The coset leader and list weight enumerator

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In Topics in Finite Fields
11th International Conference on Finite Fields and their Applications
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Corrected version, 7 November 2019

Abstract. This paper is about the coset leader and list weight enumerators
and their extensions using the theory of arrangements of hyperplanes, geometric
lattices and characteristic and Möbius polynomials.

1. Introduction

The probability of error in error-detection can be expressed in terms of the
weight enumerator of a code [23], and for error-correction the coset leader weight
enumerator is used [24]. The coset leader weight enumerator is also used in steganog-
raphy to compute the average of changed symbols [25, 26]. The computation of the
weight enumerator of a code is NP-hard [3, 4, 31]. The complexity of computing
the coset leader weight enumerator of a code is considered extremely difficult [15].
The size of lists of nearest codewords is considered in the list decoding of Reed-
Solomon codes [21, 30]. This motivates the definition of the list weight enumerator
and its extension. This research originated in [19].

Let $C$ be a linear code of length $n$ over $\mathbb{F}_q$ and let $y \in \mathbb{F}_q^n$. The weight of
the coset $y + C$ is defined by $\text{wt}(y + C) = \min\{\text{wt}(y + c) : c \in C\}$. A coset leader
is a choice of an element $y \in \mathbb{F}_q^n$ of minimal weight in its coset, that is
$\text{wt}(y) = \text{wt}(y + C)$. Let $\alpha_i$ be the number of cosets of $C$ that are of weight $i$. Let $\lambda_i$ be the number of vectors $y$ in $\mathbb{F}_q^n$ that are of minimal weight $i$ in its coset. Then
we can define the coset leader and list weight enumerator:

$$
\alpha_C(X, Y) = \sum_{i=0}^{n} \alpha_i X^{n-i} Y^i \quad \text{and} \quad \lambda_C(X, Y) = \sum_{i=0}^{n} \lambda_i X^{n-i} Y^i.
$$
There is a one-to-one correspondence between cosets and syndromes. It is a well known fact that a coset leader corresponds to a minimal way to write its syndrome as a linear combination of the columns of a parity check matrix. This idea gives rise to the derived code code $D(C)$ and its associated arrangement of hyperplanes. An inclusion/exclusion counting argument in the derived code gives us a way to determine the (extended) coset leader and (extended) list weight enumerator. A similar argument using the code itself gives a way to determine the weight enumerator, see [17] and [20]. We consider several examples, including the Hamming codes and some generalized Reed-Solomon codes.

We address the questions whether the extended coset leader weight enumerator and/or extended list weight enumerator of $C$ determine the corresponding enumerators of $C^\perp$, as is the case for the ordinary weight enumerator by the MacWilliams relations. This problem has a negative answer for the ordinary coset leader weight enumerator by [2]. See [24], Research Problem 5.1.

2. Codes, arrangements and geometric lattices

For an overview about the relation between codes, projective systems, hyperplane arrangements, and geometric lattices, see [18, 17, 20]. In this section we briefly recapture some necessary theory.

Let $G$ be the generator matrix of a linear $[n,k]$ code $C$ over $F_q$. Then we can form the $[n,k]$ code $C \otimes F_{q^m}$ over $F_{q^m}$ by taking all $F_{q^m}$-linear combinations of the codewords in $C$. We call this the extension code of $C$ over $F_{q^m}$. We can determine the weight enumerator of such an extension code by using only the code $C$. By embedding its entries in $F_{q^m}$, we find that $G$ is also a generator matrix for the extension code $C \otimes F_{q^m}$.

**Definition 2.1.** The extended weight enumerator $W_C(X,Y,T)$ of a linear code of length $n$ is a homogeneous polynomial in $X$ and $Y$ of degree $n$ with coefficients $A_w(T)$:

$$W_C(X,Y,T) = \sum_{w=0}^{n} A_w(T) X^{n-w} Y^w.$$  

The $A_w(T)$ are integral polynomials in $T$ such that $A_w(q^m)$ is the number of codewords of weight $w$ in $C \otimes F_{q^m}$.

Let $\mathbb{F}$ be a field. A projective system $\mathcal{P} = (P_1, \ldots, P_n)$ in $\mathbb{P}^r(\mathbb{F})$, the projective space over $\mathbb{F}$ of dimension $r$, is an $n$-tuple of points $P_j$ in this projective space, such that not all these points lie in a hyperplane.

An $n$-tuple $(H_1, \ldots, H_n)$ of hyperplanes in $\mathbb{F}^k$ is called an arrangement in $\mathbb{F}^k$. The arrangement is called simple if all the $n$ hyperplanes are mutually distinct. The arrangement is called central if all the hyperplanes are linear subspaces. A central arrangement is called essential if the intersection of all its hyperplanes is equal to $\{0\}$.

Let $L$ be a poset (partially ordered set) and let $x, y \in L$. Then $x$ and $y$ have a least upper bound if there is a $z \in L$ such that $x \leq z$ and $y \leq z$, and if $x \leq w$ and $y \leq w$, then $z \leq w$ for all $w \in L$. If $x$ and $y$ have a least upper bound, then such an element is unique and it is called the join of $x$ and $y$ and denoted by $x \lor y$. Similarly
the greatest lower bound of $x$ and $y$ is defined. If it exists, then it is unique and it is called the meet of $x$ and $y$ and denoted by $x \land y$. A poset $L$ is called a lattice if $x \lor y$ and $x \land y$ exist for all $x, y \in L$.

Let $L$ be a lattice with minimum 0. An atom is an element $a \in L$ that is a cover of 0. A lattice is called atomic if for every $x > 0$ in $L$ there exist atoms $a_1, \ldots, a_r$ such that $x = a_1 \lor \cdots \lor a_r$. The minimum length of a chain from 0 to $x$ is called the rank of $x$ and is denoted by $r_L(x)$ or $r(x)$ for short. A lattice is called semimodular if it has a rank function that satisfies $r(x \lor y) + r(x \land y) \leq r(x) + r(y)$ for all $x, y \in L$. A lattice $L$ is called a geometric lattice if it is atomic and semimodular and has no infinite chains. If $L$ is a geometric lattice, then it has a minimum and a maximum and $r(1)$ is called the rank of $L$ and is denoted by $r(L)$.

Via the columns of a generator matrix, we can associate a projective system – or, dually, an arrangement of hyperplanes – to a linear code. In the next examples we see that these structures give rise to a geometric lattice. These are Examples 5.45 and 5.46 from [20].

**Example 2.2.** Let $\mathbb{F}$ be a field and let $\mathcal{V} = (v_1, \ldots, v_n)$ be an $n$-tuple of nonzero vectors in $\mathbb{F}^k$, i.e., a projective system in $\mathbb{P}^{k-1}(\mathbb{F})$. Let $L = L(\mathcal{V})$ be the collection of all linear subspaces of $\mathbb{F}^k$ that are generated by subsets of $\mathcal{V}$, with inclusion as partial order. This poset is a lattice, because for all $x, y \in L$ we can determine their join and meet by $x \lor y = x + y$ and $x \land y = \bigvee\{z : z \leq x, z \leq y\}$.

The minimum of $L$ is $0_L = \{0\}$, the linear subspace generated by the empty set, and the maximum $1_L$ is given by is the subspace generated by all $v_1, \ldots, v_n$.

To show that the lattice $L$ is geometric, we have to show that it is atomic, semimodular and without infinite chains. The latter is obvious since $L$ is finite. Let $a_j$ be the linear subspace generated by $v_j$. Then $a_1, \ldots, a_n$ are the atoms of $L$. Let $x$ be the subspace generated by $\{v_j : j \in J\}$. Then $x = \bigvee_{j \in J} a_j$. If $x$ has dimension $r$, then there exists a subset $I$ of $J$ such that $|I| = r$ and $x = \bigvee_{i \in I} a_i$. Hence $L$ is atomic and $r(x) = \dim(x)$. Now $x \land y \subseteq x \cap y$, so

$$r(x \lor y) + r(x \land y) \leq \dim(x + y) + \dim(x \cap y) = r(x) + r(y).$$

Hence the semimodular inequality holds. We conclude that $L$ is a geometric lattice.

