201124 - FAI - 4

(2.4) (4)

$\|\cdot\|_1$ , $\|\cdot\|_2$ equivalent.

a. $\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$

topologies equivalent

$d_i(x,y) = \|x-y\|_i$

$$\begin{cases} B & (X, d_1) \\ B' & (X, d_2) \end{cases}$$

$$\begin{cases} B_0 \subset B_0' \\ B_0' \subset B_0 \end{cases}$$

$d_1(x,y) = \|x-y\|_1$

$B_R(x, x_0) = \{ y \in X \mid \|x_0 - y\|_1 < R \}$, $\cancel{x_0} \, x_0 \in X$

$B'_R(x, x_0) = \{ y \in \cancel{X} \mid \|x_0 - y\|_2 < R \}$, $x_0 \in X$

$? \frac{R}{a}, \quad \frac{R}{\ell} ?$

$$\|x_0 - y\|_1 \leq \ell \cdot \underbrace{\|x_0 - y\|_2}_{\leq (R/\ell)}$$

$$\left\{ B_{\frac{R}{\ell}}(X, x_0) \text{ in } (X, d_2) \right.$$

$$\Longrightarrow \|x_0 - y\|_1 \leq R$$

$$\underset{\text{in } \underline{\underline{(X, d_2)}}}{B_{\frac{R}{\ell}}(X, x_0)} \subset \underset{\text{in } \underline{\underline{(X, d_1)}}}{\underbrace{B_R}(X, x_0)}$$

$\times$

(2.5)

(2.28) $$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$$

$\underline{x \in S} \quad \xi_k(x)$

$(x_1, x_2, x_2 \ldots \quad (\xi_k(x) = x_k|)$

If not true $\rightsquigarrow$ M is not compact.

M is compact $\rightsquigarrow$

every seq has a conv. subseq.

$\Rightarrow$ $x_i = (0, \dots, 0, i, 0, \dots)$

$\qquad\qquad\qquad\qquad\qquad\qquad p^e$

$x_i(p) = i$

$$d(x_i, x_j) = \frac{1}{2^p} \boxed{\frac{|j-i|}{1+|j-i|}} \geq \frac{1}{2^p} \cdot \frac{1}{2} \not\to 0$$

$(i \neq j)$ seq. doesn't converge

$\Rightarrow$ M is not compact

$|x_i(p)| = i \to \infty$ if $i \to \infty$

$i \in \mathbb{N}$

$$x = (\xi_1, \xi_2 \cdots\cdots)$$
$$y = (\eta_1, \eta_2 \cdots\cdots)$$

$$\Delta : \; d(x,y) = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$$

$$d(0,y) = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j \underbrace{\frac{|\eta_i|}{1+|\eta_i|}}_{(\leq 1)} \leq$$

$$\frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = \underline{1}$$

---

$$(2.6)(14) \quad T : X \to Y$$

$$\dim(X) = \dim(Y) = n < \infty$$

$$R(T) = Y \Leftrightarrow T^{-1} \text{ exists.}$$

$$R(T) = Y \overset{?}{\Rightarrow} T^{-1} \text{ exists.}$$

<u>asm</u> $T^{-1}$ not exists

$$\exists X \neq 0 \quad T(X) = 0$$

$$T(\alpha \cdot X) = 0 \; , \; \forall \alpha \in \mathbb{R}$$

$$\dim(N(T)) \geq 1$$

$$T : \underline{X} \to \underline{Y}$$

$$\boxed{\underbrace{\dim(N(T))}_{\geq 1} + \dim(R(T)) = n}$$

$$\underbrace{\dim(R(T))} < n$$

(2.6-10)

$$\dim D(T) = \dim(R(T)) < n$$

$$\underbrace{x_1, \dots, x_n}$$

n vectors, lin. indep

$$N(T) \neq \{0\}$$

$$N(T) = \langle x \rangle \qquad x \neq 0$$

$$\{x, x_2, \dots, x_n\}$$

$$\underbrace{T(x) = 0, \quad T(x_2) \dots, T(x_n)}$$

$$\dim(R(T)) < n$$

$$p^{th}$$

(2.5)(4) $x_1 = (0, \dots, 0, 1, 0, \dots)$

$x_2 = (0, \dots, 0, 2, 0, \dots)$

$\vdots$

$x_i = (0, \dots, 0, i, 0, \dots)$

$$d(0, x_i) = \frac{1}{2^p} \cdot \frac{i}{1+i} \le \frac{1}{2^p}$$

$(x_i)$ a bounded seq. of sequences in $s$

$$d(x_i, x_j) = \frac{1}{2^p} \cdot \frac{|j-i|}{1+|j-i|} \ge \frac{1}{2^{p+1}}$$
$(i \ne j)$

$$\underbrace{\frac{1}{1+1}, \frac{2}{1+2}, \frac{3}{1+3}, \dots}_{\ge \frac{1}{2}} \not\to 0$$

a sequence in M that doesn't converges $\Rightarrow$ M is not compact

(so no conv. subseq.)

$\xi_k(x) \sim$ k-th coordinate of $x$.

$x_1 = (x_{11}, x_{12}, \dots, x_{1k}, \dots$

$x_2 = (x_{21}, x_{22}, \dots; x_{2k}, \dots$

$x_n = (x_{n1}, x_{n2}, \dots \ x_{nk}, \dots$

$|\xi_k(x)| \le \gamma_k$

$|\xi_2(x)| \le \gamma_2, \dots$

$|\xi_1(x)| \le \gamma_1$

in our counterexample

$$|\xi_p(x_1)| = 1$$
$$|\xi_p(x_2)| = 2$$
$$\vdots$$
$$|\xi_p(x_n)| = n$$
$$\begin{cases} |\xi_p(x_m)| \to \infty \\ \text{if } m \to \infty \end{cases}$$

<u>there exists no</u> : $\gamma_p$

So those $j$ & $k$ are necessary

but if that is enough?

That is not asked to us.