4.1 ①

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \frac{x}{L} n}$$

$$c_n = \frac{1}{L} \int_0^L g(x) e^{-2\pi i \frac{x}{L} n} dx$$

$$\int_0^L f(x) \overline{g(x)} dx = L \sum_{n=-\infty}^{\infty} c_n \overline{d_n}$$

$$\int_0^L |f(x)|^2 dx = L \sum_{n=-\infty}^{\infty} |c_n|^2$$

③

$$c_n = \frac{1}{2} \int_{-1}^{1} |x| e^{-2\pi i \frac{x}{2} n} dx = \begin{cases} \frac{(-1)^n - 1}{\pi^2 n^2} & n \neq 0 \\ \frac{1}{2} & n = 0 \end{cases}$$

⑩ a.

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

$$\sum_{k=-n}^{n} e^{ikx} = e^{-inx} \sum_{k=0}^{2n} e^{ikx} = e^{-inx} \frac{1 - e^{i(2n+1)x}}{1 - e^{ix}}$$

$$= \frac{e^{inx + \frac{1}{2}ix} - e^{-inx - \frac{1}{2}ix}}{e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix}} = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}.$$

b.

$$\int_{-\pi}^{\pi} f(x) D_n(x) dx = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{-n}^{n} f(x) e^{ikx} dx =$$

$$\sum_{-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = \sum_{-n}^{n} c_k \longrightarrow \sum_{-\infty}^{\infty} c_k \quad (n \to \infty)$$

$$\sum_{-\infty}^{\infty} c_n = f(0).$$

4.2 ① $\int_{-\infty}^{\infty} e^{-\varepsilon|t|} \cos(\omega_0 t)\, e^{-i\omega t}\, dt$

$$= \int_{-\infty}^{0} e^{\varepsilon t}\, \tfrac{1}{2}\left(e^{i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\right) dt$$

$$+ \int_{0}^{\infty} e^{-\varepsilon t} \cdot \tfrac{1}{2}\left(e^{i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\right) dt$$

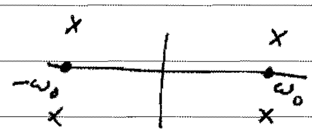
$$= \tfrac{1}{2}\int_{0}^{\infty} e^{-\varepsilon t - i(\omega_0-\omega)t} + e^{-\varepsilon t + i(\omega_0+\omega)t}$$
$$+ e^{-\varepsilon t + i(\omega_0-\omega)t} + e^{-\varepsilon t - i(\omega_0+\omega)t} \quad dt$$

$$= \tfrac{1}{2}\left\{ \frac{1}{\varepsilon + i\omega_0 - i\omega} + \frac{1}{\varepsilon - i\omega_0 - i\omega} + \frac{1}{\varepsilon - i\omega_0 + i\omega} + \frac{1}{\varepsilon + i\omega_0 + i\omega} \right]$$

$$\frac{\varepsilon + i\omega_0}{(\varepsilon + i\omega_0)^2 + \omega^2} + \frac{\varepsilon - i\omega_0}{(\varepsilon - i\omega_0)^2 + \omega^2}$$

$$= \frac{i(\omega_0 - i\varepsilon)}{\omega^2 - (\omega_0 - i\varepsilon)^2} + \frac{-i(\omega_0 + i\varepsilon)}{\omega^2 - (\omega_0 + i\varepsilon)^2}$$

poles in $\omega = \pm(\omega_0 \pm i\varepsilon)$



(not causal)

③ $\int_{-\infty}^{\infty} e^{i\alpha x} f(\lambda x + \mu)\, e^{-i\omega x}\, dx = \int_{-\infty}^{\infty} f(\lambda x + \mu)\, e^{-i(\omega - \alpha)x}\, dx.$

$z = \lambda x + \mu$, $x = \frac{z - \mu}{\lambda}$, $dx = \frac{1}{\lambda} dz$

$$= \frac{1}{|\lambda|}\, e^{i\frac{\mu}{\lambda}(\omega - \alpha)}\, \hat{f}\left(\frac{\omega - \alpha}{\lambda}\right)$$

4.2 (4) $\hat{B}(\omega) = \int_{-\infty}^{\infty} B(x) e^{-i\omega x}\, dx = \int_{-1}^{1} e^{-i\omega x} dx = 2\frac{\sin\omega}{\omega}.$

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{B}(\omega) e^{i\omega x} d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} . e^{i\omega x} d\omega.$$

$$B(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega = 1 \rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega = \pi.$$

(7) $\int_{-\infty}^{\infty} |B(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{B}(\omega)|^2 d\omega.$

$$\int_{-1}^{1} 1\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\left(\frac{\sin\omega}{\omega}\right)^2 d\omega = 2.$$

$$\rightarrow \int_{-\infty}^{\infty} \left(\frac{\sin\omega}{\omega}\right)^2 d\omega = 2 . \frac{2\pi}{4} = \pi.$$

4.3 (1). $\int_{0}^{\infty} t^{\lambda-1} e^{ct} . e^{-st} dt = \int_{0}^{\infty} t^{\lambda-1} . e^{-(s-c)t} dt.$

$$\tau = (s-c)t. , \quad d\tau = (s-c)\, dt.$$

$$= \int_{0}^{\infty} \frac{\tau^{\lambda-1}}{(s-c)^{\lambda-1}} . e^{-\tau} . \frac{1}{s-c} d\tau = \frac{1}{(s-c)^{\lambda}} \Gamma(\lambda)$$

with $\operatorname{Re} s > c$ and the principal value power function.

(2). $\int_{0}^{\infty} (a_n x^n + a_{n-1} x^{n-1} \dots + a_0) e^{-sx} dx$

$$= a_n \frac{n!}{s^{n+1}} + a_{n-1} \frac{(n-1)!}{s^n} + \dots + a_0 \frac{1}{s}$$