Complex Function Theory

(René van Hassel, translator,
September 1, 2022)

A translation[1] of the lecture notes of Wiskunde 50/52 (september 1979), written by Joop Boersma and Jan de Graaf.[2]

[1] There could still be a lot of typos in this translation.
[2] "A work of art,
or that which it pretends to be,
should never be provided with a preface."
(Ferdinand Pessoa)

Complex Function Theory - Basic

Contents

(René van Hassel, translator,
September 1, 2022)

1	Basic Concepts	4
1.1	Complex Numbers	4
1.2	Sets of complex numbers	5
1.3	Arc, curve, path in complex plane	9
2	Function Concept	12
2.1	Functions of a complex variable	12
2.2	Differentiable functions	14
2.3	Functions defined by power series	18
3	Integration in the complex plane	25
3.1	Complex integration	25
3.2	Main theorem of complex integration	30
3.3	Residue; integral formula of Cauchy	34
3.4	Construction of analytic functions by series; theorem of Weierstrass	45
4	Holomorphic functions; Series	51
4.1	Theorem of Taylor	51
4.2	Behaviour near a regular point	52
4.3	Laurent series	57
4.4	Singular points; addition of the point $z = \infty$	60
4.5	Further details about residues	63
5	Application of the residues in the integral calculus	67
5.1	Definite integrals	67
6	Analytic continuation	75
6.1	Analytic continuation	75
6.2	The logarithm	76
7	Various	80
7.1	Differential equations of Cauchy-Riemann	80
7.2	Conformal mapping	83
8	Literature	89

1 Index 90

1 Basic Concepts

1.1 Complex Numbers

The definition and properties of complex numbers are supposed to be known.
The most important points will be repeated.

In the 2-dimensional space $\mathbb{R}^2$ there can be introduced the operation "multiplying" by: $(a, b) * (c, d) = (a\,c - b\,d,\ a\,d + b\,c)$ such that we get a system of things where can be calculated just as we do with the real numbers.

These things are called complex numbers ($\mathbb{C}$). The complex numbers $(a, 0)$ are identified with the real numbers. For the number $(0, 1)$ can be introduced the abbtreviation i. So each complex number z can be written unambiguously as $x + i\,y$ where x and y are real numbers, called tbe real part and the imaginary part of z.

Notation: $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$. The complex numbers z with $\mathrm{Re}(z) = 0$ lie on the imaginary axis, those with $\mathrm{Im}(z) = 0$ lie on the real axis of the complex plane.
The number $x - i\,y$ is called the complex conjugate of z.
Notation: $\overline{z} = x - i\,y$.

The complex numbers can also be represented by the argument and modulus (absolute value), to which one is passed if in the complex plane are introduced polar coordinates: r = distance to the origin, ϕ = angle with the positive real axis. The direction of the increasing ϕ is the direction against the hands of the clock, so anti-clockwise. The angle ϕ is determined by (x, y) except for integer multiples of $2\,\pi$ and there holds

(1) $\quad x = r\cos(\phi),\ y = r\sin(\phi).$

Among modulus or absolute values of z is understood the non-negative number

(2) $\quad r = |\,z\,| = (x^2 + y^2)^{\frac{1}{2}} \geq 0.$

The angle ϕ is called the argument of z:

(3) $\quad \phi = \arg(z).$

A complex number can be written as

(4) $\quad x = x + i\,y = r\,(\cos(\phi) + \sin(\phi)) = r\exp(i\,\phi).$

With these notations there can be learned to process easy properties of complex numbers.

There is spoken about the principal value of $\arg(z)$ if the argument is limited to the interval

(5) $\pi < \arg(z) \leq \pi$.

Notation: $\mathrm{Arg}(z)$ the principal value of $\arg(z)$.

In the remainder of this course for $\arg(z)$ will be taken the principal value.
The importance of it will be discussed later.

1.2 Sets of complex numbers

The notation for the set of all those complex numbers is $\mathbb{C}$. Subsets are often described by some prescription or there is mentioned some property, which the elements z of the subset A determine. An often used notation is $A := \{z|......\}$ where in place of the dots a prescription is given where z has to satisfy. So is $\{z\,|\,|z| < 1\}$ the inside of the unit circle in $\mathbb{C}$. The imaginary axis is $\{z\,|\,z = i\,y,\, y\,\mathrm{real}\}$. In the sequel we make no distinction between the following three expressions:
"z is a complex number", "z is an element of the set $\mathbb{C}$" and
"z is a point in the complex plane".

