3.1 a. $\dfrac{z^2+1}{z(2z-1)(z+1)}$ singulier in $z=0$: Res $= -1$

$z=\frac{1}{2}$: Res $= 10/3$

$z=-1$: Res $= \frac{2}{3}$.

b. $\dfrac{e^{iz}}{z^2+4}$, sing. punten $z=2i$: $e^{-2}/4i$

$z=-2i$: $e^{+2}/(-4i)$

c. $\dfrac{\cos \pi z}{z(2z-1)(6z-1)}$, sing. p : $z=0$: 1

$z=\frac{1}{6}$: $-\frac{3}{4}\sqrt{3}$.

2. a $$\int_{|z|=\frac{3}{2}} \frac{e^z}{(z+1)^2(z-2)}\,dz = 2\pi i \,\mathrm{Res}_{-1} = -2\pi i \cdot \frac{4}{9} e^{-1}$$

$$\left.\begin{aligned} e^z &= e^{-1} \cdot e^{z+1} = e^{-1}\left(1+(z+1)+\tfrac{1}{2}(z+1)^2+\dots\right) \\ \frac{1}{z-2} &= -\frac{1}{3}\frac{1}{1-\frac{z+1}{3}} = -\frac{1}{3}\left(1+\frac{z+1}{3}+\frac{(z+1)^2}{9}+\dots\right) \end{aligned}\right\}$$

$$\frac{e^z}{z-2} = -\frac{1}{3}e^{-1}\left(1+\frac{4}{3}(z+1)+\left(\frac{1}{9}+\frac{1}{3}+\frac{1}{2}\right)(z+1)^2+\dots\right)$$

$$\frac{e^z}{(z+1)^2(z-2)} = -\frac{1}{3}e^{-1}\left(\frac{1}{(z+1)^2}+\frac{4/3}{z+1}+\frac{17}{18}+\dots\right)$$

b. $$\int_{|z|=2} \frac{\sin \pi z}{z^4-1}\,dz = 2\pi i\,\mathrm{Res}_i + 2\pi i\,\mathrm{Res}_{-i} = -\frac{1}{4}\sinh\pi - \frac{1}{4}\sinh\pi = -\frac{1}{2}\sinh\pi$$

3. $$\int_{|z|=R} \frac{1}{(z^2+1)^8(z+3)}\,dz$$

a. $0<R<1$: geen singulariteiten : $=0$

b. $1<R<3$: singulariteit in $z=i$ en $z=-i$: Residuen zijn moeilijk te bepalen

met Maple : $\mathrm{Res}_i = 10^{-8}\left(-\frac{1}{2}-\frac{28176147}{8}i\right)$

$\mathrm{Res}_{-i} = 10^{-8}\left(-\frac{1}{2}+\frac{28176147}{8}i\right)$

zodat totaal

$= -2\pi i \cdot 10^{-8}$

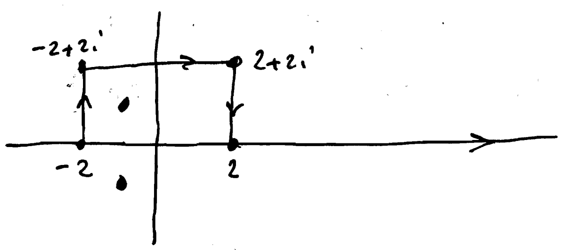
Slimmer: omdat $\left|\int_0^{2\pi} \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+1)^8(Re^{i\theta}+3)}\,d\theta\right| \leqslant \text{const.}\, 2\pi \frac{R}{R^{16}\cdot R} \to 0 \ (R\to\infty)$

volgt dat gevraagde integraal $= -2\pi i \, Res_3$

$$Res_3 = (z+3)\frac{1}{(z^2+1)^8(z+3)}\Bigg|_{z=-3} = \frac{1}{10^8} = 10^{-8}$$

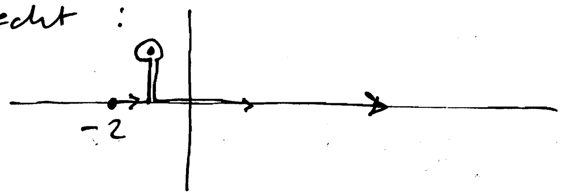
c. $R>3$: zie vorige opg (b) : $\int dz = 0$.

