t+1$. The scheme Γ^x has an embedded component for all points x of $H \cap H'$ and therefore is not *CM*. Thus the condition is not *CM*, by Lemma (6.3).

(7.5) **Example. Complete twisted cubics** according to Piene [27]. Suppose char(k) is not 2 or 3. Consider the conditions $(H,\Gamma,G(1,3))$ and $(H,\Lambda^\vee,\mathbb{P}^3)$, where for $c \in H_0$ and $l \in G(1,3)$, the Grassmann variety of lines in $\mathbb{P}^3$, we have that

$(c,l) \in \Gamma$ if and only if the line l is tangent the curve c ,

$(c,h) \in \Lambda^\vee$ if and only if h is an osculating plane of the curve c.

If $c \in H_0$ then Γ^c is the tangent curve c^* of c in $G(1,3)$, which is a rational normal curve of degree 4 and has Hilbert polynomial $4t+1$, and $(\Lambda^\vee)^c$ is the dual curve $c^\vee$ of osculating planes in $\mathbb{P}^3$, which is again a twisted cubic with Hilbert polynomial $3t+1$. Let $G = \text{Hilb}^{4t+1}(G(1,3))$. Thus we have morphisms

$$g : H_0 \to G \ \text{ defined by } \ g(c) = c^*\ ,$$

$$f : H_0 \to H \ \text{ defined by } \ f(c) = c^\vee\ .$$

The closure T of the graph of $g \times f$ in $H \times G \times H$ is called the scheme of complete twisted cubics by Piene [27]. The restrictions to T of the projections to the second and third factor we denote by g' and f' respectively. The pull backs of $\Lambda(4t+1)$ and $\Lambda(3t+1)$ under the morphisms g' and f' we denote by Γ' and $\Lambda^{\vee\prime}$ respectively, they are flattenings of Γ and $\Lambda^\vee$, by the proof of Proposition

(5.3) and Remark (5.4).

(7.6) **Example. Schubert conditions on Grassmannians.** Let $G(r,n)$ be the Grassmannian of all r-planes in $\mathbb{P}^n$, it is a smooth variety of dimension $(r+1)(n-d)$. Let $\mathbf{a}=(a_0, \ldots, a_r)$ be a sequence of integers such that $0 \leq a_0 < \cdots < a_r \leq n$. Let $F(\mathbf{a},n)$ be the flag variety of all flags $\mathbf{A}$ in $\mathbb{P}^n$, where $\mathbf{A}=(A_0, \ldots, A_r)$ and A_i is an a_i-dimensional linear subspace in $\mathbb{P}^n$ and $A_i \subset A_{i+1}$ for all $i=0, \ldots, r-1$. Let $(G(r,n), \Omega, F(\mathbf{a},n))$ be the condition where

$$(B, \mathbf{A}) \in \Omega \text{ if and only if } \dim(B \cap A_i) \geq i \text{ for all } i=0, \ldots, r,$$

for closed points B in $G(r,n)$ and $\mathbf{A}$ in $F(\mathbf{a},n)$. Then $\Omega_{\mathbf{A}}$ is called a Schubert variety and has codimension $\Sigma(a_i - i)$. For every two flags $\mathbf{A}$ and $\mathbf{B}$ in $F(\mathbf{a},n)$ there exists an invertible projective transformation ϕ of $\mathbb{P}^n$ which induces an isomorphism of $G(r,n)$ and carries $\Omega_{\mathbf{A}}$ into $\Omega_{\mathbf{B}}$, see [22]. Hence we are in the situation of Example (3.2) and (4.3) and the condition is flat $\Sigma(a_i - i)$-fold. Moreover the Schubert varieties $\Omega_{\mathbf{A}}$ are *CM*, by [15], [16] thus the condition is even *CM*, by Lemma (6.3).

(7.7) **Example. Complete quadrics,** see Laksov [24] and the references given there. Suppose char$(k) \neq 2$. Let V be a vector space of dimension $n+1$ over k, with coordinates $x_0, \ldots, x_n$. We denote the projectivization of V by $\mathbb{P}(V)$. A quadric q in $\mathbb{P}(V)$ is given by the zero locus of a quadratic form

$$\Sigma\, q_{ij} x_i x_j ,$$

where $q=(q_{ij})$ is a non-zero symmetric $(r+1) \times (r+1)$-matrix. Thus quadrics in $\mathbb{P}(V)$ are parametrized by $\mathbb{P}(S^2 V)$, where $S^2 V$ is the vector space of symmetric maps $V \to V^*$. We denote a quadric in $\mathbb{P}^n$, its symmetric matrix and the point in $\mathbb{P}(S^2 V)$ representing it, by the same q. The Grassmann variety $G(r,n)$ of r-planes in $\mathbb{P}^n$ can be embedded in $\mathbb{P}(\Lambda^{r+1} V)$ with Plücker coordinates (x_I), where $I=(i_0, \ldots, i_r)$ and $0 \leq i_0 < \cdots < i_r \leq n$. The 1-fold condition $(\mathbb{P}(S^2 V), \Gamma(r), G(r,n))$ describing the tangency between a quadric q and an r-plane x in $\mathbb{P}^n$ is defined by the zero locus of the quadratic form