**Example 2.3.** Let $\mathbb{F}$ be a field and let $\mathcal{A} = (A_1, \ldots, A_n)$ be an arrangement over $\mathbb{F}$ of hyperplanes in the vector space $V = \mathbb{F}^k$. Let $L = L(\mathcal{A})$ be the collection of all nonempty intersections of elements of $\mathcal{A}$. By definition $\mathbb{F}^k$ is the empty intersection. Define the partial order $\leq$ by

$$x \leq y \quad \text{if and only if} \quad y \subseteq x.$$ Then $V$ is the minimum element. Furthermore

$$x \lor y = x \cap y \quad \text{if} \quad x \cap y \neq \emptyset, \quad \text{and} \quad x \land y = \bigcap\{z : x \cup y \subseteq z\}.$$ Suppose that $\mathcal{A}$ is a central arrangement. Then the intersection of all hyperplanes is not empty and the maximum element. If this intersection is equal to $\{0\}$, then the arrangement is called essential. Moreover $x \cap y$ is nonempty for all $x, y \in L$, so $x \lor y$ and $x \land y$ exist for all $x, y \in L$ and $L$ is a lattice.
We will show that $L$ is atomic by showing it is isomorphic to the lattice in the previous example. Let $v_j = (v_{j1}, \ldots, v_{jk})$ be a nonzero vector such that $\sum_{i=1}^k v_{ij} X_i = 0$ is a homogeneous equation of $A_j$. Let $V = (v_1, \ldots, v_n)$ and let $x$ be the subspace generated by $\{v_j : j \in J\}$. Consider the map $\varphi : L(V) \to L(A)$ defined by

$$\varphi(x) = \bigcap_{j \in J} A_j.$$ 

Now $x \subset y$ if and only if $\varphi(y) \subset \varphi(x)$ for all $x, y \in L(V)$. Therefore, $\varphi$ is a strictly monotone map. Furthermore $\varphi$ is a bijection and its inverse map is also strictly monotone. Hence $L(V)$ and $L(A)$ are isomorphic lattices. Therefore $L(A)$ is also a geometric lattice.

We will now define some important polynomials associated to geometric lattices.

**Definition 2.4.** Let $L$ be a finite geometric lattice with Möbius function $\mu(x, y)$. The characteristic polynomial of $L$ is defined by

$$\chi_L(T) = \sum_{x \in L} \mu(0, x) T^{r(L) - r(x)}.$$ 

The two-variable characteristic polynomial or coboundary polynomial is defined by

$$\chi_L(S,T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu(x, y) S^{\alpha(x)} T^{r(L) - r(y)}$$

where $\alpha(x)$ is the number of atoms $a$ in $L$ such that $a \leq x$.

Note that $\mu(0, 1) = \chi_L(0)$ and $\chi_L(0, T) = \chi_L(T)$, because for $S = 0$ the only nonzero term has $\alpha(x) = 0$, so $x = 0_L$.

**Remark 2.5.** Let $n$ be the number of atoms of $L$. Then the following relation holds for the coboundary polynomial in terms of characteristic polynomials:

$$\chi_L(S, T) = \sum_{i=0}^n S^i \chi_i(T) \quad \text{with} \quad \chi_i(T) = \sum_{x \in L, \alpha(x) = i} \chi_{L_x}(T).$$

Here $L_x$ is the geometric lattice of all elements bigger than or equal to $x \in L$. The polynomial $\chi_i(T)$ is called the $i$-defect polynomial. See [9], [8].

**Definition 2.6.** Let $L$ be a finite geometric lattice. The two variable Möbius polynomial $\mu_L(S,T)$ is defined by

$$\mu_L(S,T) = \sum_{x \in L} \sum_{x \leq y \in L} \mu(x, y) S^{\alpha(x)} T^{r(L) - r(y)}.$$ 

Note that $\mu_L(0, T) = \chi_L(0, T) = \chi_L(T)$.

**Remark 2.7.** Let $r$ be the rank of the geometric lattice $L$. Then the following relation holds for the Möbius polynomial in terms of characteristic polynomials:

$$\mu_L(S,T) = \sum_{i=0}^r S^i \mu_i(T) \quad \text{with} \quad \mu_i(T) = \sum_{x \in L_i} \chi_{L_x}(T).$$

The Möbius polynomial was introduced by Zaslavsky [32] Section 1] for hyperplane arrangements and for signed graph colorings in [33] Section 2] where it is called the Whitney polynomial. See also [1] and [20] §5.8.2.
We give some more background on the determination of the coboundary and Möbius polynomial of an arrangement. As we will see later, the method for determining the coset leader weight enumerator has many similarities with the following.

**Proposition 2.8.** Let \( q \) be a prime power, and let \( \mathcal{A} = (A_1, \ldots, A_n) \) be a simple and essential arrangement in \( \mathbb{F}_q^k \). Then
\[
\chi_{\mathcal{A}}(q^m) = |\mathbb{F}_q^n \setminus (A_1 \cup \cdots \cup A_n)|.
\]

**Proof.** See [1 Theorem 2.2] and [5 Proposition 3.2] and [11 Sect. 16] and [27 Theorem 2.69] and [20 Proposition 5.45].

**Remark 2.9.** Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be a simple and essential arrangement in \( \mathbb{F}^k \). Let \( x \) be an element of \( L = L(\mathcal{A}) \), i.e., an intersection of hyperplanes, with dimension \( l \). The *restriction* \( \mathcal{A}_x \) is the arrangement in \( \mathbb{F}^l \) of all hyperplanes \( x \cap A_j \) such that \( x \cap A_j \neq \emptyset \) and \( x \cup A_j \neq x \). Then \( L(\mathcal{A}_x) = L_x \). Let \( \cup A_x \) be the union of the hyperplanes of \( \mathcal{A}_x \). Then by Proposition 2.8 we have
\[
\chi_{\mathcal{A}_x}(q^m) = |(x \setminus (\cup A_x))(\mathbb{F}_q^m)|.
\]

So \( \chi_{\mathcal{A}_x}(q^m) \) counts the number of vectors that are in \( x \subseteq \mathbb{F}^k \) but not in any element of \( L \) bigger than \( x \). Or, equivalently, it counts the number of vectors that are in \( x \) but not in any other element of \( L_x \).

**Definition 2.10.** From now on \( \{1, \ldots, n\} \) will be abbreviated by \( [n] \). Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be an essential simple arrangement over the field \( \mathbb{F} \) in \( \mathbb{F}^k \) and let \( J \subseteq [n] \). Define \( A_J = \cap_{j \in J} A_j \). Then \( A_J \) are elements of a geometric lattice with rank function \( r \) as in Example 2.3 Consider the decreasing sequence
\[
N_k \subseteq N_{k-1} \subseteq \cdots \subseteq N_1 \subseteq N_0
\]
of algebraic subsets of the affine space \( \mathbb{A}^k \), defined by
\[
N_i = \bigcup_{\substack{J \subseteq [n] \\ r(A_J) = i}} A_J.
\]

Define \( \mathcal{M}_i = (N_i \setminus N_{i+1}) \).

Note that \( N_0 = \mathbb{A}^k \), \( N_1 = \cup_{j=1}^n A_j \), \( N_k = \{0\} \) and \( N_{k+1} = \emptyset \). Furthermore, \( N_i \) is a union of linear subspaces of \( \mathbb{A}^k \) of all dimension \( k - i \).

**Proposition 2.11.** Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be an essential simple arrangement over the field \( \mathbb{F} \) in \( \mathbb{F}^k \). Let \( z(x) = \{j \in [n] : x \in A_j\} \) and \( r(x) = r(A_{z(x)}) \) the rank of \( x \) for \( x \in \mathbb{A}^k \). Then
\[
N_i = \{x \in \mathbb{A}^k : r(x) \geq i\} \quad \text{and} \quad \mathcal{M}_i = \{x \in \mathbb{A}^k : r(x) = i\}.
\]

**Proof.** See [20 Proposition 5.47].

**Proposition 2.12.** Let \( \mathcal{A} \) be an essential simple arrangement over \( \mathbb{F}_q \) and let \( L = L(\mathcal{A}) \) be the geometric lattice of \( \mathcal{A} \). Then
\[
\mu_i(q^m) = |\mathcal{M}_i(\mathbb{F}_q^m)|.
\]

**Proof.** See [1 Theorem 6.3] and [20 Proposition 5.48].
3. Coset leader and list weight enumerator

We repeat the definitions of the coset leader and list weight enumerators before we generalize them for extension codes.

**Definition 3.1.** Let $C$ be a linear code of length $n$ over $\mathbb{F}_q$. Let $y \in \mathbb{F}_q^n$. The weight of the coset $y + C$ is defined by

$$\text{wt}(y + C) = \min\{\text{wt}(y + c) : c \in C\}.$$ 

A coset leader is a choice of an element $y \in \mathbb{F}_q^n$ of minimal weight in its coset, that is, $\text{wt}(y) = \text{wt}(y + C)$. Let $\alpha_i$ be the number of cosets of $C$ that are of weight $i$. Let $\lambda_i$ be the number of vectors $y$ in $\mathbb{F}_q^n$ that are of minimal weight $i$ in its coset. Then $\alpha_C(X, Y)$, the coset leader weight enumerator of $C$, and $\lambda_C(X, Y)$, the list weight enumerator of $C$, are polynomials defined by

$$\alpha_C(X, Y) = \sum_{i=0}^{n} \alpha_i X^{n-i} Y^i \quad \text{and} \quad \lambda_C(X, Y) = \sum_{i=0}^{n} \lambda_i X^{n-i} Y^i.$$ 

See [14, 24]. The covering radius $\rho(C)$ of $C$ is the maximal $i$ such that $\alpha_i(C) \neq 0$.