4. $$\int_K \frac{dz}{z^2+2z+2}$$



polen : $z^2+2z+2=0 \rightarrow z=-1\pm i$

Trek de contour recht :



zodat

$$\int_K dz = \int_{-2}^{\infty}\frac{dx}{(x+1)^2+1} - 2\pi i \, Res(-1+i)$$

$$= \operatorname{arctg}(x+1)\Big|_{-2}^{\infty} - 2\pi i\frac{1}{2i} = \frac{\pi}{2}+\frac{\pi}{4}-\pi = -\frac{1}{4}\pi.$$

3.2 . 1 a $$\int_0^{2\pi}\frac{1+2\cos\varphi}{5+4\cos\varphi}\,d\varphi = \int_{|z|=1}\frac{1+z+z^{-1}}{5+2(z+z^{-1})}\cdot\frac{1}{iz}\,dz$$

$$= \frac{1}{2i}\int_{|z|=1}\frac{z^2+z+1}{z(z+2)(z+\frac{1}{2})}\,dz = \frac{2\pi i}{2i}\left(1-\frac{3/4}{3/4}\right) = 0$$

1 b . $$\int_0^{2\pi}\frac{2\cos\varphi}{(5+4\cos\varphi)^2}\,d\varphi = \frac{1}{4i}\int_{|z|=1}\frac{1+z^2}{(z+2)^2(z+\frac{1}{2})^2}\,dz$$

$$= \frac{1}{4i}\int \frac{32/27}{z+2}+\frac{20/9}{(z+2)^2}-\frac{32/27}{z+\frac{1}{2}}+\frac{5/9}{(z+\frac{1}{2})^2}\,dz$$

$$= \frac{2\pi i}{4i}\left(\frac{-32}{27}\right) = -\frac{16}{27}\pi.$$

1c. $$\int_0^{2\pi} \frac{\sin\varphi}{4+3i\sin\varphi}\,d\varphi = \int_{|z|=1} \frac{\frac{1}{2i}(z-z^{-1})}{4+\frac{3i}{2i}(z-z^{-1})}\cdot\frac{1}{iz}\,dz$$

$$= \int_{|z|=1} \frac{1-z^2}{3(z+3)(z-\frac{1}{3})z}\,dz = \frac{2\pi i}{3}\left(Res_{1/3}+Res_0\right)$$

$$= \frac{2}{3}\pi i\left(\frac{1-1/9}{(3+\frac{1}{3})\cdot\frac{1}{3}} + \frac{1}{3\cdot -1/3}\right)$$

$$= \frac{2}{3}\pi i\left(\frac{4}{5}-1\right) = -\frac{2}{15}\pi i$$

1d. $$\int_0^{2\pi} \frac{\cos 2\varphi}{4+3i\cos\varphi}\,d\varphi = \int_{|z|=1} \frac{\frac{1}{2}(z^2+z^{-2})}{4+3i\frac{1}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}\,dz = -\frac{1}{3}\int_{|z|=1}\frac{z^4+1}{z^2(z-3i)(z+\frac{1}{3}i)}\,dz$$

$$Res_{-1/3i} = -\frac{1}{3}\,\frac{1+\frac{1}{81}}{-\frac{1}{9}(-\frac{1}{3}i-3i)} = \frac{41}{45}i$$

$$Res_0 = -\frac{1}{3}\cdot\frac{8}{3}i \quad \text{want} \quad \frac{1+z^4}{(1-\frac{z}{3i})(1+\frac{z}{\frac{1}{3}i})} \simeq (1+z^4)(1+3iz-\tfrac{1}{3}zi+\dots) = 1+\tfrac{8}{3}iz+\dots$$

$$\text{int} = +2\pi i\left(\frac{41}{45}i - \frac{8}{9}i\right) = -\frac{2}{45}\pi$$

2a. $$\int_0^{2\pi} \frac{\cos 2\theta}{12+5i\cos\theta}\,d\theta = \int_{|z|=1} \frac{\frac{1}{2}(z^2+z^{-2})}{12+\frac{5i}{2}(z+\frac{1}{z})}\cdot\frac{1}{iz}\,dz$$

$$= -\int_{|z|=1}\frac{z^4+1}{z^2(5z^2-24iz+5)}\,dz = -\frac{1}{5}\int\frac{z^4+1}{z^2(z-5i)(z+\frac{1}{5}i)}\,dz$$

$$Res_{-\frac{1}{5}i} = -\frac{1}{5}\cdot\frac{1+\frac{1}{625}}{-\frac{1}{25}(-\frac{1}{5}i-5i)} = i\,\frac{626}{25\cdot 26}$$

$$Res_0 = -\frac{24}{25}i \quad \text{omdat} \quad \frac{1}{(z-5i)(z+\frac{1}{5}i)} = (1-\tfrac{1}{5}iz+\dots)(1+5iz+\dots) = 1+\tfrac{24}{5}iz+\dots$$

$$\text{int} = 2\pi i\left(\frac{626}{25\cdot 26}i - \frac{24}{25}i\right) = -\frac{4\pi}{25\cdot 26}$$