Let V be some set of complex numbers. To express that a certain complex number a belongs to V, is used the notation $a \in V$; we say: a is an element of V. If we want to express that a does not belong to V, then we write $a \notin V$.

Finite set: a set with a finite number of elements.
Infinite set: a set with infinitely many elements.
Bounded set: A set V is called bounded if we can find a constant M such that $|z| < M$ for all $z \in V$. Such a bounded set can be covered by a circle around the origin and a finite radius; also by a square of finite dimensions.

Subset: A is a subset of B if every element of A is an element of A. The notation is $A \subset B$ (A is contained in B) or $B \supset A$ (B includes A). The possibility that $A = B$, is not excluded in this notation $A \subset B$. Is $A \neq B$, then A is a proper subset of B.
Intersection of A and B is the set of elements which belong both to A and B. The notation is $A \cap B$, or $A.B$, or AB. If $A \cap B = \emptyset$ ($\emptyset =$ lege verzameling), then A and B are called disjoint.

Union of A and B, notation $A \cup B$ or $A + B$, is the set of complex numbers which belong to A, to B, or to both.

The concepts of intersection and union can be applied to an arbitrary collection of sets.

Complement of A: the set of complex numbers that don't belong to A. Notation for the complement of A is A', or A^c, also $\mathbb{C} \setminus A$.

Surrounding:
Let a be a complex number and p positive. The numbers z with the property $|z - a| < p$ are located within the circle with center a and radius p. If p is small, then they say that this z lies in the surrounding of a.
Therefore we define:
Surrounding of a is the set of numbers z which satisfy to $|z - a| < p$.
The surrounding depends on a and p.
Therefore: p-surrounding of a. The point a belongs to each of its surroundings.
One speaks of a reduced surrounding if a is excluded. So: a reduced p-surrounding of a is the set of numbers z with $0 < |z - a| < p$.
Limit point The number a is called a limit point of the set V if in every surrounding of a lie infinitely many points of V.

Exercise: a is limit point of V if in every reduced surrounding of a lies at least one point of V.

Remark: a limit point of V is not necessarily a point of V.

Closed set:
A set V is called closed if every limit point of V is also a point of V. So a closed set contains all its limit points.
Internal point:
The number a is an internal point of V if there can be found a surrounding of a which belongs completely to V.
Open set:
A set which consists of only internal points, is called an open set.

Exercise:
Familiarize yourself with these concepts by sketching pictures.

Connected set:
A set V is called connected if every pair points P, Q of V can be connected by a

curve, which lies inside V. (definition of a curve, see page ??)

Region:
This is for us a very important concept. A region G inside the complex plane is a set of complex numbers, which is

1. not empty,
2. open and
3. connected.

Every point of G is an internal point of G, because a region is by definition open. Two arbitrary points of G can be connected by a polygon, which lies completely inside G, because G is connected. A limit point of G belongs to G or does not belong to G. In the first case the limit point is a point of G, and so it is an internal point of G. In the second case the limit point of G is a so-called boundary point of G. The set of limit points of G, which doesn't belong to G, form together the boundary of G. (The boundary of a set V is defined by the set of points such that in every surrounding lies a point of V and a point of V^c.) The union of G with its boundary is a closed set, wich is be notated by $\overline{G}$. $\overline{G}$ is called a closed region. We will for that choose the word domain. So domain = region + boundary.

Two important theorems:

Theorem 1

A. The cover theorem of Heine and Borel:

> If a bounded closed set A of complex numbers is contained in the union of a collection of open sets, than is A already contained in (can be covered by) a finite number of those open sets.