$$\Sigma(\Lambda^{r+1} q)_{I,J}\, x_I x_J ,$$

where I and J are multi-indices $I=(i_0, \ldots, i_r)$, $J=(j_0, \ldots, j_r)$ such that $0 \leq i_0 < \cdots < i_r \leq n$, $0 \leq j_0 < \cdots < j_r \leq n$ and $(\Lambda^{r+1} q)_{I,J}$ is the determinant of the $(r+1) \times (r+1)$-submatrix of q consisting of the rows $i_0, \ldots, i_r$ and columns $j_0, \ldots, j_r$. One can view $\Lambda^{r+1} q$ as an element of $\mathbb{P}(S^2 \Lambda^{r+1} V)$, the latter we will denote by M_r. So the condition $\Gamma(r)$ is the pull back of $\Lambda(r)$, where $(M_r, \Lambda(r), G(r,n))$ is the 1-fold condition defined by the zero locus of the quadratic form

$$\Sigma\, Q_{I,J}\, x_I x_J ,$$

where $(Q_{I,J}) \in \mathbb{P}(S^2 \Lambda^{r+1} V) = M_r$.
Let U_r be the open dense set of quadrics which have rank at least $r+1$. Then we have a morphism

$$\lambda_r : U_r \to M_r ,$$

defined by $\lambda_r(q) = \Lambda^{r+1} q$. Let $U = U_n$ and let

$$\lambda : U \to M_1 \times \cdots \times M_{n-1} ,$$

be the morphism defined by $\lambda = \lambda_1 \times \cdots \times \lambda_{n-1}$. Define the variety B of complete quadrics to be the closure of the graph of λ in $M_0 \times M_1 \times \cdots \times M_{n-1}$. Let π_r be the projection of B to M_r and $\Gamma(r)'$ the pull back to B of $\Lambda(r)$ under π_r. The variety B can be obtained by a sequence of *blowing ups* with smooth centers starting with M_0 which is smooth, hence B is smooth, see Vainsencher [37].

A complete n-quadric is some k-tuple $\mathbf{q} = (q_1, \ldots, q_k)$, where q_1 is a quadric in $\mathbb{P}^n$ of rank r_1, and q_i is a quadric in the singular locus of q_{i-1} of rank r_i, for all $i = 2, \ldots, k$ and $r_1 + \cdots + r_k = n+1$. The closed points of B are in one to one correspondence with complete n-quadrics. Thus if $\mathbf{q} = (q_1, \ldots, q_k)$ is a complete n-quadric then $\mathbf{q}' = (q_2, \ldots, q_k)$ is a complete $(n+1-r_1)$-quadric in the singular locus of q_1, see Finat [7]. The condition $\Gamma(r)'$ can be expressed inductively as follows. Let $\mathbf{q}$ be a complete n-quadric with ranks $(r_1, \ldots, r_k)$ and x an r-plane in $\mathbb{P}^n$. Then $(\mathbf{q},x) \in \Gamma(r)'$ if and only if

(i) x is tangent to q_1 in case $r < r_1$

(ii) x intersects $\mathrm{Sing}(q_1)$ non-transversally or x intersects $\mathrm{Sing}(q_1)$ transversally and $(\mathbf{q}', x \cap \mathrm{Sing}(q_1)) \in \Gamma(r-r_1)'$ in case $r \geq r_1$.

From this description it follows by induction that all the fibres $\Gamma(r)_{\mathbf{q}}'$ at the closed point $\mathbf{q}$ of B have codimension 1 in $G(r,n)$. Hence $(B, \Gamma(r)', G(r,n))$ is a proper 1-fold condition. Moreover the condition $\Gamma(r)'$ is a hypersurface in $B \times G(r,n)$ and B and $G(r,n)$ are smooth, hence the condition is even *CM*, by Proposition (6.5).

Note that the complete quadrics are not obtained by the use of Hilbert scheme flattening as in the proof of Proposition (5.3), although it is very similar to it. It would be interesting to know whether these two completions are isomorphic, see also Kleiman's question [21] page 362.

§8 Schubert calculus.

We sketch how to get an intersection theory on singular varieties.