2b. $\displaystyle\int_0^{2\pi} \frac{\cos 5\theta}{3+2\sqrt{2}\cos\theta}\,d\theta = \int_{|z|=1} \frac{\frac{1}{2}(z^5+z^{-5})}{(3+\sqrt{2}(z+z^{-1}))\,iz}\,dz$

$\displaystyle = -\tfrac{1}{4}\sqrt{2}\,i \int_{|z|=1} \frac{1+z^{10}}{z^5(z+\sqrt{2})(z+\frac{1}{2}\sqrt{2})}\,dz$

Beter: $\displaystyle \mathrm{Re}\left[\int_0^{2\pi} \frac{e^{5i\theta}}{3+2\sqrt{2}\cos\theta}\,d\theta\right]$

$\displaystyle \mathrm{Res}_{-\frac{1}{2}\sqrt{2}} = -\tfrac{1}{4}\sqrt{2}\,i\; \frac{1+2^{-5}}{-2^{-\frac{5}{2}}\cdot(\frac{1}{2}\sqrt{2})} = \tfrac{1}{4}\sqrt{2}\,i\cdot\tfrac{33}{4}.$

$\displaystyle \mathrm{Res}_0 = -\tfrac{1}{4}\sqrt{2}\,i\cdot\tfrac{31}{4}$ omdat $\displaystyle \frac{1}{(z+\sqrt{2})(z+\frac{1}{2}\sqrt{2})} =$

$\displaystyle = \frac{1}{(1+2^{-1/2}z)(1+2^{1/2}z)} = (1-a+a^2-a^3+a^4)(1-b+b^2-b^3+b^4)$

met $a = 2^{-1/2}$, $b = 2^{1/2}$

$= 1 + \dots + (a^4+a^3b+a^2b^2+ab^3+b^4) + \dots$

$= 1 + \dots + (4+2+1+\frac{1}{2}+\frac{1}{4})z^4 + \dots$

$= 1 + \dots + \frac{31}{4}z^4.$

int $= 2\pi i\cdot\frac{1}{4}\sqrt{2}\,i\left(\frac{33}{4}-\frac{31}{4}\right) = -\frac{1}{4}\sqrt{2}\,\pi$

2d. $\displaystyle\int_0^{2\pi} \frac{e^{-i\theta}}{(3+4i\cos\theta)^2}\,d\theta = \int_{|z|=1} \frac{z^{-1}}{(3+4i\cdot\frac{1}{2}(z+z^{-1}))^2}\cdot\frac{1}{iz}\,dz =$

$\displaystyle = \int_{|z|=1} \frac{1}{-4i\,(z-2i)^2(z+\frac{1}{2}i)^2}\,dz$

$\displaystyle \left(\frac{1}{z-2i}\right)^2 = \left(\frac{1}{z+\frac{1}{2}i-\frac{5}{2}i}\right)^2 = \left(\frac{1}{-\frac{5}{2}i\left(1-\frac{z+\frac{1}{2}i}{\frac{5}{2}i}\right)}\right)^2$

$\displaystyle = -\frac{4}{25}\left(1-\frac{4}{5}i(z+\tfrac{1}{2}i)+\dots\right)$

dus $\displaystyle \mathrm{Res}_{-\frac{1}{2}i} = -\frac{1}{4i}\cdot\frac{-4}{25}\cdot\frac{-4}{5}i = -\frac{4}{125}$

int $\displaystyle = -2\pi i\,\frac{4}{125} = -\frac{8}{125}\pi i$

3. $I_n = \int_0^{\pi/2} \cos^{2n}\varphi \, d\varphi$ (later)

3.3 1b. $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\, dx =$

$$\int_{\curvearrowleft} \frac{1}{(z^2+4)^2}\, dz = \int_{-R}^{R} \frac{1}{(z^2+4)^2}\, dz + \int_0^{\pi} \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+4)^2}\, d\theta$$

$$\left| \int_0^{\pi} \frac{iRe^{i\theta}}{|R^2e^{2i\theta}+4|^2}\, d\theta \right| \underset{\text{marge}}{\le} 2\pi \cdot \left| \frac{R}{R^4} \right| = \frac{2\pi}{R^3} \to 0 \quad (R \to \infty)$$

dus $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\, dx = \lim_{R\to\infty} \int_{C_R} \frac{1}{(z^2+4)^2}\, dz = 2\pi i \, \mathrm{Res}_{2i} = \frac{\pi}{16}$.

