Entanglement-assisted Quantum Codes from Algebraic Geometry Codes

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Abstract. Quantum error correcting codes play the role of suppressing noise and decoherence in quantum systems by introducing redundancy. Some strategies can be used to improve the parameters of these codes. For example, entanglement can provide a way for quantum error correcting codes to achieve higher rates than the one obtained via traditional stabilizer formalism. Such codes are called entanglement-assisted quantum (QUENTA) codes. In this paper, we use algebraic geometry codes to construct three families of QUENTA codes, where one of them has maximal entanglement and is maximal distance separable. At the end, we show that for any asymptotically good tower of algebraic function fields there is an asymptotically good family of maximal entanglement QUENTA codes with nonzero rate, relative minimal distance, and relative amount of entanglement.

Keywords: Quantum Codes · Algebraic Geometry Codes · Maximal Distance Separable · Maximal Entanglement · Asymptotically Good.

1 Introduction

It is generally accepted that the prospect of practical large-scale quantum computers and the use of quantum communication are only possible with the implementation of quantum error correcting codes. Quantum error correcting codes play the role of suppressing noise and decoherence by introducing redundancy. The capability of correcting errors of such codes can be improved if it is possible to have pre-shared entanglement states. This class of codes is known as Entanglement-Assisted Quantum (QUENTA)codes, also denoted by EAQECC's in the literature. Additionally, this class of codes can achieve the hashing bound [24,13] and violate the quantum Hamming bound [14]. The first QUENTA codes were proposed by Bowen [1] followed by the work from Fattal, *et al.* [9]. The stabilizer formalism of QUENTA codes was created by Brun *et al.* [2], where they showed that QUENTA codes paradigm does not require the dual-containing constraint as the standard quantum error-correcting code does [15].

After this landmark paper from Brun *et al.*, many works have focused on the construction of QUENTA codes based on classical linear codes [23,5,18,12,17]. However, the analysis of q-ary QUENTA codes was taken into account only recently [8,5,20,6,12,16,11,17]. The majority of them utilized constacyclic codes [8,6,20] or negacyclic codes [5,20] as the classical counterpart. Since the length of the classical codes is normally proportional to the square of the size of the field, most of the quantum codes from the previous works have a length that is proportional to the square of the size of the finite field. On the other hand, Liu *et al.* used k-Galois dual codes [17] and Galois LCD codes [16], which allow quantum codes with length lower than the previous ones mentioned. So, there is no result in the literature with QUENTA codes having length proportional to a greater power of the cardinality of the finite field. In addition, it has not been shown previously that there exists a family of asymptotically good maximal entanglement QUENTA codes. Such a family can be used to achieve the hashing bound. A possible approach to solve both questions is using algebraic geometry codes (AG) codes as the classical counterpart to construct QUENTA codes.

The AG codes were invented by Goppa [10] and have several properties. Two properties important for this paper are that its parameters can be calculated via the degree of a divisor, which allows a direct description of the code, and the intersection of two AG codes can be associated to a linear code that is also an AG code. As will be shown, using AG codes we can derive quantum codes with interesting properties. Before presenting such results, some constructions have been made.

First of all, we introduce the idea of relative hull, which is a generalization of the concept of hull utilized in the construction of Linear Complementary Dual (LCD) codes [4]. With this tool, it is possible to quantify the amount of entanglement in a QUENTA codes in a more direct way if the intersection of two classical codes is known. As will be shown, this is the case for AG codes. A analysis of the lower bound for the minimal distance of some quantum codes constructed demonstrates that this bound differs from the Singleton bound up to one unit, leading to the conclusion that it is possible to construct QUENTA codes from AG codes that are almost maximal distance separable (MDS) codes or maximal distance separable (MDS) codes; i.e., they have Singleton defects equal to one or zero. Furthermore, these codes also have maximal entanglement, which can be employed to achieve entanglement-assisted quantum capacity of a depolarizing channel [1,7,19]. We give three examples of families of QUENTA codes with these properties.

The paper is organized as follows. In Section 2, we describe what needs to be known about AG codes so that they can be applied to the construction method of Wilde and Brun [23]. Afterwards, a few basic description of how to utilize AG codes to construct QUENTA codes are given in Section 3. In this same section, we apply the method proposed to AG codes constructed from rational, Hermitian, and Elliptic function fields; the first function field allows QUENTA codes that are MDS and latter function field allows almost MDS ones. In Section 4, we show that there exists families of QUENTA codes that are asymptotically good in its rate, relative distance and entanglement-assisted rate. Lastly, the conclusion is carried out in Section 5.