Proof

If the collection itself is finite, there is nothing to prove.
Further by contradiction. A is bounded, A can be covered by a square V_0 (boundary included) completely lying in the finite z-plane.
Suppose that the theorem was incorrect. Divide V_0 in four equal squares. Then is under these squares at least one (say V_1, boundary included) with the property: the part of A located in V_1 can not be covered by a finite number of open sets out of

the collection of open sets, referred to.
Repeat: V_1 is divided in four equal squares. At least one of them (V_2, boundary included) has the property that $A \cap V_2$ (the intersection of A and V_2) can not be covered by a finite number of those partical open sets. And so on.
We get so a sequence of squares: V_0, V_1, V_2, ..., with $v_0 \supset V_1 \supset V_2 \supset$ ...., which shrinks to one common point P. This point belongs to every V_n, and is a limit point of A (why?). Because A is closed, belongs P to A.

In the given collection of open sets there is at least one, which contains P, and so in internally contains. Than there is also a p-surrounding of P (if we take p small enough) with the property that, the intersection with A, entirely is covered by just that paricular open set.
This leads to a contradiction. In the long run (n sufficiently large) lie all Vn inside that p-surrounding. On the one hand could V_n not be covered by a finite number and on the other hand she is (from a certain rank number) covered by just of these open sets. Out of the contradiction follows that the thereom is correct.

□

Exercise: Let G be a region, D a closed subeset of G, then there exists a positive number δ such that for every $z_1 \in D$ and $z_2 \in G^c$ holds $|z_1 - z_2| > \delta$; in other words: the boundary of D doesn't come arbitrary close to the boundary of G.
(Hint: cover D with the collection of circles $|z - m| < r$, at which $m \in D$ and $r = (\frac{1}{2}) \cdot$ (distance of m to the boundary of G).)

Theorem 2

B. The theorem of Bolzano and Weierstrass (B.W.):

> A bounded infinity set has at least one limit point.

Proof

Give the proof yourself by using once again the in theorem 1 used square-methode.

□

Definition 1

A sequence of numbers z_n ($n = 1, 2, 3, ...$; there may be equal among them) has the limit a if by an arbitrary $\epsilon > 0$ there can be found some rank number N such that $|z_n - a| < \epsilon$ for $n \geq N$.

A convergence criterium, where the limit a does not occur, holds:
A sequence $\{z_n\}$ has a limit, if for every $\epsilon > 0$ there exists a N such that for $n > N$ and $m > N$ holds $|z_n - z_m| < \epsilon$ (Cauchy).
Proof. There is some M such that for all $m > M$: $|z_m - z_{(M+1)}| < 1$. Let a be the limit point of z_m (B.W!). Then $|z_n - a| \leq |z_n - z_m| + |z_m - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ to make.

Definition 2

Is $\{a_n\}$ a sequence of real numbers then we define:

$$\limsup_{n\to\infty} a_n = \begin{cases} L & \text{if for every } \epsilon > 0 \text{ only finitely many elements of the sequence} \\ & \text{are } > L + \epsilon, \text{ while infinitely many elements are } > L - \epsilon, \\ \infty & \text{if the sequence is not bounded above,} \\ -\infty & \text{if } a_n \to -\infty \text{ for } n \to \infty. \end{cases}$$

Remark: If a sequence $\{a_n\}$ has a limit L then $\limsup_{n\to\infty} a_n = L$.

Theorem 3

Every real sequence $\{a_n\}$ has a $\limsup$.

Exercise: Prove this.
(Hint: if $\limsup \neq \pm\infty$ then all $a_n < C$ and ∞ many $> D$; bisect $[D, C]$.....)

1.3 Arc, curve, path in complex plane

By a smooth arc we mean in this lecture: the continuous differentiable image in $\mathbb{R}^2$ of a linesegment. With this is meant the following: if for all $a \leq t \leq b$ the

functions $x(t)$ and $y(t)$ have continuous derivatives then we call the set of points $z = x(t) + i\,y(t)$ $(a \leq t \leq b)$ a smooth arc. The point $z_1 = f(a)$ is called the starting point of the arc, $z_2 = f(b)$ the end point. The variable t is called the parameter and $z = x(t) + i\,y(t)$ a parametric representation of the arc. The same arc can have many different parametric representations.
(Often we still require that $\{x'(t)\}^2 + \{y'(t)\}^2 > 0$.)
If we write arc in the sequel we mean still: smooth arc.

A curve is a connection of a finite number of smooth arcs; it has a starting and a end point. A curve is a curve without double points. If the starting and the end point fall together we speak of a closed curve.