$$\frac{1}{z^2+4} = \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i} \right).$$

$$\left(\frac{1}{z^2+4}\right)^2 = -\frac{1}{16}\left(\left(\frac{1}{z-2i}\right)^2 - \frac{2}{(z-2i)(z+2i)} + \frac{1}{(z+2i)^2} \right).$$

$$\to \mathrm{Res}_{2i} = -\frac{1}{16} \cdot -2 \cdot (z-2i) \cdot \frac{1}{(z-2i)(z+2i)} = \frac{1}{8} \cdot \frac{1}{4i} = -\frac{i}{32}$$

3. $I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\, dx = 2\pi i \, \mathrm{Res}_i$ (analoog aan 1b)

$$\left(\frac{1}{x^2+1}\right)^n = \left(\frac{1}{2i}\right)^n \left(\frac{1}{x-i} - \frac{1}{x+i} \right)^n = \left(\frac{1}{2i}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x-i}\right)^{n-k} \left(\frac{-1}{x+i}\right)^k$$

$$\mathrm{Res}_i = \left[\left(\frac{1}{2i}\right)^n \binom{n}{n-1} \left(\frac{1}{x-i}\right)^{n-n+1} \left(\frac{-1}{x+i}\right)^{n-1} \right] \cdot (x-i)$$

$$= \left(\frac{1}{2i}\right)^n n \left(\frac{-1}{2i}\right)^{n-1} = \frac{-2ni}{(+4)^n}$$

zodat $2\pi i \cdot \frac{2ni}{4^n} = \frac{4\pi n}{4^n} = \frac{\pi n}{4^{n-1}}$

$\cancel{\frac{1}{(x^2+1)^n} = \frac{1}{(2i)^n}\left(\frac{1}{x-i}\right)^n}$

$$\frac{1}{(x^2+1)^n} = \frac{1}{(x-i)^n}\cdot\frac{1}{(x+i)^n} = \frac{1}{(x-i)^n}\,\frac{1}{(2i)^n}\left(1+\frac{x-i}{2i}\right)^{-n}$$

$$= \frac{1}{(2i)^n}\,\frac{1}{(x-i)^n}\cdot\sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!\,k!}(-1)^k\frac{(x-i)^k}{(2i)^k}$$

$$= \sum_{k=0}^{\infty}\frac{(n+k-1)!}{(n-1)!\,k!}(-1)^k\,\frac{(x-i)^{k-n}}{(2i)^{n+k}}.$$

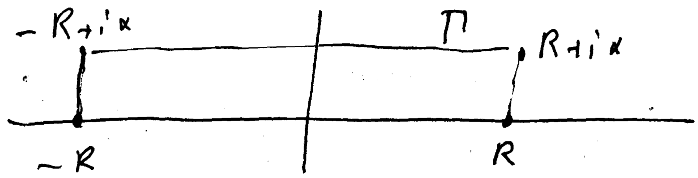
residu in $k=n-1$:

$$\frac{(2n-2)!}{(n-1)!\,(n-1)!}\cdot(-1)^{n-1}\frac{1}{(2i)^{2n-1}} =$$

$$= -2i\binom{2n-2}{n-1}\frac{1}{4^n}.$$

dus $I_n = 2\pi i\cdot \mathrm{Res} = \dfrac{\pi}{4^{n-1}}\dbinom{2n-2}{n-1}.$

4. $I(\alpha) = \int_{-\infty}^{\infty} e^{-x^2}\cos(2\alpha x)\,dx.$, $\alpha > 0$



$$\int_\Gamma e^{-z^2}\,dz = 0 = \int_{-R}^{R} e^{-x^2}\,dx + \int_{R}^{R+i\alpha} e^{-z^2}\,dz$$

$$-\int_{-R+i\alpha}^{R+i\alpha} e^{-z^2}\,dz - \int_{-R}^{-R+i\alpha} e^{-z^2}\,dz.$$

$$\left|\int_R^{R+i\alpha} e^{-z^2}\,dz\right| \leq \alpha \cdot \max\left|e^{-(R+iy)^2}\right| = \alpha e^{-R^2} \max . e^{y^2}$$