Notation. Throughout this paper, p denotes a prime number and q is a power of p. Let $F/\mathbb{F}_q$ be an algebraic function field over $\mathbb{F}_q$ of genus g, where $\mathbb{F}_q$ denotes the finite field with q elements. A linear code C with parameters $[n, k, d]_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum distance d. Lastly, an $[[n, k, d; c]]_q$ quantum code is a q^k-dimensional subspace of $\mathbb{C}^{q^n}$ with minimum distance d that utilizes c pre-shared entanglement pairs.

2 Preliminaries

In this section, we introduce some ideas related to linear complementary dual (LCD) codes, algebraic geometry (AG) codes and entanglement-assisted quantum (QUENTA) codes. The first description to be given is that of LCD codes, but first we need to give the definition of the Euclidean dual of a code.

Definition 1. *Let C be a $\mathbb{F}_q$-linear code of length n. The dual of C is defined as $C^{\perp} := \{\ \mathbf{x} \in \mathbb{F}_q^n \mid \mathbf{x} \cdot \mathbf{c} = 0 \text{ for all } \mathbf{c} \in C\ \}$.*

When the intersection between a code and its dual gives only the vector $\mathbf{0}$, the code is called LCD. A formal description can be seen below.

Definition 2. *The hull of a linear code C is given by $hull(C) := C^{\perp} \cap C$. The code is called* linear complementary dual (LCD) code if the hull is trivial; i.e, $hull(C) = \{\mathbf{0}\}$.

The class of LCD codes is a possible way to construct QUENTA codes that have maximal entanglement and asymptotically good families (see Sections 3 and 4).

When we consider two linear codes instead of one, the idea of relative hull and linear complementary pairs emerge. The next definition gives such a description.

Definition 3. *Let C_1 and C_2 be two $\mathbb{F}_q$-linear code. The* relative hull *of C_1 over C_2 is defined by $hull(C_1, C_2) := C_1^{\perp} \cap C_2$. If $hull(C_1, C_2) = \{\mathbf{0}\}$, then C_1 is called a linear C_2-complementary dual code.*

For the present paper, the relative hull between two codes will have a direct relation with the amount of entanglement used by a QUENTA code (see Theorem 1).

2.1 Algebraic-Geometry codes

Let $F/\mathbb{F}_q$ be an algebraic function field of genus g. A place P of $F/\mathbb{F}_q$ is the maximal ideal of some valuation ring $\mathcal{O}_P$ of $F/\mathbb{F}_q$. We also define $\mathbb{P}_F := \{P | P \text{ is a place of } F/\mathbb{F}_q\}$.

A divisor of $F/\mathbb{F}_q$ is a formal sum of places given by $D := \sum_{P\in\mathbb{P}_F} n_P P$, with $n_P \in \mathbb{Z}$, where almost all $n_P = 0$. The support and degree of D are defined as $supp(D) := \{P \in \mathbb{P}_F | n_p \neq 0\}$ and $\deg(D) := \sum_{P\in\mathbb{P}_F} n_P \deg(P)$, respectively, where $\deg(P)$ is the degree of the place P. When a place has degree one, it is called a rational place.

The discrete valuation corresponding to a place P is written as ν_P. For every element x of $F/\mathbb{F}_q$, we can define a principal divisor of x by $(x) := \sum_{P\in\mathbb{P}_F} \nu_P(x)P$. For $x \in \mathcal{O}_P$, we define $x(P) \in \mathcal{O}_P/P$ to be the residue class of x modulo P; for $x \in F\setminus\mathcal{O}_P$, we put $x(P) := \infty$. For a given divisor G, we denote the Riemann-Roch space associated to G by $\mathcal{L}(G) = \{x \in F^*|(x) \geq -G\} \cup \{0\}$.

The given description of Riemann-Roch spaces shows that when we are talking about such spaces we deal with functions that obey a set of rules which are described by the defining divisor. One natural question that could arise is the relation between the intersection of two Riemann-Roch spaces and the respective divisor that defines such a space. Such a result was shown by Munuera and Pellikaan[21]. Before showing it, we need to define the intersection of two divisors, which is done in the following.