A Jordan curve is a simple closed curve.

Properties: a Jordan curve divides the complex plane into two disjoint parts, the inner area and the outer area. Both parts are regions, which have the curve as boundary. The Jordan curve is called positive oriented (can be indicated by an arrow) if we move in the direction of the arrow and the inner regions is to our left hand side (counterclockwise).

Path
Later we need the concept of path in the complex plane. A path is connection of a sequence of arcs not necessarily a finite number. A path can walk to infinity, or it can come from there (for instance out of a cerain direction), etc..

We recall a theorem out of the lecture Wiskunde 20.

Theorem 4

Let K be an arc with parametrisation $z = x(t) + i\,y(t)$, $(a \leq t \leq b)$. Then is the length of K equal to

$$\boxed{L = \int_a^b \sqrt{((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2)}\,dt = \int_a^b |\frac{dz}{dt}|\,dt.}$$

Single connected region A region G is called simple connected, if with every Jordan curve J which belongs to G also the inner area of J belongs to G.
If G is connected but not simple connected, dan is G called multiple connected.

Examples:

1. $\{z \mid |z| < 1\}$ is simply connected.
2. $\{z \mid |z| > 1\}$ is multiple connected.
3. $\{z \mid 1 < |z| < 2\}$ is multiple connected.
4. Let out of the in (3) described region the line segment
   $\{ z \mid \mathrm{Re}(z) < 0 \,\mathrm{and}\, \mathrm{Im}(z) = 0\}$ then arises a simple connected region.

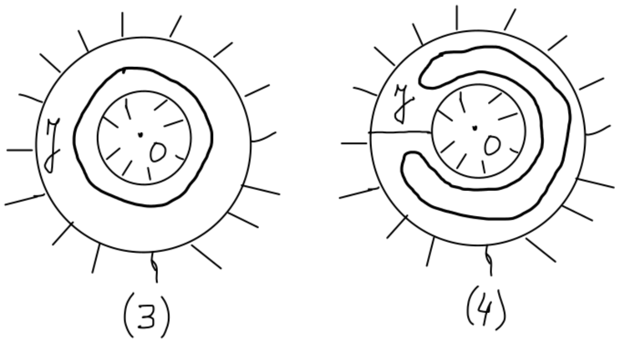

Figure 1 See examples (3) and (4) .

Here beneath follow some other examples with associated figures.

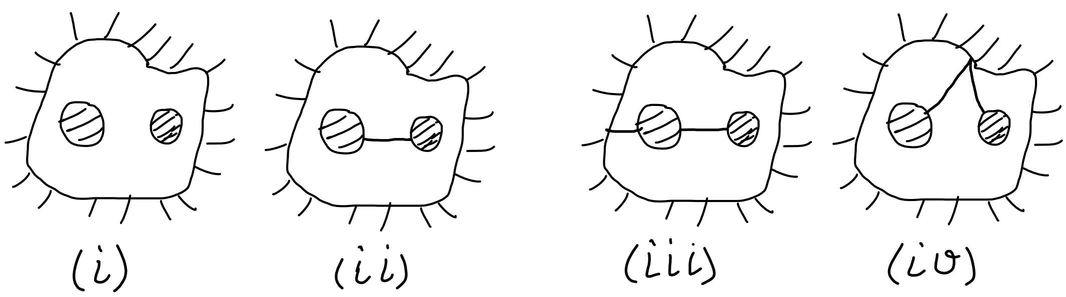

Figure 2 (i) and (ii) multiple connected, (iii) and (iv) simple connected.

The not to G belonging "islands" may also be points: just as (3) is for instance $\{z \mid 0 < |z| < 1\}$ multiple connected.

2 Function Concept

2.1 Functions of a complex variable

Let's given:

1. a set A of complex numbers,

2. a prescription, such that to every $z \in A$ a certain complex number $w \in \mathbb{C}$ is added.

We write $w = f(z)$, and represent the situation with: f (sometimes one writes $f(z)$) is a (single-valued) complex function of z defined on A. To mention the set, where the function is defined, is essential. Compare with the former used term of "permissible" values of the independent variable.