$$\leq \alpha e^{-R^2+\alpha^2} \to 0 \quad \text{als} \quad R \to \infty$$

$$\left|\int_{-R}^{-R+i\alpha} e^{-z^2}\,dz\right| \to 0 \quad \text{idem.}$$

$$\int_{-R+i\alpha}^{+R+i\alpha} e^{-z^2}\,dz = \int_{-R}^{R} e^{-(x+i\alpha)^2}\,dx = \int_{-R}^{R} e^{-\cancel{R}^2-2ixa+\alpha^2}\,dx$$

$$= e^{\alpha^2}\int_{-R}^{R} e^{-x^2} \cdot \cos(2x\alpha)\,dx \quad \text{want imag. deel is symmetrisch.}$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

dus $\quad e^{\alpha^2}\int_{-\infty}^{\infty} e^{-x^2}\cos(2x\alpha)\,dx = \sqrt{\pi}.$

$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2x\alpha)\,dx = \sqrt{\pi}\, e^{-\alpha^2}.$$

3.4. 2c. $\displaystyle \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} e^{iax}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i(1+a)x}}{1+x^2}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i(-1+a)x}}{1+x^2}\,dx$

als $1+a>0$ dan bovenlangs sluiten:

$$\frac{1}{2} \cdot 2\pi i \cdot (x-i) \cdot \frac{1}{(x-i)(x+i)} e^{i(1+a)x}\Big|_{x=i} = \frac{\pi \cancel{i}}{2\cancel{i}} e^{-(1+a)}$$

$1+a<0$ dan onderlangs sluiten:

$$-\frac{1}{2} 2\pi i\,(x+i)\frac{1}{(x-i)(x+i)} e^{i(1+a)x}\Big|_{x=-i} = \frac{\pi}{2} e^{(1+a)}$$

samen 1e integraal: $\frac{\pi}{2} e^{-|1+a|}$.

2e integraal 't zelfde: $\frac{\pi}{2} e^{-|-1+a|}$

$$\left.\begin{matrix} \frac{\pi}{2} e^{-|1+a|} \\ \frac{\pi}{2} e^{-|-1+a|} \end{matrix}\right\} \frac{\pi}{2}\left(e^{-|1+a|} + e^{-|a-1|}\right)$$

3d $\displaystyle\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^4+x^2+1}\,dx = \mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x^4+x^2+1}\,dx\right]$

$$x^4+x^2+1 = \left(x^2-e^{\frac{2}{3}\pi i}\right)\left(x^2-e^{-\frac{2}{3}\pi i}\right)$$

$$= \left(x-e^{\frac{1}{3}\pi i}\right)\left(x+e^{\frac{1}{3}\pi i}\right)\left(x-e^{-\frac{1}{3}\pi i}\right)\left(x+e^{-\frac{1}{3}\pi i}\right)$$

polen in bovenhalfvlak: $x = e^{\frac{1}{3}\pi i}$ en $x = -e^{-\frac{1}{3}\pi i}$

$\pi > 0$ dus sluit bovenlangs:

$$2\pi i\left(\mathrm{Res}_{e^{\frac{1}{3}\pi i}} + \mathrm{Res}_{-e^{-\frac{1}{3}\pi i}}\right) =$$

$$= 2\pi i\left[\frac{\exp\left(i\pi e^{\frac{1}{3}\pi i}\right)}{2e^{\frac{1}{3}\pi i}\cdot 2i\sin\left(\frac{1}{3}\pi\right)2\cos\left(\frac{1}{3}\pi\right)} + \frac{\exp\left(-i\pi e^{-\frac{1}{3}\pi i}\right)}{-2\cos\left(\frac{1}{3}\pi\right)\cdot 2i\sin\left(\frac{1}{3}\pi\right)\cdot -2e^{-\frac{1}{3}\pi i}}\right]$$

$$= \frac{2\pi i}{8i\sin\left(\frac{1}{3}\pi\right)\cos\left(\frac{1}{3}\pi\right)}\cdot\left[\frac{\exp\left(\frac{1}{2}\pi i-\frac{1}{2}\sqrt{3}\pi\right)}{e^{\frac{1}{3}\pi i}} + \frac{\exp\left(-\frac{1}{2}\pi i-\frac{1}{2}\sqrt{3}\pi\right)}{e^{-\frac{1}{3}\pi i}}\right]$$