We have $\alpha_i = \lambda_i = \left(\binom{n}{i}\right)(q - 1)^i$ for all $i \leq (d - 1)/2$, where $d$ is the minimum distance of $C$, since coset leaders are unique for these $i$. The coset leader weight enumerator gives a formula for the probability of error, that is the probability that the output of the decoder is the wrong codeword. In this decoding scheme the decoder uses the chosen coset leader as the error vector. See [24, §1.5]. The list weight enumerator is of interest in case the decoder has as output the list of all nearest codewords [21, 30].

Consider the functions $\alpha_i(T)$ and $\lambda_i(T)$ such that $\alpha_i(q^m)$ and $\lambda_i(q^m)$ are equal to the number of cosets of weight $i$ and the number of elements in $\mathbb{F}_{q^m}$, of minimal weight $i$ in its coset, respectively, with respect to the extended coded $\mathbb{F}_q \otimes \mathbb{F}_{q^m}$.

**Definition 3.2.** The extended coset leader weight enumerator and the extended list weight enumerator are defined by:

$$\alpha_C(X, Y, T) = \sum_{i=0}^{n} \alpha_i(T) X^{n-i} Y^i \quad \text{and} \quad \lambda_C(X, Y, T) = \sum_{i=0}^{n} \lambda_i(T) X^{n-i} Y^i.$$ 

In [14, Theorem 2.1] it is shown that the function $\alpha_i(T)$ is determined by finitely many data for all extensions of $\mathbb{F}_q$. This shows by Lagrange interpolation that the $\alpha_i(T)$ are polynomials in the variable $T$. In fact, let $C$ be an $[n, k]$ code over $\mathbb{F}_q$. Then there are well defined nonnegative integers $F_{ij}$ such that

$$\alpha_C(X, Y, T) = 1 + \sum_{i=1}^{n-k} \sum_{j=1}^{n-k} F_{ij} (T - 1)(T - q) \cdots (T - q^{i-1}) X^{n-i} Y^i.$$ 

This is similar to the following expression of the extended weight enumerator in terms of the generalized weight enumerator, see [14, 17, 18, 20, 22].

$$A_w(T) = \sum_{r=0}^{k} \sum_{j=1}^{r} (T - 1)(T - q) \cdots (T - q^{j-1}) A_w^r.$$
Remark 3.3. We have \( \alpha_i(T) = \lambda_i(T) = \left( \begin{smallmatrix} i \\ r 
 \end{smallmatrix} \right) (T - 1)^i \) for all \( i \leq (d - 1)/2 \), where \( d \) is the minimum distance of \( C \). An information set of \( C \) is a subset of size \( k \) of the positions of \( C \) where all possible \( q^k \) combinations of symbols occur. The columns of a generator matrix that correspond to an information set are linearly independent. Let \( i(C) \) be the number of information sets of \( C \). Then \( \lambda_{n-k}(T) = i(C) \alpha_{n-k}(T) \).

Remark 3.4. Note that the cosets of \( Y \) and \( \lambda Y \) have the same weight for all \( \lambda \in \mathbb{F}_q^* \). Hence \( \alpha_i = (q - 1) \alpha_i \) and \( \lambda_i = (q - 1) \alpha_i \) for all \( i > 0 \). Therefore the polynomials \( \alpha_i(T) \) and \( \lambda_i(T) \) are divisible by \( T - 1 \) for all \( i > 0 \), that is there exist polynomials \( \tilde{\alpha}_i(T) \) and \( \tilde{\lambda}_i(T) \) such that \( \alpha_i(T) = (T - 1) \tilde{\alpha}_i(T) \) and \( \lambda_i(T) = (T - 1) \tilde{\lambda}_i(T) \) for all \( i > 0 \).

Remark 3.5. Let \( d \) be the minimum distance of the code. If \( i > (d - 1)/2 \) then there exists an error of weight \( i \) that is not uniquely correctable, so \( \alpha_i < \lambda_i \). See [14]. The Newton radius is the maximal weight of a uniquely correctable error.

For some easy examples it is straightforward to determine the coset leader and list weight enumerators.

Example 3.6. Let \( C = \mathbb{F}_q^n \). Then \( \lambda_C(X,Y,T) = \alpha_C(X,Y,T) = X^n \).

Example 3.7. Let \( C = \{0\} \). Then \( \lambda_i(T) = \alpha_i(T) = \left( \begin{smallmatrix} i \\ r \n \end{smallmatrix} \right) (T - 1)^i X^{n-i}Y^i \) and \( \lambda_C(X,Y,T) = \alpha_C(X,Y,T) = (X + (T - 1)Y)^n \).

Example 3.8. Let \( C \) be the dual of an \([n, 1, d]\) code. Then there are \( T \) cosets, corresponding to the elements of the ground field of the code. One is the code itself, the other \( T - 1 \) cosets have each \( d \) elements of minimal weight \( 1 \). This gives \( \lambda_C(X,Y,T) = X^n + d(T - 1)X^{n-1}Y \) and \( \alpha_C(X,Y,T) = X^n + (T - 1)X^{n-1}Y \).

Example 3.9. Let \( C \) be the \([n, 1, n]\) repetition code. Then this code has not such an easy description of \( \lambda_C(X,Y,T) \) and \( \alpha_C(X,Y,T) \) as Example 3.8. Apart from the known expressions for \( \lambda_i(T) \) and \( \alpha_i(T) \) for \( i \) up to half the minimum distance \( \lfloor(n - 1)/2 \rfloor \) that hold for every code, we can determine \( \alpha_{n-1} \) and \( \lambda_{n-1} \). Let \( a = (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots, a_n) \) be a coset leader of weight \( n - 1 \), then \( a_1 \mathbf{h}_1 + \ldots + a_{i-1} \mathbf{h}_{i-1} + a_i \mathbf{h}_i + \ldots + a_n \mathbf{h}_n = \mathbf{s} \). Since also \( a_1 \mathbf{h}_1 + \ldots + a_n \mathbf{h}_n = 0 \), we can subtract \( a_j \) times that equation from the first one to obtain the coordinates of another coset leader. This coset leader also has to have weight \( n - 1 \), because \( a \) is a coset leader. This means all coordinates of the coset leader have to be different, and every coset has \( n \) coset leaders. So we have that \( \lambda_{n-1}(T) = n \alpha_{n-1}(T) \) and \( \alpha_{n-1}(T) = (T - 1)(T - 2) \cdots (T - n + 1) \). An alternative proof will be given in Example 5.5.

Although the extended weight enumerator contains a lot of information about a code, it does not determine the coset leader weight enumerator or even the covering radius of a code. See [6] for a counterexample. Also, all \([n, k, n - k + 1]\) codes over \( \mathbb{F}_q \) are called MDS (Maximum Distance Separable) and have the same generalized weight enumerator, but the covering radius varies for fixed \( n, k \) and \( q \). See Example 5.30 of [19] and Example 4.10 in this paper.

Definition 3.10. Let \( H \) be a parity check matrix of a linear \([n, k]\) code \( C \) over \( \mathbb{F}_q \) and let \( y \) be a vector in \( \mathbb{F}_q^n \). Let \( s = Hy^T \) be the syndrome of this word with respect to \( H \). The weight of \( s \) with respect to \( H \), also called the syndrome weight of \( s \), is defined by \( \text{wt}_H(s) = \text{wt}(y + C) \).
Note that $\alpha_i$ is the number of syndromes in $F_q^{n-k}$ with respect to $H$ that are of weight $i$. See [14, Definition 2.1]. This number does not depend on the choice of $H$.

There is a one-to-one correspondence between cosets and syndromes. It is a well known fact that a coset leader corresponds to a minimal way to write its syndrome as a linear combination of the columns of a parity check matrix. The geometric interpretation of the weight of a coset and the syndrome weight is as follows. Let $h_j$ be the $j$-th column of $H$ and let $J \subseteq [n]$. Let $V_J$ be the subspace of $F_q^{n-k}$ that is generated by the vectors $h_j$ with $j \in J$. Then we define the set

$$V_t = \bigcup_{|J|=t} V_J.$$ 

**Proposition 3.11.** Let $s$ in $F_q^{n-k}$ be a syndrome with respect to $H$. Then

$$\text{wt}_H(s) = t \text{ if and only if } s \in V_t \setminus V_{t-1}.$$ 

**Corollary 3.12.** Let $C$ be a linear $[n,k]$ code with parity check matrix $H$. Then $\alpha_t$ is the number of vectors that are in the span of $t$ columns of $H$ but not in the span of $t-1$ columns of $H$.