If we put $z = x + i\,y,\ w = u + i\,v$, we can split $f(z)$ into a real and an imaginary part: $f(z) = u(x,\,y) + i\,v(x,\,y)$. The functions u and v are real valued functions of two variables, defined for $(x,\,y) \in A$. A complex function is nothing but a pair of two real functions of two real variables. It is clear that we to narrow the function concept, if we want to get something interesting; otherwise we commit real function theory.

Limit concept

The definition of limit is formal the same as used in the real analysis.
A function f defined on A has the limit L for $z \to a$ if to every $\epsilon > 0$ there exist a $\delta > 0$ such that for $0 < |z - a| < \delta$ and $z \in A$ holds $|f(z) - L| < \epsilon$.
We write:

$\lim\limits_{z\to a} f(z) = L,$ or also $f(z) \to L\ (z \to a)$.

Other definition (check equivalence): f has the limit L for z to a if $\lim_{n\to\infty} f(z_k) = L$ for every sequence $\{z_k\}$ with $z_k \in A$ and $z_k \to a\ (k \to \infty)$.

Theorems about summation, difference, product and quotient of limits the same as in the real analysis.

Continuity concept

The function f is called continuous in the point $a \in A$ if $f(a) = \lim_{z\to a} f(z)$.

Equivalent definition: f is continuous in $a \in A$ if to every $\epsilon > 0$ there exists a δ such that $|f(z) - f(a)| < \epsilon$ for $|z - a| < \delta$ and $z \in A$.

f is called continuous on (or in) A if $f(z)$ is continuous in every point of A. In most cases is A a region, a domain, or a curve.

Out of the real analysis we know the concepts of left- and right-continuity: $\lim_{x \uparrow a} f(x) = f(a)$, resp. $\lim_{x \downarrow a} f(x) = f(a)$. In the complex plane there a lot more possibilities to approach a point. Let f be defined on A and B a subset of A. Then f is called continuous in the point a with respect to B if

1. $a \in B$,
2. to every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for $|z - a| < \delta$ and $z \in B$.

Example: by $f(z) = z^{-1} - (\overline{z})^{-1}$ $(z \neq 0)$, $f(0) = 0$ is f defined on $A =$ whole complex plane; f is discontinuous in 0, but continuous in 0 with respect to $B =$ real axis.

Remark: If there is spoken about "continuity in a", without a doubt, then is assumed that the function is defined in a full surrounding of a.

Remark: Summation, difference,product and quotient of continuous functions are continuous functions (with the well-known exception: divide not by zero), even as in the real-function theory.

Is $f(z)$ a continuous function of z, then are the real functions $u(x, y)$ and $v(x, y)$, defined by $f(z) = u(x, y) + i\, v(x, y)$, continuous functions of x and y. The reverse holds also (check this). To get really interesting complex-function theory, we need to require more than only continuity.

Exercise: Is f continuous, then are also $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, en $|f|$ continous functions of z. The principal value of $\arg(z)$ is not continuous in the neighbourhood of the negative real axis; at that part of that axis the function is continuous (because there is $\arg(z)$ equal to π).

Theorem 5

Is f defined and continuos on a closed and bounded set A, then is f on A bounded, and $|f(z)|$ has both a maximum and a minimum on A.

Proof

Analoguous to the theorems 1 and 2.

□

2.2 Differentiable functions

We have already noticed that we won't get very far with continuous functions. Therefore we will require differentiability. A priori is not clear that there will come something interesting, because in the real analysis we also have functions of two variables, which you can differentiate to these variables.

We shall define the derivative f' of f analoguously to the way which is used in the real analysis of functions of one real variable. The results turn out to be very surprising. It turns out that a functions which is differentiable, is automatically arbitrary often differentiable.

We shall the derivative only define in an internal point of the definition set A of the function. The set of all internal points of A is an open set. One can prove that an open set is the union of a collection of disjoint regions. Therefore it is enough, to define differentiability of a function which is defined in a region (open + connected).

Definition 3

A function f, defined on a region G, is called differentiable in the point a of G as the difference quotient

$$\frac{f(a+h)-f(a)}{h} \quad (h \neq 0)$$

has a limit $L(a)$ for $h \to 0$.