$$= \frac{\pi}{4\cdot\frac{1}{2}\sqrt{3}\cdot\frac{1}{2}}\cdot\exp\left(-\frac{1}{2}\sqrt{3}\pi\right)\left[e^{\frac{1}{6}\pi i}+e^{-\frac{1}{6}\pi i}\right]$$

$$= \frac{\pi}{\sqrt{3}}\cdot\exp\left(-\frac{1}{2}\sqrt{3}\pi\right)\cdot 2\cos\left(\frac{1}{6}\pi\right) =$$

$$\frac{\pi}{\sqrt{3}}\exp\left(-\frac{1}{2}\sqrt{3}\pi\right)\cdot\sqrt{3} = \pi\exp\left(-\frac{1}{2}\sqrt{3}\pi\right).$$

4. $I(a) = \int_0^\infty \frac{(\sin x)^2}{x^2+a^2}\,dx \qquad a>0$

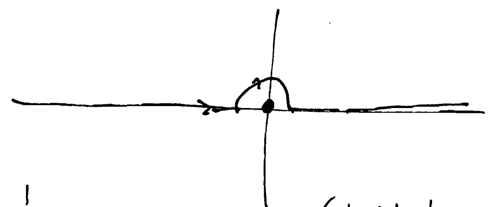
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2+a^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx - \frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+a^2}\,dx$$

$$= \frac{1}{4a}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} - \frac{1}{4a}\int_{-\infty}^{\infty} \frac{\cos(2at)}{1+t^2}\,dt.$$

$$= \frac{1}{4a}\cdot\pi - \frac{1}{4a}\cdot\pi\exp(-2a).$$

8.a. $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1-\cos(2x)}{x^2}\,dx.$

deformeer contour:



$\frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{4}\int_{\frown} \frac{1}{z^2}\,dz = 0$ (sluit bovenlangs) met het dat $\frac{R}{R^2}\to 0$

$$\frac{1}{4}\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2}\,dx = \frac{1}{8}\int_{-\infty}^{\infty} \frac{e^{2ix}+e^{-2ix}}{x^2}\,dx.$$

$$= \frac{1}{8}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2}\,dx + \frac{1}{8}\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2}\,dx$$

sluit bovenlangs   sluit onderlangs $\to$ pool in $z=0$.

$$= 0 + \frac{1}{8}\cdot -2\pi i\, Res_0 = -\frac{1}{2}\pi$$

$\frac{1-2ix+\dots}{x^2} = \frac{1}{x^2} - \frac{2i}{x} + \dots \to Res_0 = -2i$

totaal: $\frac{1}{2}\pi$.

Merk op: zie opg. 4: $\lim_{a\to 0} \frac{\pi}{4a}\left(1-e^{-2a}\right) = \frac{\pi}{4a}(1-1+2a) = \frac{\pi}{2}$

b. $\displaystyle\int_0^\infty \frac{\sin x}{x(1+x^2)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\cdot\frac{1}{1+x^2}\,dx.$

deformeer om $x=0$:

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\frac{1}{1+x^2}\,dx = \frac{1}{4i}\int_{-\infty}^{\infty} \frac{e^{ix}-e^{-ix}}{x(1+x^2)}\,dx$$

sluit 1e bovenlangs en merk op dat $\frac{e^{iz}}{x(1+x^2)}\cdot R \to 0$

$$\frac{1}{4i}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(1+x^2)}\,dx = \frac{2\pi i}{4i}(x-i)\frac{e^{ix}}{x(x+i)(x-i)} = \frac{1}{2}\pi\frac{e^{-1}}{i\cdot 2i} =$$

$$= -\frac{1}{4}\pi e^{-1}$$

sluit 2e onderlangs;

$$-\frac{1}{4i}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x(1+x^2)}\,dx = +\frac{2\pi i}{4i}\cdot\left[\frac{e^{-i\cdot 0}}{1+0} + \frac{e^{-i\cdot -i}}{-i\cdot(-i-i)}\right]$$

$$= \frac{1}{2}\pi\left(1+\frac{e^{-1}}{-2}\right).$$

samen: $\frac{1}{2}\pi - \frac{1}{4}\pi e^{-1} - \frac{1}{4}\pi e^{-1} = \frac{1}{2}\pi - \frac{1}{2}\pi e^{-1}$

c. $\displaystyle\int_0^\infty \frac{x-\sin x}{x^3}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x-\sin x}{x^3}\,dx.$

deformeer om $x=0$: $\displaystyle\frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\frac{1}{x^2}}_{I_1} - \underbrace{\frac{1}{2i}\frac{e^{ix}}{x^3}}_{I_2} + \underbrace{\frac{1}{2i}\frac{e^{-ix}}{x^3}}_{I_3}\,dx$

I_1: sluit bovenlangs; $I_1 = 0$.