**Proof.** Both the Proposition and the Corollary follow directly from the definitions of the syndrome weight and $V_t$. □

**Remark 3.13.** Note that Corollary 3.12 implies that $\alpha_i(T) = 0$ for all $i > n-k$, since $V_{n-k}$ contains all vectors of $F_q^{n-k}$. Furthermore

$$\sum_{i=0}^{n-k} \alpha_i = q^{n-k} \text{ and } \sum_{i=0}^{n-k} \alpha_i(T) = T^{n-k}.$$ 

Hence

$$\sum_{i=1}^{n-k} \bar{\alpha}_i = \sum_{j=0}^{n-k-1} q^j \text{ and } \sum_{i=1}^{n-k} \bar{\alpha}_i(T) = \sum_{j=0}^{n-k-1} T^j.$$ 

**Example 3.14.** Let $C$ be the binary Hamming code of length 7. Its parity check matrix consists of all possible nonzero vectors in $F_2^3$, and the corresponding arrangement is shown in Figure 1. We can determine the extended coset leader weight enumerator by Corollary 3.12. As always, we have $\alpha_0(T) = 1$, this is the code itself. There are seven projective points in the arrangement, so $\bar{\alpha}_1(T) = 7$. On each of the seven lines there are $T+1$ points, of which we counted already three per line, so $\bar{\alpha}_3(T) = 7(T-2)$. Since $\bar{\alpha}_1(T) + \bar{\alpha}_2(T) + \bar{\alpha}_3(T) = T^2 + T + 1$, we find that $\bar{\alpha}_3(T) = (T-2)(T-4)$.

We see that $\rho(C) = 1$, $\rho(C \otimes F_4) = 2$ and $\rho(C \otimes F_{2^m}) = 3$ for $m \geq 3$. The list weight enumerator is equal to

$$\lambda_C(X,Y,T) = X^7 + 7(T-1)X^6Y + 21(T-1)(T-2)X^5Y^2 + 28(T-1)(T-2)(T-4)X^4Y^3.$$
Figure 1. The hyperplane arrangement of the parity check matrix of the binary $[7, 4]$ Hamming code.

Figure 2. Two projective systems that induce the same lattice, but induce codes with different coset leader weight enumerators.

From the previous example, one might be tempted to think that the geometric lattice associated to the dual of a code is sufficient for determining the extended coset leader weight enumerator. But this is not true: we already mentioned the case of MDS codes. In Figure 2, we see two projective systems of six points. The points are on two lines in both cases, so the configurations give rise to equivalent geometric lattices. However, the corresponding dual codes do not have the same extended coset leader weight enumerator. In the left configuration, there is one syndrome that corresponds to a coset that has three different coset leaders of weight 2. In the right configuration, we do not have such a syndrome: instead, there are three cosets of weight 2, each with two coset leaders.

4. The derived code

To differentiate between codes with the same geometric lattice, we need the notion of a derived code and the corresponding derived arrangement: this is the arrangement with as hyperplanes all hyperplanes spanned by at least $k - 1$ points of the projective system $\mathcal{P}$. A similar notion of a derived configuration is given in [10], §5.10.
We will define the sets $V_i$ again in a more formal way for an arbitrary $k \times n$ matrix $G$ of rank $\rho$ over a field $\mathbb{F}$ instead of a parity check matrix $H$ of a code over $\mathbb{F}_q$.

**Definition 4.1.** Let $G$ be a $k \times n$ matrix of rank $\rho$ over a field $\mathbb{F}$. Let $J \subseteq \{1, \ldots, n\}$. Let $g_j$ be the $j$-th column of $G$. Let $G_J$ be the $k \times |J|$ submatrix of $G$ consisting of the columns $g_j$ with $j \in J$. Let $G_J(X)$ be the $k \times (|J|+1)$ matrix obtained by adding the column $X = (X_1, \ldots, X_k)^T$ to $G_J$, where $X_1, \ldots, X_k$ are variables. Let $G_{J\setminus \{j\}}$ be the $k \times |J| - 1$ matrix obtained by deleting the $i$-th row of $G_J$. Let $\Delta_J(X) = \det(G_J(X))$ and $\Delta_{i,j} = \det(G_{i,j})$ in case $|J| = k - 1$. Let $D(G)$ be the \textit{derived matrix} of $G$ of size $k \times \binom{n}{k-1}$ with entries $\Delta_{i,j}$ with $i = 1, \ldots, k$ and $J \subseteq \{1, \ldots, n\}$ of size $k - 1$ ordered lexicographically. Let $D_1(G)$ be the matrix obtained from $D(G)$ by deleting the zero columns. Let $D_2(G)$ the matrix obtained from $D_1(G)$ by deleting all columns that are a scalar multiple of a previous column.

Note that $D_2(G) = \bar{D}(G)$ is the \textit{simplification} of $D(G)$.

**Remark 4.2.** Suppose $|J| = k - 1$. Then
\[
\Delta_J(X) = \sum_{i=1}^{k} \Delta_{i,J}X_i \quad \text{and} \quad \Delta_J(g_j) = \sum_{i=1}^{k} \Delta_{i,J}g_{ij} = 0 \quad \text{for all } j \in J.
\]
The columns of $G_J$ are independent if and only if $\Delta_{i,j} \neq 0$ for some $i = 1, \ldots, k$. Moreover $\Delta_J(X)$ is the equation of the hyperplane that is generated by the columns of $G_J$ if these are independent.

**Lemma 4.3.** Let $G$ and $G'$ be two $k \times n$ matrices of rank $\rho$ over a field $\mathbb{F}$. If $G' = SG$ for some $k \times k$ invertible matrix $S$, then
\[
D(G') = \det(S) \cdot S^{-T}D(G),
\]
where $S^{-T}$ is the transpose of the inverse of $S$.

**Proof.** Let $\Delta_{i,J}(X) = \det(G_{i,J}(X))$ and $\Delta'_{i,J} = \det(G'_{i,J})$ for $|J| = k - 1$. Then
\[
G_J'(X) = (G_J'X) = (SG_JX) = S(G_JS^{-1}X).
\]
It follows that
\[
\sum_{j=1}^{k} \Delta'_{i,j}X_j = \Delta_{i,J}(X) = \det(S) \cdot \Delta_J(S^{-1}X) = \det(S) \cdot \sum_{j=1}^{k} \left( \sum_{i=1}^{k} \Delta_{i,J}(S^{-1})_{ij} \right) X_j.
\]

**Definition 4.4.** Let $G$ be a generator matrix of an $[n, k]$ code $C$ over $\mathbb{F}_q$. The \textit{derived codes} $D(C)$, $D_1(C)$ and $D_2(C)$ are defined by the generator matrices $D(G)$, $D_1(G)$ and $D_2(G)$, respectively.

**Remark 4.5.** The definition of the derived codes $D(C)$, $D_1(C)$ and $D_2(C)$ does not depend on the chosen generator matrix by Lemma 4.3, since $\det(S) \cdot S^{-T}$ is an invertible matrix. Notice that $D(C)$ has length $\binom{n}{k-1}$. The code $D_1(C)$ has the same length as $D(C)$ if and only if every subset of $k - 1$ columns of $C$ are independent, that means $d(C^\perp) \geq k$ or, equivalently, $C^\perp$ is almost MDS.
**Example 4.6.** Let $C = \mathbb{F}_q^n$ be the trivial code with generator matrix $G = I_n$. Then $D(G)$ has entries $(-1)^{i+j} \delta_{i,n+1-j}$. Hence $D(C) = C$.

**Example 4.7.** Let $C$ be a code of dimension 2 with generator matrix $G$. Then $D(G)$ has as its first row $(-g_{21} \cdots -g_{2n})$ and in its second row $(g_{11} \cdots g_{1n})$. Hence $D(G)$ is also a generator matrix of $C$ and $D(C) = C$.

**Example 4.8.** Let $C$ be the binary simplex code of dimension 3 and generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$  

Then $D(G) = D_1(G)$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & \end{pmatrix}$$

and $D_2(G)$ is equal to

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$  

**Remark 4.9.** In Example 4.8 we see that the binary simplex code $C$ of dimension 3 is generalized equivalent with $D_2(C)$. This property holds for all $q$-ary simplex codes of any dimension. A converse is also true. Suppose that $C$ is an $[n,k]$ code such that $k \geq 3$ and $C$ and $D_2(C)$ are generalized equivalent codes. Consider the projective system $\mathcal{P}$ of the code. Assume moreover that on every line through two points of $\mathcal{P}$ there is a third point of $\mathcal{P}$. Then $\mathcal{P}$ defines a Desarguesian projective space of dimension $k-1$ that is embedded in the projective space of dimension $k-1$ over $\mathbb{F}_q$. Hence $\mathcal{P}$ is isomorphic with $\mathbb{F}^{k-1}(F)$, where $F$ is a subfield of $\mathbb{F}_q$. See for example [16, §4.3].