Definition 4. *Let G and H be divisors over $F/\mathbb{F}_q$. If $G = \sum_{P\in\mathbb{P}_F} \nu_P(G)P$ and $H = \sum_{P\in\mathbb{P}_F} \nu_P(H)P$, where $P \in \mathbb{P}_F$ is a place, then the intersection $G \cap H$ of G and H over $F/\mathbb{F}_q$ is defined as follows*

$$G \cap H = \sum_{P\in\mathbb{P}_F} \min\{\nu_P(G), \nu_P(H)\}P. \tag{1}$$

In addition, the union is given by

$$G \cup H = \sum_{P\in\mathbb{P}_F} \max\{\nu_P(G), \nu_P(H)\}P. \tag{2}$$

Proposition 1. *[21, Lemma 2.6] Let G and H be divisors over $F/\mathbb{F}_q$. Then $\mathcal{L}(G) \cap \mathcal{L}(H) = \mathcal{L}(G \cap H)$.*

In the Section 3 it will be shown that when AG codes are used to construct QUENTA codes, the amount of entanglement used is equal to the dimension of the intersection of the two Riemann-Roch spaces.

For the exactly value of the dimension of a Riemann-Roch space and the construction of the dual code of a AG code, it is necessary to introduce the ideas of differential spaces and canonical divisors. $\Omega_F := \{\omega|\omega \textit{ is a Weil differential of } F/\mathbb{F}_q\}$ be the differential space of $F/\mathbb{F}_q$. Given a nonzero differential ω, we denote by $(\omega) := \sum_{P\in\mathbb{P}_F} \nu_P(\omega)P$ the canonical divisor of ω. All canonical divisors are equivalent and have degree equal to $2g - 2$. Furthermore, for a divisor G we define $\Omega_F(G) := \{\omega \in \Omega_F | \omega = 0 \textit{ or } (\omega) \geq G\}$, and its dimension as an $\mathbb{F}_q$-vector space is denoted by $i(G)$.

The dimension of a Riemann-Roch space can be calculated through its defining divisor, the divisor of a Weil differential and the genus of a curve.

Proposition 2. *[22, Thm. 1.5.15](Riemann-Roch Theorem) Let W be a canonical divisor of $F/\mathbb{F}_q$. Then for each divisor G, the dimension of $\mathcal{L}(G)$ is given by $\ell(G) = \deg(G) + 1 - g + \ell(W - G)$, where $\deg(G)$ is the degree of the divisor G.*

Now we can define the first AG code utilized in this paper, see Definition 5, and its parameters, see Proposition 3. As can be seen, these parameters are related to the degrees of divisors, genus and number of rational places. So, with simple arithmetic we can create families of codes, even when the algebraic function field is fixed.

Definition 5. *Let $P_1, \cdots, P_n$ be pairwise distinct rational places of $F/\mathbb{F}_q$ and $D = P_1+\ldots+P_n$. Choose a divisor G of $F/\mathbb{F}_q$ such that $supp(G) \cap supp(D) = \varnothing$. The algebraic-geometry (AG) code $C_{\mathcal{L}}(D, G)$ associated with the divisors D and G is defined as $C_{\mathcal{L}}(D, G) := \{(x(P_1), \ldots, x(P_n)) | x \in \mathcal{L}(G)\}$.*

Proposition 3. *[22, Cor. 2.2.3]Let $F/\mathbb{F}_q$ be a function field of genus g. Then the AG code $C_{\mathcal{L}}(D, G)$ is a $[n, k, d]$-linear code over $\mathbb{F}_q$ with parameters $k = \ell(G) - \ell(G - D)$ and $d \geq n - \deg(G)$. If $2g - 2 < \deg(G) < n$, then $k = \deg(G) - g + 1$.*

Lemma 1. *Let $F/\mathbb{F}_q$ be a function field of genus g and let D be a divisor as in Definition 5. If G_1 and G_2 are two divisors such that $suppG_1 \cap suppD = \varnothing$, resp. $suppG_2 \cap suppD = \varnothing$, and $\deg(G_1 \cup G_2) < n$, then $C_{\mathcal{L}}(D, G_1) \cap C_{\mathcal{L}}(D, G_2) = C_{\mathcal{L}}(D, G_1 \cap G_2)$.*

Another important type of AG code is given in the following.