For $|h|$ small enough lies $a+h$ also in G. The above-mentioned limit $L(a)$ we denote also with $f'(a)$, and $f'(a)$ is called the derivative of $f(z)$ in the point a. This is fully analoguous to the case of the real analysis. (Remember that h is complex here!)

Equivalent definitions

1. $f(z)$ defined in G has derivative $L(a)$ in point $a \in G$ if for every $\epsilon > 0$ there is a δ such that $|\frac{f(a+h)-f(a)}{h} - L(a)| < \epsilon$ for every z with $0 < |z-a| < \delta$.

2. If in the surrounding of a holds

   $f(z) = f(a) + (z-a)\,L(a) + (z-a)\,\eta(z, a),$

   with $\eta(z, a) \to 0$ for $z \to a$, then we can define $L(a)$ as the derivative of $f(z)$ in the point a.

Exercise. Prove the equivalence of the definitions (1) and (2).

Theorem 6

Is f differentiable in the point a, then is f continuous in a.

Proof

f differentiable in a, so f is defined in a full surrounding for a. We can then find numbers $L = f'(a)$ and δ_1 such that (take $\epsilon = 1$)

$$|\frac{f(z)-f(a)}{z-a} - L| < 1 \quad \text{for all } z \text{ with } 0 < |z-a| < \delta_1.$$

So also

$$|f(z) - f(a) - (z-a)\,L| < |z-a| \quad \text{for all } z \text{ with } 0 < |z-a| < \delta_1.$$

Under these circumstances

$$|f(z) - f(a)| = |f(z) - f(a) - L\,(z-a) + L\,(z-a)|$$
$$\le |f(z) - f(a) - L\,(z-a)| + |L|\,|(z-a)|$$
$$\le (1 + |L|)\,|z-a| \le (1 + |L|)\,\delta_1.$$

Let $\epsilon > 0$ be given. Choose $\delta = \min(\delta_1, \frac{\epsilon}{(1+|L|)\,\delta_1})$. Then holds

$$|f(z) - f(a)| < \epsilon \quad \text{provided that} \quad |z-a| < \delta.$$

So f is continuous in a.

$\square$

Here we have written down the proof in all precision. The understanding that the theorem is correct, we can get faster by: if $z \to a$ then has $\dfrac{f(z) - f(a)}{z - a}$ a finite limit. Because the denominator approaches 0 the nominator has also to approach 0.

Remark: of course the reverse of the above mentioned theorem holds not.

Example. $\mathrm{Re}(z)$ is continuous function of z, but it has no derivative at any point. Verify that $f(z) = \overline{z} = x - i\,y$ is not differentiable.

Remark: de known rules for differentiating of sum, difference, product and the quotient (denominator not zero) of functions just go through.

Exercise: Prove the chain-rule.
Is $\phi(z)$ differentiable in z_0, $f(\phi)$ differentiable in $\phi_0 = \phi(z_0)$, then is $F(z) = f(\phi(z))$ differentiable in z_0 and

$F'(z_0) = f'(\phi_0)\,\phi'(z_0)$.

Recommendation: use definition (2) of equivalent definitions of differentiability (why?).

Definition 4

Is f defined in a region G, and is f in every point differentiable, then is f called differentiable in G. Then f is called an (one valued) analytic function, or also holomorphic function, in G. Also the name regular function is used instead of holomorf.

Analytic in a point, on a curve
Analytic in a point = differentiable in that point and in a full surrounding of that point.

Analytic on a curve = analytic in every point of that curve.

So: by analytic in a point or on a curve we need to imbed that point or that curve into a region, and require that the function is differentiable in that region.

Exercise. Consider $f(z) = x^2\, y\, +\, i\, x\, y^2$.
Prove: $f(z)$ is differentiable in $z = 0$, but $f(z)$ is not analytic in $z = 0$.

The real function defined by

$$f(x) = \begin{cases} 0 & \text{if x=0,} \\ x^2 \sin \dfrac{1}{x} & \text{if x} \neq 0 \end{cases}$$

is for all x differentiable, but f' is not continuous in $x = 0$, so not differentiable. Such a thing is not possible in complex function theory. This is shown by the following

Fundamental property ("property*").

Is $f(z)$ holomorfphic in G, then is also $f'(z)$ holomorphic in G.