Let X be an irreducible scheme of finite type over k of dimension n. Now $\mathbf{F}^r(X)$ is the collection of flat r-fold conditions (X, Γ, Y) such that Y is irreducible. Then $\mathbf{F}^r(X)$ satisfies the intersection property, by Proposition (4.7). Let $F^r(X)$ be the free abelian group generated by $\mathbf{F}^r(X)$ and

$$F^*(X) = \oplus \, \{F^r(X) \mid r = 0, \ldots, n\}.$$

Define the map

$$\int : F^n(X) \to \mathbb{Z} ,$$

by $\int(X, \Gamma, Y) = \int \Gamma_Y$, the generic number of solutions,, see (1.15), on generators of $F^n(X)$ and extend by linearity. The map $\int$ is a morphism of groups.

Define an intersection

$$\circ : F^r(X) \otimes F^s(X) \rightarrow F^{r+s}(X)$$

by

$$(X, \Gamma, Y) \otimes (X, \Lambda, Z) \rightarrow (X, \Gamma \circ \Lambda, Y \times Z)$$

on generators of $F^r(X)$ and $F^s(X)$, and extend by linearity.
Define the following numerical equivalence relation ~ on $F^r(X)$ by

$$\Sigma a_i(X, \Gamma_i, Y_i) \sim \Sigma b_j(X, \Lambda_j, Z_j)$$

if and only if

$$\Sigma a_i \int (Z, f^{-1}(\Gamma_i), Y_i) = \Sigma b_j \int (Z, f^{-1}(\Lambda_j), Z_j)$$

for all closed embeddings $f : Z \rightarrow X$ such that Z is irreducible of dimension r. This is well-defined by the pull back property, see Proposition (4.9). Let $N^r(X)$ be the subgroup of $F^r(X)$ of elements numerically equivalent to zero and let

$$N^*(X) = \oplus \{N^r(X) \mid r = 0, \ldots, n\}.$$

Define

$$S^r(X) = F^r(X) / N^r(X)$$

and

$$S^*(X) = \oplus \{S^r(X) \mid r = 0, \ldots, n\}.$$

Then the intersection $\circ$ is well-defined on $S^*(X)$ and this product is distributive with respect to +, associative, commutative and has unit the class of $(X, X \times Y, Y)$, where $Y = \mathrm{Spec}(k)$. Hence $S^*(X)$ is a commutative ring with a unit. We call $S^*(X)$ the *Schubert ring* of X.
If (X, Γ, Y), (X, Λ, Z) and $(X, \Gamma + \Lambda, Y \times Z)$ are elements of $F^r(X)$ then

$$(X, \Gamma, Y) + (X, \Lambda, Z) \equiv (X, \Gamma + \Lambda, Y \times Z) \bmod N^r(X).$$

So if we denote (X, Γ, Y) by Γ then the two meanings of $\Gamma + \Lambda$ are equal modulo $N^r(X)$.
In case X is a smooth quasi projective variety we can associate to every condition (X, Γ, Y) the cycle $[\Gamma_y]$, with y some element of $V(\Gamma)$, which gives a well-defined cycle class modulo algebraic equivalence, for every choice y in $V(\Gamma)$. An element of $F^*(X)$ which is numerically equivalent to zero gives a cycle class in the Chow group $A^*_{alg}(X)$ modulo algebraic equivalence, which is numerically equivalent to zero. Thus we have a well defined morphism of rings

$$S^*(X) \rightarrow A^*_{num}(X).$$

It would be interesting to know what its image is.
For every r-fold condition (X, Γ, Y) and for every embedding $f : Z \rightarrow X$ of a variety Z of dimension r we have that

$$[Z] \cdot [\Gamma_y] = [f^{-1}(\Gamma)_y]$$

in $A^0_{alg}(X)$, for all $y \in Y$ such that $Z \cap \Gamma_y$ consists of finitely many points. Hence the morphism $S^*(X) \to A^*_{num}(X)$ is injective.

We call a class in $S^r(X)$ effective if it has a representative $\Sigma a_i(X, \Gamma_i, Y_i)$ such that $a_i \geq 0$ for all i. Let $S^r_{eff}(X)$ be the set of elements of $S^r(X)$ which are effective. It follows from the definition of the product that if $a \in S^r_{eff}(X)$ and $b \in S^s_{eff}(X)$ then $a \circ b \in S^{r+s}_{eff}(X)$.

Consequently, if X is the blow up of $\mathbb{P}^2$ at a point and E is the exceptional divisor then E has self intersection -1, so its class in $A^1_{num}(X)$ does not lie in the image of the map $S^1(X) \to A^1_{num}(X)$.

It is not always possible to define a ring structure on the Chow group $A^*(X)$ in case X is singular, see [13] appendix A, (1.1.2).

Thus we have defined a Schubert ring for every variety X over k, even in the case X is singular. This is of some importance in enumerative geometry, since after flattening (completion) of geometrical figures one may end up with a singular variety. It is for instance not known whether the variety T of complete twisted cubics, example (7.5), is smooth.

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