Definition 6. *Let $F/\mathbb{F}_q$ be a function field of genus g and let G and D be divisors as in Definition 5. Then we define the code $C_{\Omega}(D, G)$ as $C_{\Omega}(D, G) := \{(res_{P_1}(\omega), \ldots, res_{P_n}(\omega) | \omega \in \Omega_F(G - D)\}$, where $res_{P_i}(\omega)$ denotes the residue of ω at P_i, with parameters $[n, k', d']$, where $k' = i(G - D) - i(G)$ and $d' \geq \deg(G) - (2g - 2)$.*

Proposition 4. *[22, Thm. 2.2.7]Let $C_{\Omega}(D, G)$ be the AG code from Definition 6. If $2g - 2 < \deg(G) < n$, then $C_{\Omega}(D, G)$ is an $[n, k', d']$-linear code over $\mathbb{F}_q$, where $k' = n + g - 1 - \deg(G)$ and $d' \geq \deg(G) - (2g - 2)$.*

The relationship between the codes $C_{\mathcal{L}}(D, G)$ and $C_{\Omega}(D, G)$ is given in the next proposition.

Proposition 5. *[22, Prop. 2.2.10] Let $C_{\mathcal{L}}(D, G)$ be the AG code described in Definition 5. Then $C_{\Omega}(D, G)$ is its Euclidean dual, i. e., $C_{\mathcal{L}}(D, G)^{\perp} = C_{\Omega}(D, G)$. Additionally, if we have a Weil differential η such that $\nu_{P_i}(\eta) = -1$ and $\eta_{P_i} = 1$ for all $i = 1, \ldots, n$, then $C_{\Omega}(D, G) = C_{\mathcal{L}}(D, D - G + (\eta))$.*

It is not hard to see that the divisor of a Weil differential from Proposition 5 can be decomposed as a sum of a divisor proportional of D with one that has its support different from the support of D; i. e., $(\eta) = -D + (\eta')$, where $supp(D) \cap supp((\eta')) = \varnothing$.

2.2 Entanglement-assisted quantum codes

Definition 7. *A quantum code $\mathcal{Q}$ is called an $[[n,k,d;c]]_q$ entanglement-assisted quantum (QUENTA) code if it encodes k logical qudits into n physical qudits using c copies of maximally entangle states and can correct $\lfloor (d-1)/2 \rfloor$ quantum errors. The rate of a QUENTA code is given by k/n, relative distance by d/n, and entanglement-assisted rate by c/n. Lastly, a QUENTA code is said to have maximal entanglement when $c = n-k$.*

Formulating a stabilizer paradigm for QUENTA codes gives a way to use classical codes to construct this quantum codes [3]. In particular, we have the next procedure by Wilde and Brun [23].

Proposition 6. *[23, Corollary 1] Let H_1 and H_2 be parity check matrices of two linear codes with parameters $[n,k_1,d_1]_q$ and $[n,k_2,d_2]_q$, respectively. Then there is a QUENTA code with parameters $[[n,k_1+k_2-n+c,\min\{d_1,d_2\};c]]_q$ that requires $c = \mathrm{rank}(\mathrm{H_1H_2^T})$ maximally entangled states.*

A measurement of goodness for a QUENTA code is the quantum Singleton bound. Let $[[n,k,d;c]]_q$ be a QUENTA code, then the quantum Singleton bound is given by $d \leq \frac{n-k+c}{2}+1$. The difference between d and the Singleton bound is called Singleton defect. When the Singleton defect is equal to zero (resp. one) the code is called maximum distance separable code (resp. almost maximum distance separable code) and it is denoted MDS code (resp. almost MDS code).

3 New Construction Method for QUENTA Codes

It is shown in Proposition 6 the connection between the entanglement in a QUENTA code and the rank of a matrix that is the product of the two parity check matrices of the classical codes utilized to construct such quantum code. However, such rank can be difficult to calculate in some cases. As it will be shown in Theorem 2, it is possible to, instead of calculating such rank, relate the entanglement with the relative hull between the two classical codes.

Lemma 2. *Let C_1 and C_2 be $[n,k_1,d_1]_q$ and $[n,k_2,d_2]_q$ linear codes with parity-check matrices H_1 and H_2, respectively. If $\dim(hull(C_1,C_2)) = l_1$ and $\dim(hull(C_2,C_1)) = l_2$, then $\mathrm{rank}(H_1H_2^T) = n - \max\{k_1+l_1, k_2+l_2\}$.*

Observe that for the result in Theorem 2 it was not needed to make any previous consideration. So, the the previous result can be seen as a new way to calculate the amount of entanglement used in a QUENTA code.