Proof is postponed. We refer to this result by calling it property*. Consequence of property*: is $f(z)$ holomorphic in G, then exist all the derivatives $f'(z)$, $f"(z)$, ..., in G and these are all holomorphic in G.
Although for now we don't need to use this property it is good to see, already now, that differentiability of a function is such a heavy requirement, that the function exists a lot more nice properties.

Examples of analytic functions

$f(z) = \text{constant}, \quad f'(z) = 0.$

$f(z) = z, \qquad f'(z) = 1.$

$f(z) = z^n, \qquad f'(z) = n\, z^{(n-1)} \quad (n\,\text{whole number}).$

$$f(z) = \sum_{n=0}^{N} a_n\, z^n, f'(z) = \sum_{n=1}^{N} n\, a_n\, z^{(n-1)} \quad \text{(polynomial).}$$

A polynomial is an analytic functions in G = whole z-plane.

Exercise. Where are the functions z^{-2}, $(1\, +\, z)^{-1}$, holomorphic?
Idem for the quotient of two polynomials in z.
Verify that $|z|$ is nowhere analytic.

2.3 Functions defined by power series

Power series are very important to the complex function theory.

Powerserie around the point z_0 is of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Powerserie around the origin is $\sum_{n=0}^{\infty} a_n z^n$.

Theorem 7

The root test:

The power series $\sum_{n=0}^{\infty} a_n z^n$ is absulute and uniform convergent for $|z| \leq \rho$ if $0 < \rho < R$ and divergent for $|z| > R$. Hereby is R a number which is determined by the sequence $\{a_n\}$, and according to Cauchy-Hadamard holds:

$$\frac{1}{R} = \limsup_{n \to \infty} (|a_n|^{(\frac{1}{n})}).$$

(Thereby if $R = 0$ if $\limsup_{n \to \infty} (|a_n|^{(\frac{1}{n})}) = \infty$ and $R = \infty$ if this lim sup is 0.)

Proof

First the trival case that $R = 0$

For $z = 0$ is the power series always convergent, with sum a_0. That is then in the case that $R = 0$ also the only point of convergence, because under these circumstances the general term of the series doesn't even go to zero (what is needed for convergence). After all: $z \neq 0$

$$\limsup_{n \to \infty} |a_n z^n|^{(\frac{1}{n})} = |z| \limsup_{n \to \infty} |a_n|^{(\frac{1}{n})} = \infty.$$

So $a_n z^n$ doesn't approach to zero if $n \to \infty$.

Second case, $R > 0$

Take $0 < \rho < R$ and $\rho < r < R$. Then is $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{(\frac{1}{n})} < \frac{1}{r}$. At the right of $\frac{1}{r}$ is therefore only a finite number of numbers $|a_n|^{(\frac{1}{n})}$. In other words: there is a N such that $|a_n|^{(\frac{1}{n})} < \frac{1}{r}$ for $n > N$. So for $n > N$ and for all z with $|z| \leq \rho$:

$|a_n\, z^n| \leq (\frac{\rho}{r})^n$.

Because $(\frac{\rho}{r})^n$ not depends on z and $\sum_{n=0}^{\infty} (\frac{\rho}{r})^n$ converges is $\sum_{n=0}^{\infty} a_n\, z^n$ absolute convergent and uniform convergent for $|z| \leq \rho$.

Is $|z| > R$, then is $\limsup_{n\to\infty} |a_n\, z^n|^{(\frac{1}{n})} = \frac{|z|}{R} > 1$, such that the general term of the series doesn't go to zero for $n \to \infty$. This means that the series diverges. Herewith is the theorem proved.

$\square$

Remark: The quantity R is called the radius of convergence; the point z with $|z| = R$ form the circle of convergence.

Exercise: a power series convergence uniformly on every bounded and closed set located inside the circle of convergence.

Examples:

$$R = 0: \quad \sum_{n=0}^{\infty} (-1)^n\, n!\, z^n$$

$$R = \infty: \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$R = 1: \quad \sum_{n=0}^{\infty} z^n, \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

The behaviour of the power series at the radius of convergence can be anything. So is $\sum_{n=0}^{\infty} z^n$ divergent for $|z| = R = 1$.