I_2; sluit bovenlangs:

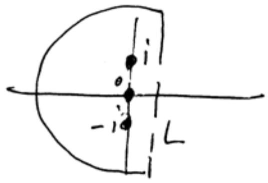
geen residu bijdrage $\to I_2 = 0$

I_3: sluit onderlangs: $\text{Res} = \frac{1 - ix - \frac{1}{2}x^2 \ldots}{x^3} \to -\frac{1}{2}$

$I_3 = -\frac{1}{4i}\cdot 2\pi i\cdot -\frac{1}{2} = \frac{1}{4}\pi$, Samen: $\frac{1}{4}\pi + \cancel{\frac{1}{4}\pi}^{\,0} = \frac{1}{4}\pi$

3.5

1c $\frac{1}{2\pi i}\int_L \frac{1}{s(s^2+1)} e^{st}\, ds$ ~~met~~ met



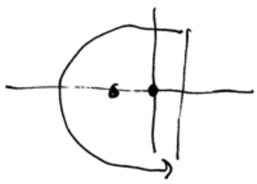
polen in $s=0$, $s=i$, $s=-i$

$Res_0 = 1 \cdot e^0 = 1$, $Res_i = (s-i)\frac{1}{s(s-i)(s+i)} e^{st} = \frac{1}{i \cdot 2i} e^{it} = -\frac{1}{2} e^{it}$

$Res_{-i} = (s+i)\frac{1}{s(s+i)(s-i)} e^{st} = \frac{1}{-i \cdot -2i} e^{-it} = -\frac{1}{2} e^{-it}$

samen: $1 - \frac{1}{2} e^{it} - \frac{1}{2} e^{-it} = 1 - \cos t.$

1d. $\frac{1}{2\pi i}\int_L \frac{1}{s^3+4s^2+4s} e^{st}\, ds$ met



polen in $s=0$, $s=-2$

$Res_0 = \frac{1}{4} e^0 = \frac{1}{4}.$

Res_{-2}: $\frac{1}{s(s+2)^2} e^{st} = \frac{e^{(y-2)t}}{y^2(y-2)} = \frac{e^{-2t} e^{yt}}{-2y^2(1-\frac{1}{2}y)} =$

$-\frac{1}{2} e^{-2t} \frac{1}{y^2}(1+yt)(1+\frac{1}{2}y) = -\frac{1}{2} e^{-2t}\left(\frac{1}{y^2} + \frac{t+\frac{1}{2}}{y} + \dots\right).$

dus $Res_{-2} = -\frac{1}{2}(t+\frac{1}{2}) e^{-2t}$

totaal: $\frac{1}{4} - \frac{1}{2}(t+\frac{1}{2}) e^{-2t}.$

3.6

1a $\displaystyle \underset{\displaystyle I}{\underset{\parallel}{\int_0^\infty \frac{x^{\mu-1}}{(x^2+1)^2}\,dx}}$ convergentie voor $x \downarrow 0$: $-1 < \mu-1$, dus $\mu > 0$

convergentie voor $x \to \infty$: $\mu-1-4 < -1$, dus $\mu < 4$.

beschouw $z^{\mu-1} = e^{(\mu-1)\ln z}$ met $\ln z$ hoofdwaarde.

$$\int_{-\infty}^{\infty} \frac{z^{\mu-1}}{(z^2+1)^2}\,dz = \int_{-\infty}^{0} \frac{e^{(\mu-1)\ln|z| + \pi i(\mu-1)}}{(z^2+1)^2}\,dz + \int_0^\infty \frac{z^{\mu-1}}{(z^2+1)^2}\,dz = \left(1+e^{\pi i(\mu-1)}\right) I = \left(1-e^{\pi i \mu}\right) I$$

↳ bovenlangs vert. snede

$$\mathrm{Res}_i = \left[\frac{d}{dz}\,\frac{e^{(\mu-1)\ln z}}{(z+i)^2}\right]_{z=i} = \left[\frac{(z+i)^2\,\frac{\mu-1}{z}\,e^{(\mu-1)\ln z} - e^{(\mu-1)\ln z}\cdot 2(z+i)}{(z+i)^4}\right]_{z=i}$$

$$= e^{(\mu-1)\frac{\pi}{2}i}\,\frac{\mu-2}{-4i} = e^{\frac{1}{2}\pi i\mu}\,\frac{\mu-2}{4}.$$