If the relative hull of a code with respect to other is known, then Theorem 2 can be used instead of Proposition 6 to construct new QUENTA codes in a more direct way. We show in the following that this is the case when considering classical AG codes.

Theorem 1. *Let C_1 and C_2 be two linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively. If $l_1 = \dim(hull(C_1, C_2))$ and $l_2 = \dim(hull(C_2, C_1))$, with $k_1 + l_1 \geq k_2 + l_2$, then there is an $[[n, k_2 - l_1, \min\{d_1, d_2\}; n - k_1 - l_1]]_q$ QUENTA code.*

Corollary 1. *Let C_1 and C_2 be two linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively, with $hull(C_1, C_2) = \{\mathbf{0}\}$. Then there is a QUENTA code with parameters $[[n, k_2, \min\{d_1, d_2\}; n - k_1]]_q$.*

Corollary 2. *Let C be a MDS LCD code with parameters $[n, k, d]_q$. Then there is a MDS maximal entanglement QUENTA code with parameters $[[n, k, d; n - k]]_q$.*

It is shown in Corollary 2 that for any MDS LCD code in the literature we can construct a QUENTA code that is, simultaneously, MDS and maximal entanglement.

For AG codes, the property needed in Corollary 1 can be translated to a relation between the divisors used to construct them. The following theorem presents this description and a more general result.

Theorem 2. *Let $F/\mathbb{F}_q$ be an algebraic function field and η be the Weil differential of Proposition 5 with divisor $(\eta) = -D + (\eta')$. Consider that D, $G_1 = H_1 - (\eta')$, and $G_2 = H_2 - (\eta')$ are divisors following the construction of Definition 5, with $H_1 \geq H_2 > 0$. Then there exists a QUENTA code with parameters $[[n, \deg(G_2) - g + 1, n - \deg(G_1); n - \deg(G_1) + g - 1]]_q$.*

Proof. Let $C_1 = C_{\mathcal{L}}(D, G_1)$ and $C_2 = C_{\mathcal{L}}(D, G_2)$. Since that $G_1 = H_1 - (\eta')$ and $G_2 = H_2 - (\eta')$, we have $l_1 = l_2 = 0$. Additionally, using $H_1 \geq H_2 > 0$, we have that $k_1 \geq k_2$. So, the remaining statements follow from the application of Proposition 3 to the Corollary 1.

The first family of QUENTA codes constructed in this paper is shown in the following theorem. The rational function field $\mathbb{F}_q(z)/\mathbb{F}_q$ is used to derive this family.

Theorem 3. *Let q be a power of a prime. So, if a_1, a_2, b_1, b_2 are positive integers such that $a_1 \geq b_1$ and $a_2 \geq b_2$, with $b_1 + b_2 \geq q - 3$ and $a_1 + a_2 < 2q - 5$, then exists a QUENTA code with parameters $[[q - 1, b_1 + b_2 + 4 - q, 2q - 4 - (a_1 + a_2); 2q - 5 - (a_1 + a_2)]]_q$. In particular, if $a_1 + a_2 - 1 \leq b_1 + b_2 \leq a_1 + a_2$, then there is an MDS maximal entanglement QUENTA code with parameters $[[q - 1, b_1 + b_2 + 4 - q, 2q - 4 - (a_1 + a_2); 2q - 5 - (a_1 + a_2)]]_q$.*

The following theorem shows a construction of QUENTA codes derived from Hermitian function field. Next, Elliptic function field will be used to obtain maximal entanglement QUENTA codes with Singleton defect at most one.

Theorem 4. *Let q be a power of a prime and $F/\mathbb{F}_{q^2}$ be the Hermitian function field defined by the equation*

$$y^q + y = x^{q+1}.$$

Let a_1, a_2, b_1, b_2 be positive integers such that $a_1 \geq b_1$, $a_2 \geq b_2$, with $b_1 + b_2 > q^3 + 2q(q-1) - 5$ and $a_1 + a_2 < 2q^3 + q(q-1) - 4$. Then there exists a QUENTA code with parameters $[[q^3 - 1, b_1 + b_2 - q^3 - 3q(q-1)/2 + 4, 2q^3 + q(q-1) - 4 - (a_1 + a_2); 2q^3 + 3q(q-1)/2 - 5 - (a_1 + a_2)]]_{q^2}$.