**Example 4.10.** Let $G$ be the generator matrix of an MDS code with parameters $[n,k,n-k+1]$. Then all $k \times k$ submatrices of $G$ have rank $k$, so all $k \times (k-1)$ submatrices of $G$ have rank $k-1$. Hence $D(G)$ has no zero columns and $D(G) = D_1(G)$. We show that no two columns of $D(G)$ are multiples of each other. Suppose the contrary: there are two equal columns. Then there exist distinct subsets $I$ and $J$ of $[n]$ such that the hyperplanes $A_I$ and $A_J$ are the same. So the $g_i$, $i \in I$ and $g_j$, $j \in J$ all lie in the same hyperplane. Hence there exists a $j \in J \setminus I$ such that the $k$ vectors $g_i$, $i \in I \cup \{j\}$ lie in the same hyperplane. But that contradicts the fact that all $k \times k$ submatrices of $G$ have rank $k$. Hence $D(G) = D_2(G)$ and the code $D(C)$ is simple and has length $\binom{n}{k-1}$. The converse is also true: $D_2(C)$ and $D(C)$ have the same length if and only if any $k$ columns of $C$ are independent, that means that $C$ is MDS.

Now consider the arrangement $\mathcal{A} = \mathcal{A}_{D(G)}$ defined by $D(G)$. Let $\mathcal{P} = (P_1, \ldots, P_n)$ be the projective system defined by $G$. Consider $P_j$ for some $j$. There are $\binom{n-1}{k-2}$ subsets $I$ of $[n]$ of size $k-2$ such that $I \cup \{j\}$ has size $k-1$, hence there are $\binom{n-2}{k-2}$ hyperplanes of $\mathcal{A}$ going through $P_j$.

Let $P$ be a point of the $k-1$ dimensional projective space that is distinct from $P_j$ for all $j$. If $P$ lies in $l$ hyperplanes of $\mathcal{A}$, then there exist $l$ subsets $I_1, \ldots, I_l$ of
[\n] such that \( P \in A_{I_j} \) for all \( j = 1, \ldots, l \). Then the \( I_j \) are mutually disjoint, since otherwise there are \( k \) points of \( P \) in a hyperplane. Hence \( P \) lies in at most \( n/(k-1) \) hyperplanes. Then \((n-1)_{k-1}\) is the maximal number of hyperplanes of \( A \) going through a given point, since \((n-1)_{k-1} \geq n/(k-1)\). Therefore \( D(C) \) has minimum distance \( (n-1)_{k-1} - (n-1)_{k-2} \) by [20] Proposition 5.15.

If there exists an \([n, 3, n-2]\) code, then its derived code has parameters \([N, 3, d]\) with \( N = \binom{n}{2} \) and \( d = \binom{n-1}{2} \). Let \( d_m \) be the maximal value of the minimum distance of an \( \mathbb{F}_q \)-linear code of length \( N \) and dimension 3. In the following table we give the parameters \( q, n, N, d \) and \( d_m \) in the columns using [7, 12, 13]:

<table>
<thead>
<tr>
<th>( q )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( 7 )</th>
<th>( 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>( N )</td>
<td>15</td>
<td>10</td>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>( d )</td>
<td>10</td>
<td>( 6, 3 )</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>( d_m )</td>
<td>11</td>
<td>( 6, 3 )</td>
<td>11</td>
<td>7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( q )</th>
<th>( 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>10</td>
</tr>
<tr>
<td>( N )</td>
<td>45</td>
</tr>
<tr>
<td>( d )</td>
<td>36</td>
</tr>
<tr>
<td>( d_m )</td>
<td>39</td>
</tr>
</tbody>
</table>

Only in two cases we obtain a code with maximum known \( d_m \) as a derived code.

**Example 4.11.** This example is Example 2.28 from [17]. We consider two \([6, 3, 4]\) MDS codes \( C_1 \) and \( C_2 \) over \( \mathbb{F}_9 \), generated by the matrices

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & \alpha^5 & \alpha^6 \\
0 & 0 & 1 & \alpha^3 & \alpha & \alpha^3
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & \alpha^7 & \alpha^4 & \alpha^6 \\
0 & 0 & 1 & \alpha^5 & \alpha & 1
\end{pmatrix},
\]

where \( \alpha \in \mathbb{F}_9 \) is the primitive element such that \( \alpha^2 + \alpha + 1 = 0 \). The derived codes \( D(C_1) \) and \( D(C_2) \) are generated by

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & \alpha^4 & \alpha^6 & \alpha^3 & \alpha^7 & \alpha & 1 & \alpha^2 & 1 & \alpha & 1 & \alpha^7 & 1 \\
0 & 1 & 0 & \alpha^7 & 1 & 0 & \alpha^4 & 1 & 1 & \alpha^6 & \alpha & 1 & \alpha & 1 & \alpha^3 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0 & 0 & \alpha^7 & \alpha^2 & \alpha^3 & \alpha & 0 & \alpha^7 & \alpha^4 & \alpha^7 & \alpha & 0 & 0 \\
0 & 1 & 0 & 1 & \alpha^3 & 0 & \alpha^6 & \alpha^6 & 0 & \alpha^7 & \alpha & \alpha^6 & \alpha^3 & \alpha & \alpha^5 \\
0 & 0 & 1 & \alpha^5 & \alpha^5 & \alpha^3 & \alpha^7 & \alpha^4 & \alpha^3 & \alpha^5 & \alpha^4 & \alpha & \alpha^5 & \alpha^5
\end{pmatrix}.
\]

The weight distributions of \( D(C_1) \) and \( D(C_2) \) are, respectively,

\[
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 48, 0, 16, 312, 288, 64)
\]

and

\[
(1, 0, 0, 0, 0, 0, 0, 0, 0, 48, 0, 32, 264, 336, 48).
\]

So the latter two codes are not generalized equivalent, and therefore not all \([6, 3, 4]\) MDS codes over \( \mathbb{F}_9 \) are generalized equivalent.

**Remark 4.12.** It is not possible to define the notion of a derived matroid \( D(M) \) for a given matroid \( M \) such that \( D(M_c) \) is isomorphic with \( M_{D(C)} \) for all codes \( C \). The reason for this lies in the representation of matroids. It is possible to represent a matroid by various matrices. Even if those matrices are defined over the same field, it is possible that they generate nonequivalent codes. These nonequivalent
codes are likely to have different derived codes. This is the case in Example 4.11 where the two codes $C_1$ and $C_2$ have the same uniform matroid, but the matroids of the derived codes $D(C_1)$ and $D(C_2)$ are not isomorphic.

We can now give a more formal definition of the sets $V_i$.

**Definition 4.13.** Let $G$ be a $k \times n$ matrix of rank $k$ over a field $\mathbb{F}$. Let $J \subseteq \{1, \ldots, n\}$. Let $V_J$ be the vector subspace of $\mathbb{F}^k$ that is generated by the vectors $g_j^T$, $j \in J$. Then we define

$$V_i = \bigcup_{|J|=t} V_J.$$ 

**Remark 4.14.** Let $L_G$ be the partial order with as elements the subspaces $V_J$ and the inclusion as partial order. Then $L_G$ is a geometric lattice, see Example 2.2. Let $A_G$ be the arrangement of $G$. Let $L(A_G)$ be the geometric lattice of the arrangement $A_G$. Consider the map $\varphi : L_G \rightarrow L(A_G)$ defined by

$$\varphi(V_J) = A_J.$$ 

Recall from Definition 2.10 that $A_J = \bigcap_{i \in J} A_i$. Then $\varphi$ is an isomorphism from $L_G$ to $L(A_G)$ as geometric lattices, see Example 2.3.

Consider the arrangement $A_{D_2(G)}$. This arrangement is essential and simple. $A_{D_2(G)}$ is the collection of all hyperplanes $A_J$ with equation $\Delta_J(X) = 0$, where $J$ consists of $k-1$ elements such that the $g_i$, $i \in J$ are independent. Hence $A_J = V_J$ in this case.

For arbitrary $J$, if $V_J$ has dimension $l$, then there is a $J' \subseteq J$ consisting of $l$ elements such that $g_i$, $i \in J'$ are independent. So $V_J = V_{J'}$. Now $V_J$ is a subspace of the column space of $G$, which has dimension $k$. Hence there is an $I \subseteq [n]$ consisting of $k$ elements such that $J' \subseteq I$ and the vectors $g_i$, $i \in I$ are independent. So $V_{I \setminus \{i\}}$ is equal to the hyperplane $A_{I \setminus \{i\}}$ of the arrangement $A_{D_2(G)}$ and

$$V_J = \bigcap_{i \in (I \setminus J')} A_{I \setminus \{i\}}.$$ 

is an intersection of the $k-l$ hyperplanes of the arrangement $A_{D_2(G)}$.