Totaal: $2\pi i\, e^{\frac{1}{2}\pi i \mu}\,\frac{\mu-2}{4} = \left(1-e^{\pi i\mu}\right) I$

$$\tfrac{1}{4}\pi(\mu-2) = \tfrac{1}{2i}\left(e^{-\frac{1}{2}\pi i\mu} - e^{\frac{1}{2}\pi i\mu}\right) I$$

$$I = \frac{\frac{1}{4}\pi(2-\mu)}{\sin(\frac{1}{2}\pi\mu)}.$$

2d $\displaystyle \int_0^\infty \frac{1}{x^4+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{z^4+1}\,dz = \pi i\left(\mathrm{Res}_{e^{\frac{1}{4}\pi i}} + \mathrm{Res}_{e^{\frac{3}{4}\pi i}}\right).$

$$\mathrm{Res}_{e^{\frac{1}{4}\pi i}} = \left[(z-e^{\frac{1}{4}\pi i})\,\frac{1}{(z-e^{\frac{1}{4}\pi i})(z-e^{\frac{3}{4}\pi i})(z+e^{\frac{1}{4}\pi i})(z+e^{\frac{3}{4}\pi i})}\right]_{z=e^{\frac{1}{4}\pi i}} = \frac{e^{-\frac{1}{4}\pi i}}{4i}$$

$$\mathrm{Res}_{e^{\frac{3}{4}\pi i}} = \left[(z-e^{\frac{3}{4}\pi i})\,\frac{1}{(z-e^{\frac{1}{4}\pi i})(z-e^{\frac{3}{4}\pi i})(z+e^{\frac{1}{4}\pi i})(z+e^{\frac{3}{4}\pi i})}\right]_{z=e^{\frac{3}{4}\pi i}} = \frac{e^{+\frac{1}{4}\pi i}}{4i}$$

totaal: $\displaystyle \frac{\pi i}{4i}\left(e^{-\frac{1}{4}\pi i} + e^{\frac{1}{4}\pi i}\right) = \tfrac{1}{2}\pi\cos\tfrac{1}{4}\pi = \tfrac{1}{4}\pi\sqrt{2}$

3d $\quad I = \int_0^\infty \frac{\ln x}{x^4+1}\,dx.$

beschouw $\ln z$ hoofdwaarde

$$\int_{-\infty}^{\infty} \frac{\ln z}{z^4+1}\,dz = \int_{-\infty}^{0} \frac{\ln|z|+\pi i}{z^4+1}\,dz + \int_0^\infty \frac{\ln z}{z^4+1}\,dz =$$

Ls bovenlangs vert. sn.

$$\pi i \int_0^\infty \frac{1}{z^4+1}\,dz + 2\int_0^\infty \frac{\ln z}{1+z^4}\,dz$$

Sluit bovenlangs.

$$\mathrm{Res}_{e^{\frac{1}{4}\pi i}} = \ln(e^{\frac{1}{4}\pi i}) \cdot \frac{1}{4i} e^{-\frac{1}{4}\pi i} = \frac{\pi}{16} e^{-\frac{1}{4}\pi i} \quad (\text{zie opg. 2d})$$

$$\mathrm{Res}_{e^{\frac{3}{4}\pi i}} = \ln(e^{\frac{3}{4}\pi i}) \frac{1}{4i} e^{\frac{1}{4}\pi i} = \frac{3}{16}\pi e^{\frac{1}{4}\pi i}$$

samen:

$$\int_{-\infty}^{\infty} \frac{\ln z}{z^4+1}\,dz = 2\pi i\left(\frac{\pi}{16} e^{-\frac{1}{4}\pi i} + \frac{3}{16}\pi e^{\frac{1}{4}\pi i}\right) = \frac{1}{8}\pi^2\left(-2\sin\frac{\pi}{4} + 4i\cos\frac{\pi}{4}\right)$$

$$= -\frac{1}{8}\pi^2\sqrt{2} + \frac{1}{4}\pi^2 i\sqrt{2}$$

zodat $\quad \displaystyle\int_0^\infty \frac{\ln z}{1+z^4}\,dz = -\frac{1}{16}\pi^2\sqrt{2}$

$$\int_0^\infty \frac{1}{1+z^4}\,dz = \frac{1}{4}\pi\sqrt{2} \quad (\text{check opg. 2d}).$$