Theorem 5. *Let $q = 2^m$, with $m \geq 1$ an integer, and $F/\mathbb{F}_q$ be an Elliptic function field with n rational points and genus $g = 1$ defined by the equation*

$$y^2 + y = x^3 + bx + c, \tag{3}$$

where $b, c \in \mathbb{F}_q$. Let a_1, a_2, b_1, b_2 be positive integers such that $a_1 \geq b_1$, $a_2 \geq b_2$, with $b_1 + b_2 > n$ and $a_1 + a_2 < 2n - 2$. Then there exists a QUENTA code with parameters $[[e-2, b_1 + b_2 - e, 2e - 2 - (a_1 + a_2); 2e - 2 - (a_1 + a_2)]]_q$, where e is the number of rational places of the Elliptic curve. In particular, if $a_1 = b_1$ and $a_2 = b_2$, then there is an almost MDS maximal entanglement QUENTA code with parameters $[[e-2, a_1 + a_2 - e, 2e - 2 - (a_1 + a_2); 2e - 2 - (a_1 + a_2)]]_q$.

4 Asymptotically Good Maximal Entanglement QUENTA Codes

In this section, we show that from any family of (classical) asymptotically good AG codes, we can construct a family of asymptotically good maximal entanglement QUENTA codes. This is a consequence of the use of the result from Carlet, *et al.* [4] applied to Theorem 2. Before showing it, we need to define the concept of (classical) asymptotically good codes.

Definition 8. *Let q be a prime power and $\alpha_q := \sup\{R \in [0,1] \colon (\delta, R) \in U_q\}$, for $0 \leq \delta \leq 1$. Here U_q denotes the set of all ordered pair $(\delta, R) \in [0,1]^2$ for which there is a family of linear codes that are indexed as C_t, with parameters $[n_t, k_t, d_t]_q$, such that $n_t \to \infty$ as $t \to \infty$ and $\delta = \lim_{t\to\infty} d_t/n_t$, $R = \lim_{t\to\infty} k_t/n_t$. If $\delta, R > 0$, then the family is called asymptotically good.*

Proposition 7. *[4, Corollary 5.5] Let $q \geq 3$ be a power of a prime and $A(q) = \limsup_{g\to\infty} \frac{N_q(g)}{g}$, where $N_q(g)$ denotes the maximum number of rational places that a global function field of genus g with full constant field $\mathbb{F}_q$ can have. Then there exists a family of LCD codes with*

$$\alpha_q^{LCD}(\delta) \geq 1 - \delta - \frac{1}{A(q)}, \text{ for } \delta \in [0,1]. \tag{4}$$

Theorem 6. *Let $q \geq 3$ be a power of a prime and $A(q)$ as defined in Proposition 7. Then there exists a family of maximal entanglement QUENTA codes with parameters $[[n_t, k_t, d_t; c_t]]_q$, such that*

$$\lim_{t\to\infty} \frac{d_t}{n_t} \geq \delta, \qquad \lim_{t\to\infty} \frac{k_t}{n_t} \geq 1 - \delta - \frac{1}{A(q)}, \quad \text{and} \qquad \lim_{t\to\infty} \frac{c_t}{n_t} \in [\delta, \delta + 1/A(q)]. \tag{5}$$

for $\delta \in [0, 1 - 1/A(q)]$.

Although the Gilbert-Varshamov bound for QUENTA codes over qudits has not been defined, we conjecture that the codes constructed in Theorem 6 exceeds the asymptotic Gilbert-Varshamov for q sufficiently large.

5 Conclusion

This paper has been devoted to the use of AG codes in the construction of QUENTA codes. We firstly showed that the intersection of two AG codes is also a AG code. This was used in a new description of how to compute the entanglement in a QUENTA code. As a consequence, we constructed three new families of QUENTA codes where one of them is MDS and the other is almost MDS. Lastly, it was shown that for any asymptotically good classical family of AG code, there is a family asymptotically good maximal entanglement QUENTA codes. It is worth mentioning that all the results presented in this paper can be generalized to QUENTA codes derived from the Hermitian case.

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