**Definition 4.15.** Let $G$ be a $k \times n$ matrix of rank $k$ over a field $\mathbb{F}$. Let $L_G$ be the geometric lattice of $G$. Let $L(A_{D_2(G)})$ be the geometric lattice of the arrangement $A_{D_2(G)}$. Define the map

$$\psi : L_G \rightarrow L(A_{D_2(G)}) \text{ by } \psi(V_J) = V_J.$$ 

**Remark 4.16.** The map $\psi$ is well defined, since $V_J$ is an intersection of hyperplanes of the arrangement $A_{D_2(G)}$. The map $\psi$ is not a morphism of posets but an anti-morphism, since if $x \leq y$ in $L_G$, then $x \subseteq y$, hence $\psi(y) \leq \psi(x)$ in $L(A_{D_2(G)})$. The map $\psi$ is clearly injective, but not always surjective. Furthermore $\psi$ is not an anti-morphism of lattices, that is the rules $\psi(x \wedge y) = \psi(x) \vee \psi(y)$ and $\psi(x \vee y) = \psi(x) \wedge \psi(y)$ do not always hold. See Example 4.18.

**Remark 4.17.** The matrix $G$ gives rise to the decreasing sequence

$$N_k \subset N_{k-1} \subset \cdots \subset N_1 \subset N_0$$

of algebraic subsets of the affine space $\mathbb{A}^k$ as explained in Section 2. Furthermore we have the decreasing sequence

$$V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k.$$
Figure 3. The projective system of $G$ (the points $a, b, c, d$) and the arrangement of $D(G)$ (the lines 1, 2, 3, 4, 5, 6).

Figure 4. The Hasse diagram of the code in Example 4.18.

of the algebraic subsets $V$ defined before. If $x \in V_t \setminus V_{t+1}$, then $x \in V_J$ for some $J$ such that $t = |J| = \dim(V_J)$. Hence $\psi(V_J) = V_J \subseteq (N_{k-t} \setminus N_{k-t-1})$. Hence $(V_t \setminus V_{t+1}) \subseteq (N_{k-t} \setminus N_{k-t-1})$ and $V_t \subseteq N_{k-t}$ for all $t$. But it is not always the case that $V_t = N_{k-t}$, as we will see in Example 4.18.

Example 4.18. Let $G$ be the $3 \times 4$ matrix given by

$$G = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \text{ then } D(G) = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$ 

A picture of the projective system is shown in Figure 3. The Hasse diagrams of the code and the derived code are given in Figures 3 and 5.

5. Coset leader weights and the derived code

We will now use the derived code, as defined in the previous section, to find a formal way to determine the extended coset leader weight enumerator and extended list weight enumerator.
Figure 5. The Hasse diagram of the derived code in Example 4.18. On the second level, there are the points a, b, e, c, f, g, d respectively.

**Definition 5.1.** Let $H$ be a parity check matrix of the code $C$ of an $[n, k]$ code. Let $L_H$ be the geometric lattice of $H$. Let $A$ be the arrangement of $D_2(H)$ and let $L = L(A)$ be the geometric lattice of $A$ with rank function $r$. Let $\psi : L_H \rightarrow L$ be the map defined in Definition 4.15. Define for $x \in L$:

$$\hat{r}(x) = \max \{ r(y) : y \in \psi(L_H), y \leq x \}.$$  

**Remark 5.2.** Note that $\hat{r}(x) = r(x)$ for $x \in L$ if and only if $x \in \psi(L_H)$. Hence $\hat{r}(x) = r(x)$ for all $x \in L$ if and only if $L = \psi(L_H)$.

We can now determine the extended coset leader weight enumerator. The formula for $\alpha_i(T)$ is analogous to the one for $\mu_i(T)$ of Proposition 2.12.

**Theorem 5.3.** The coefficients of the extended coset leader weight enumerator are given by

$$\alpha_i(T) = \sum_{x \in L, \hat{r}(x) = n-k-i} \chi_{L_x}(T).$$

**Proof.** Suppose we have a syndrome vector $x \in \mathbb{A}^{n-k}$. We want to determine its syndrome weight $wt_H(x)$. We do this by looking at the elements of $L$, which are subspaces of $\mathbb{A}^{n-k}$. Let $x \in L$ be the element in $L$ with the highest rank that contains $x$. Such an element exists, because every syndrome vector is contained in $0_L = \mathbb{A}^{n-k}$ and if a vector is contained in some $y \in L$, then it is also contained in all elements covered by $y$. Furthermore, this element $x$ is unique, since if there were more, $x$ would be in there intersection, which is an element of $L$ of higher rank. In fact, $x$ is contained in all elements of the sublattice $L^x$.

We distinguish between the two cases $x \in \psi(L_H)$ and $x \notin \psi(L_H)$. First, assume $x \in \psi(L_H)$ and $\dim x = t$. Then $r(x) = n-k-t$ and also $\hat{r}(x) = n-k-t$. Now $x$ can be written as a linear combination of $t$ vectors of $H$: the columns corresponding to the coatoms bigger than $x$ in $L$, or, equivalently, the atoms smaller than $x$ in $L_H$. But since $x$ has maximal rank such that $x \in x$, we have that $x$ can not be written as a linear combination of $t-1$ columns of $H$. This means $x \in V_t \setminus V_{t-1}$ and thus $wt_H(x) = t$.

Now suppose $x \notin \psi(L_H)$. Then there is a $y \in \psi(L_H)$ such that $y \leq x$ and there is
no element of higher rank with that property. This \( y \) is not necessarily unique, but its rank is: it is equal to \( \hat{r}(x) \). Let \( \hat{r}(x) = n - k - t = r(y) \), then \( y \) is of dimension \( t \) and \( x \in V_I \setminus V_{i-1} \) because the rank of \( y \) is maximal.

Combining this, we see that for all vectors \( x \in x \) that are not in any element bigger than \( x \), we have \( \text{wt}_H(x) = t \). The number of such vectors is given by \( \chi_{L_v}(q^m) \), as was explained in Remark 2.9. This proves the given formula. \( \square \)

**Remark 5.4.** If \( L = \psi(L_H) \), then \( \alpha_i(T) = \mu^*_n-k-1(T) \), where \( \mu^*_n(T) \) is the Möbius polynomial of the dual code.

**Example 5.5.** Consider the \([n,1,n]\) repetition code with parity check matrix \( H = (I_{n-1} | h_n) \) with \( h_n = (-1, \ldots, -1)^T \). Let \( A_{ij} \) be the hyperplane spanned by all column vectors of \( H \) except \( h_i \) and \( h_j \). Then \( A_{ij} \) is given by the equation \( X_i - X_j = 0 \) if \( 1 \leq i < j \leq n \) and by \( X_i = 0 \) if \( 1 \leq i < j = n \). The arrangement of \( A_{D_H} \) is simple and consists of the \( A_{ij} \).

The geometric lattice \( L(A_{D_H}) \) has the following alternative combinatorial description by collections \( P \) of subsets of \( \{1, \ldots, n\} \). The minimum element and the maximum element both consist of one subset, the empty set and the whole set, respectively. Otherwise \( P \) is a collection of mutual disjoint subsets of \( \{1, \ldots, n\} \) with at least 2 elements. Then this \( P \) corresponds one-to-one to the intersection of all hyperplanes \( A_{ij} \) such that there exists a subset \( I \in P \) such that \( i, j \in I \). If \( P \) is not the minimum element, then the rank of \( P \) is equal to \( | \cup P | - | P | \), that is if \( P \) consists of the mutual disjoint subsets \( I_1, \ldots, I_t \), then

\[
    r(P) = \sum_{i=1}^{t} | I_i | - t.
\]

The partial order is given by \( P \leq P' \) if and only if for all \( I \in P \) there exists (a unique) \( I' \in P' \) such that \( I \subseteq I' \).

The vector subspace \( V_J \) of \( \mathbb{F}^{n-1} \) is generated by the vectors \( h_i^T \), \( j \in J \). If \( | J | = n-1 \), then \( V_J = \mathbb{F}^{n-1} \). Now \( L(H) \) consists of all \( V_J \) with \( J \) a subset of \( \{1, \ldots, n\} \) such that \( | J | \neq n - 1 \). The \( P \) that corresponds to \( \psi(V_J) \) is the singleton consisting of the unique subset \( J^c = \{1, \ldots, n\} \setminus J \).

Let \( P \) consist of the mutual disjoint subsets \( I_1, \ldots, I_t \). Let \( Q \) corresponds to an element \( \psi(V_J) \). Then \( Q \leq P \) if and only if there exists an \( i \) such that \( J^c \subseteq I_i \). Hence

\[
    \hat{r}(P) = \max_{i=1,\ldots,t} | I_i | - 1
\]

if \( P \) is not the minimum element and \( \hat{r}(P) = 0 \) if and only if \( P \) is the minimum element. So \( \alpha_{n-1}(T) = \chi_{L(P_{D_H})}(T) = (T - 1)(T - 2) \cdots (T - n + 1) \) by [28] Corollary 2.2 and [1, 29]. This is in accordance with Example 3.9.

For the determination of the extended list weight enumerator, we can take a similar approach. We now need to calculate not only the coset weight of a syndrome vector, but also the number of different coset leaders. We do this with the following multiplicity function, defined for all \( x \in L \):

\[
    m(x) = | \{ y \in \psi(L_H) : y \leq x, r(y) = \hat{r}(x) \} |.
\]

**Remark 5.6.** The number \( m(x) \) counts the number of elements of maximal rank in \( L \) such that they are below \( x \) and elements of \( \psi(L_H) \). By the definition of \( \hat{r} \), we have that \( m(x) = 1 \) if \( r(x) = \hat{r}(x) \).
The formula for $\lambda_i(T)$ is now as follows.

**Theorem 5.7.**

$$
\lambda_i(T) = \sum_{x \in L, \hat{r}(x) = n - k - i} m(x) \cdot \chi_L(T).
$$

**Proof.** Let $x$ and $\hat{r}$ be defined as in the proof of Theorem 5.3. Again, we distinguish between the two cases $x \in \psi(L_H)$ and $x \notin \psi(L_H)$.

If $x \in \psi(L_H)$, then from the proof of Theorem 5.3 we know that $x$ is a linear combination of the atoms smaller than $x$ in $L_H$. Because $L_H$ is an atomic lattice, this linear combination is unique. So $x$ is a syndrome that corresponds to a coset with a unique coset leader. For these cosets, their contribution to the extended list weight enumerator is the same as their contribution to the coset leader weight enumerator. This means we should have $m(x) = 1$, which is indeed the case as we noticed in Remark 5.6.

Now let $x \notin \psi(L_H)$. From the proof of Theorem 5.3 we know there is a $y \in \psi(L_H)$ such that $y \leq x$ and there is no element of higher rank with that property. As noticed, this $y$ does not need to be unique. In fact, the number of $y$’s with this property is exactly $m(x)$. Every $y$ is a unique join of elements of $L_H$, so every $y$ corresponds to a way of writing $x$ as a linear combination of columns of $H$. Therefore, the contribution of $x$ to the extended list weight enumerator is $m(x)$ times its contribution to the extended coset leader weight enumerator. 

We illustrate the previous theorems by several examples.

**Example 5.8.** Let $C$ be an $[n, n-2, 3]$ code. Let $H$ be a parity check matrix of $C$. Then $D(H)$ and $H$ generate the same code $C^\perp$ by Example 4.7. Furthermore, $C^\perp$ is an MDS code. Hence $D_2(H) = D(H)$ and the extended coset leader weight enumerator of $C$ is equal to the M"obius polynomial of $C^\perp$:

$$
\alpha_1(T) = \mu_1(T) = n(T-1) 
$$

and

$$
\alpha_2(T) = \mu_2(T) = (T-1)(T-n).
$$

**Example 5.9.** Let $C$ be the $q$-ary Hamming code of redundancy $r$ and length $(q^r - 1)/(q-1)$. Then $C^\perp$ and $D_2(C^\perp)$ are both general equivalent to the simplex code, see Example 4.8. Hence by Remark 5.3 the extended coset leader weight enumerator of $C$ is equal to the M"obius polynomial of the simplex code, which is computed in [20] Example 5.53:

$$
\alpha_i(T) = \mu_{r-i}(T) = \left[\begin{array}{c} r \\ i \end{array}\right]_q (T - 1)(T - q) \cdots (T - q^{i-1}),
$$

where $\left[\begin{array}{c} r \\ i \end{array}\right]_q$ is the Gaussian binomial.

**Example 5.10.** Let $C$ be an MDS code with parameters $[n,n-3,4]$ with parity check matrix $H$ over the field $\mathbb{F}_q$. Then the projective system $P$ of $H$ consists of $n$ points in the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ such that no three points of $P$ are on a line. This means that $P$ is an arc. See for example Chapter 8 of [16]. Lines that intersect $P$ in 0, 1 or 2 points are called exterior lines, tangents and secants, respectively.

Suppose $q$ is odd and $P$ is a complete arc. Then $n = q + 1$ and $P$ is a conic. It is a classical result that the points and lines of $\mathbb{P}^2(\mathbb{F}_q)$ can be divided into three types. See for example §8.1 of [16]. For the points, it is as follows:

- There are $\left[\begin{array}{c} q+1 \\ 2 \end{array}\right]$ external points of $C$. Through such a point are two tangents of $C$, $\frac{1}{2}(q-1)$ secants of $C$, and $\frac{1}{2}(q-1)$ exterior lines of $C$.  

\begin{itemize}
\item There are \(q + 1\) points on \(C\). Through such a point there is one tangent of \(C\) and \(q\) secants of \(C\).
\item There are \(\binom{q}{2}\) internal points of \(C\). Through such a point are no tangents of \(C\), \(\frac{1}{2}(q + 1)\) secants of \(C\), and \(\frac{1}{2}(q + 1)\) exterior lines of \(C\).
\end{itemize}

Dually, we have the following types of lines:

\begin{itemize}
\item There are \(\binom{q + 1}{2}\) secants of \(C\). On such a line, two points are on \(C\), \(\frac{1}{2}(q - 1)\) points are internal points of \(C\) and \(\frac{1}{2}(q - 1)\) points are external points of \(C\).
\item There are \(q + 1\) tangents to \(C\).
\item There are \(\binom{q}{2}\) exterior lines of \(C\). On such a line, \(\frac{1}{2}(q + 1)\) points are internal points of \(C\) and \(\frac{1}{2}(q + 1)\) points are external points of \(C\).
\end{itemize}

The derived arrangement \(\mathcal{A}\) of \(H\) consists of the \(\binom{q + 1}{2}\) secants of \(C\). All points of \(\mathbb{P}^2(\mathbb{F}_q)\) are on at least two lines of \(\mathcal{A}\), i.e., are intersections of the lines of \(\mathcal{A}\), if \(q > 3\). Assume \(q > 3\). The lattice \(L = L(\mathcal{A})\) has on the first level \(L_1\) all lines of \(\mathcal{A}\) and \(\hat{r}(x) = r(x) = 1\) if \(x\) is such a line. For such a line we have \(\chi_{L_x}(T) = T^2 - (q + 1)T + q = (T - 1)(T - q)\) by Definition 2.4.

The second level \(L_2\) consists of all the points of the plane \(\mathbb{P}^2(\mathbb{F}_q)\). For these points, \(\hat{r}(x) = 2\) if \(x\) is a point of \(\mathcal{P}\) and \(\hat{r}(x) = 1\) otherwise. For all these points we have \(\chi_{L_x}(T) = T - 1\) by Definition 2.4. As usual, \(\alpha_0(T) = 1\). For \(\alpha_1(T)\), we look at all \(x\) with \(\hat{r}(x) = 2\). These are exactly the points of \(\mathcal{P}\) and there are \(q + 1\) of them, so

\[\alpha_1(T) = \sum_{\hat{r}(x)=2} \chi_{L_x}(T) = (q + 1)(T - 1).\]

For \(\alpha_3(T)\), we look at all \(x\) with \(\hat{r}(x) = 1\). These are the points not in \(\mathcal{P}\) and the lines of \(\mathcal{A}\), so

\[\alpha_2(T) = \sum_{\hat{r}(x)=1} \chi_{L_x}(T) = (q^2 + q + 1 - (q + 1))(T - 1) + \binom{q + 1}{2}(T - 1)(T - q) = (T - 1)\left((\binom{q + 1}{2}T + q^2 - q\binom{q + 1}{2})\right).\]

We can determine \(\alpha_3(T)\) by Remark 3.13

\[\alpha_3(T) = (T - 1)(T^2 + T + 1 - (q + 1) - \binom{q + 1}{2}) = (T - 1)(T^2 + (1 - \binom{q + 1}{2})T - q(q + 1) + q\binom{q + 1}{2})\]

In case \(q = 3\) we get similarly:

\[\begin{align*}
\alpha_0(T) &= 1 \\
\alpha_1(T) &= 4(T - 1) \\
\alpha_2(T) &= 3(T - 1) + 6(T - 1)(T - 2) \\
&= (T - 1)(6T - 9) \\
\alpha_3(T) &= (T^2 + T + 1 - (6T - 9))(T - 1) \\
&= (T^2 - 5T + 10)(T - 1)
\end{align*}\]

We will now determine the extended list weight enumerator for \(q > 3\). Again, \(\lambda_0(T) = 1\). For \(\lambda_1(T)\), note that \(m(x) = 1\) for all \(x\) with \(\hat{r}(x) = 2\) because of
Remark 5.6 So
\[ \lambda_1(T) = \alpha_1(T) = (q + 1)(T - 1). \]

For \( \lambda_2(T) \), look at the \( x \) such that \( r(x) = 1 \). Then \( x \) is a line of \( \mathcal{A} \) or a point not on \( \mathcal{P} \). In the first case, \( m(x) = 1 \). In the second case, we have to count how many lines of \( \mathcal{A} \) go through the point \( x \). This depends on \( x \) being an internal or external point of \( \mathcal{P} \).

\[ \lambda_2(T) = \begin{cases} (q^2 + 1)(T - q) + \frac{1}{2}q(q - 1) + \frac{1}{2}(q + 1) & \text{if } q \text{ even and } n > q, \\ (q^2 + 1)(T - 1) & \text{otherwise}. \end{cases} \]

The code is MDS. So every triple is an information set and the number of information sets of \( C \) is \( i(C) = \binom{q+1}{3} \). Furthermore \( \lambda_3(T) = i(C)\alpha_3(T) \) by Remark 3.3. Hence
\[ \lambda_3(T) = \left( \binom{q+1}{3} \right) \left( T^2 + T + 1 - (q^2 + q + 1 + \binom{q+1}{2})(T - q) \right) = \left( \binom{q+1}{3} \right)(T - q)(T - \frac{1}{2}(q - 2)(q + 1)). \]

**Example 5.11.** Let \( C \) be an MDS code with parameters \([n, n - 3, 4]\) with parity check matrix \( H \) over the field \( \mathbb{F}_q \). We saw in Example 5.10 that the points of \( \mathcal{P} \) form an arc in the projective plane. An arc in \( \mathbb{P}^2(\mathbb{F}_q) \) is complete if and only if every point of \( \mathbb{P}^2(\mathbb{F}_q) \) lies on a secant of the arc. If \( q \) is even and \( n > q - \sqrt{q} + 1 \), then \( \mathcal{P} \) is contained in a complete arc of \( q + 2 \) elements. If moreover \((n, q) \neq (2, 2)\) then this complete arc is unique. See [16, Corollary 2 of Theorem 10.3.3]. In case \( q \) is odd and \( n > q - \frac{1}{4}\sqrt{q} + \frac{7}{4} \), then \( \mathcal{P} \) is contained in a unique complete arc of \( q + 1 \) elements, that is a conic. See [16, Theorem 10.4.4] and for recent improvements: “Planar arcs” by S. Ball and M. Lavrauw, arXiv:1705.10940v4.

Assume that \( q \) is odd and \( n > q - \frac{1}{4}\sqrt{q} + \frac{7}{4} \). Then \( \mathcal{P} \) is contained in a unique conic \( C \) of \( q + 1 \) elements. A point \( Q \) in the projective plane over \( \mathbb{F}_q \) lies on a secant of \( \mathcal{P} \) if and only if \( Q \notin C \cap \mathcal{P} \).

The points of \( \mathcal{P} \) correspond one-to-one to the projective syndromes of weight 1. So \( \alpha_1(T) = n \).

All points outside \( C \) correspond one-to-one to projective syndromes of weight 2 over \( \mathbb{F}_q \). The projective syndromes of weight 2 over \( \mathbb{F}_q \) but not over \( \mathbb{F}_q \) correspond one-to-one to a point on a unique line between two distinct points of \( \mathcal{P} \), since a point that lies on two secant lines of \( \mathcal{P} \) is defined over \( \mathbb{F}_q \). Hence
\[ \bar{\alpha}_2(T) = q^2 + q + 1 - (q + 1) + \binom{n}{2}(T + 1 - (q + 1)) = \binom{n}{2}T + q^2 - q\binom{n}{2}. \]

Finally \( \bar{\alpha}_1(T) + \bar{\alpha}_2(T) + \bar{\alpha}_3(T) = T^2 + T + 1 \) by 3.13. Therefore
\[ \bar{\alpha}_3(T) = T^2 + (1 - \binom{n}{2})T + q\binom{n}{2} - q^2 - n + 1. \]

Note that when substituting \( n = q + 1 \), we get the formulas from Example 5.10. If \( q \) even and \( n > \frac{1}{2}(q + 2) \), then similarly
\[ \bar{\alpha}_2(T) = \binom{n}{2}T + q^2 - 1 - q\binom{n}{2}, \]
\[ \bar{\alpha}_3(T) = T^2 + (1 - \binom{n}{2})T + q\binom{n}{2} - q^2 - n + 2. \]

The derived arrangement is more difficult from the case in Example 5.10. Therefore determining the list weight enumerator is also not easy.
6. MacWilliams type property for duality

Research Problem 5.1 in [24, Chapter 5] asked whether the coset leader weight enumerator of $C$ determines the coset leader weight enumerator of $C^\perp$, as is the case for the ordinary weight enumerator by the MacWilliams relations. This problem has a negative answer by [2]. The authors give three binary [15,3,7] codes that have the same coset leader weight enumerator, but the dual codes have mutually distinct coset leader weight enumerators. In fact a much smaller counterexample will do.

Example 6.1. The two codes of length 3 with parity check matrices $H_1 = (110)$ and $H_2 = (111)$ both have the same extended coset leader weight enumerator $X^3 + (T - 1)X^2Y$. But their dual codes have distinct extended coset leader weight enumerator, since

$$\alpha_{C^+}(X,Y,T) = X^3 + 2(T - 1)X^2Y + (T - 1)XY^2$$

$$\alpha_{C^-}(X,Y,T) = X^3 + 3(T - 1)X^2Y + (T - 1)(T - 2)XY^2.$$  

Remark that the code $C^+_1$ is degenerate. A non degenerate counterexample is obtained as follows.

Example 6.2. Let $C_3$ and $C_4$ be the two [6,3] codes over $\mathbb{F}_2$ with generator matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$  

The next table shows the coefficients of the extended coset leader weight enumerator and the extended list weight enumerator of the codes and their duals. The values for $i = 0$ are left out, since they are all equal to 1 because of the zero word.

<table>
<thead>
<tr>
<th></th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{C,i}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$5(T - 1)$</td>
<td>$5(T - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$2(T - 1)(3T - 5)$</td>
<td>$2(T - 1)(3T - 5)$</td>
</tr>
<tr>
<td>3</td>
<td>$(T - 1)(T - 2)(T - 3)$</td>
<td>$(T - 1)(T - 2)(T - 3)$</td>
</tr>
<tr>
<td>$\alpha_{C^+,-i}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$4(T - 1)$</td>
<td>$5(T - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$3(T - 1)(2T - 3)$</td>
<td>$2(T - 1)(3T - 5)$</td>
</tr>
<tr>
<td>3</td>
<td>$(T - 1)(T - 2)(T - 3)$</td>
<td>$(T - 1)(T - 2)(T - 3)$</td>
</tr>
<tr>
<td>$\lambda_{C,i}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$6(T - 1)$</td>
<td>$6(T - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$2(T - 1)(7T - 12)$</td>
<td>$2(T - 1)(7T - 11)$</td>
</tr>
<tr>
<td>3</td>
<td>$12(T - 1)(T - 2)(T - 3)$</td>
<td>$13(T - 1)(T - 2)(T - 3)$</td>
</tr>
<tr>
<td>$\lambda_{C^+,-i}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$6(T - 1)$</td>
<td>$6(T - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$13(T - 1)^2$</td>
<td>$2(T - 1)(7T - 11)$</td>
</tr>
<tr>
<td>3</td>
<td>$12(T - 1)(T - 2)(T - 3)$</td>
<td>$13(T - 1)(T - 2)(T - 3)$</td>
</tr>
</tbody>
</table>

We see that the extended coset leader weight enumerator of the two codes are equal, but none of the other polynomials, so they are not defined by the extended coset leader weight enumerator.
7. Directions for further work

We have given a formal way to determine the extended coset leader and list weight enumerator, via the derived code. Several examples were considered. We also studied to which extend the several polynomials define each other. The theory and examples give rise to new research problems.

The structure of the derived arrangement is actually studying dependencies between dependencies of points in the projective space. In computational geometry, these is known as second order syzygies. See [10] for an introduction to the topic. For determining the coset leader and list weight enumerator of a given code, the language of computational geometry could be suitable.

It was mentioned in Remark 4.12 that the derived code is not a matroid invariant. However, one could define a ‘derived matroid’ as the most general derived arrangement possible. For example, in Figure 2, the situation on the left should never happen: three lines in the derived arrangement should be in the most general possible position, so they can not intersect in a point. Studying this object might give us more information between invariants of a code that are matroid invariants, such as the extended weight enumerator, and the extended coset leader weight enumerator.

In Examples [5, 10] we considered a Reed-Solomon code of codimension 3. Can we generalize the methods in this example to other dimensions? Example 5.5 calculates \( \alpha_i(T) \) and \( \lambda_i(T) \) for some cases, but more calculations are necessary to complete the determination of the extended coset leader and list weight enumerator. Are all \( \alpha_i \) a product of linear factors?

If we repeat the procedure of taking the derived code, we will eventually end up with the simplex code, i.e., all hyperplanes of the projective space belong to the derived code. We wonder how fast this process converges, and what it tells us about the code.

In Section 6, we showed by counterexample that the extended coset leader weight enumerator is not determined by the same polynomial from the dual code. It seems unlikely that such a relation exists for the extended list weight enumerator, but so far there is no counterexample known.

References