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{is abslute and uniform convergent for } |z| = R = 1.$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{is convergent for } z = -1, \text{ divergent for } z = 1.$$

Theorem 8

The function $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ is inside the circle of convergence $|z| < R$ a holomorphic function of z, and its derivative is found by termswise differentiation: $f'(z) = \sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$. This last series has the same radius of convergence as the first series.

Proof

$\limsup_{n\to\infty} (n\,|a_n|)^{(\frac{1}{n})} = \limsup_{n\to\infty} (|a_n|)^{(\frac{1}{n})}$ because $n^{(\frac{1}{n})} \to 1$ for $n \to \infty$. The series $\sum_{n=0}^{\infty} n\, a_n\, z^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n\, z^n$. So also has $\sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$ this radius of convergence.
Is the thus obtained function $g(z) = \sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$ really the derivative of the function $f(z)$?
Take z inside the circle of convergence of the two series. Choose $\rho > 0$ such that all points of the circle $|\zeta - z| \leq \rho$ lie inside the circle of convergence. Set $\zeta = z + h$, with so $|h| \leq \rho$, and $h \neq 0$, then has the next sense:

$$\frac{f(z + h) - f(z)}{h} - g(z) = \sum_{n=0}^{\infty} a_n\, (\frac{(z + h)^n - z^n}{h} - n\, z^{(n-1)}).$$

Now is:

$$|\frac{(z\,+\,h)^n\,-\,z^n}{h}\,-\,n\,z^{(n-1)}|\,=$$

$$|h|\,|\binom{n}{2}\,z^{(n-2)}\,+\,\binom{n}{3}\,h\,z^{(n-3)}\,+\,...\,+\,\binom{n}{n}\,h^{(n-2)}|$$

$$\leq\,|h|\,(\binom{n}{2}\,|z|^{(n-2)}\,+\,\binom{n}{3}\,\rho\,|z|^{(n-3)}\,+\,...\,+\,\binom{n}{n}\,\rho^{(n-2)})$$

$$\leq\,\frac{|h|}{\rho^2}\,(\binom{n}{2}\,\rho^2\,|z|^{(n-2)}\,+\,\binom{n}{3}\,\rho^3\,|z|^{(n-3)}\,+\,...\,+\,\binom{n}{n}\,\rho^{(n)})$$

$$\leq\,\frac{|h|}{\rho^2}\,(|z|^n\,+\,\binom{n}{1}\,\rho\,|z|^{(n-1)}\,+\,\binom{n}{2}\,\rho^2\,|z|^{(n-2)}\,+\,...\,+\,\binom{n}{n}\,\rho^{(n)})$$

$$=\,\frac{|h|}{\rho^2}\,(|z|\,+\,\rho)^n.$$

Therefore

$$|\frac{f(z\,+\,h)\,-\,f(z)}{h}\,-\,g(z)|\,\leq\,\frac{|h|}{\rho^2}\,\sum_{n=0}^{\infty}|a_n|\,(|z|\,+\,\rho)^n.$$

The series in the right-hand side converges because $|z|\,+\,\rho\,<\,R$. The series is independent of h. The right-hand side goes to zero for $h\,\rightarrow\,0$. So the limit of $\frac{f(z\,+\,h)\,-\,f(z)}{h}$ exists and is equal to $g(z)$. So f is differentiable and has $g(z)$ as its derivative. With this is the proof ready.

$\square$

Remark: This proof and many more to follow have all the same structure. Of a function f we suspect that $f' = g$. Take a point z. Write then

$$\frac{f(z\,+\,h)\,-\,f(z)}{h}\,-\,g(z) = h\,\phi(z,\,h).$$

Show that $|\phi(z,\,h)|$ is bounded for $|h|$ small enough, which means that the right-hand side $\rightarrow 0$ if $|h| \rightarrow 0$. Then is, with the definition of the derivative, proved that $f'(z) = g(z)$. Since z was randomly chosen, the proof is finished.

Remarks

So a power series defines a function which is holomorphic in the circular domain of convergence. For this special holomorphic function is the existence of all higher order derivatives clear. These derivatives are obtained by term wise differentiation;

all these series have the same radius of convergence. For powerseries is with this property* proved. We shall later see that a function, which is holomorphic in a surrounding of z_0 is the sum of a power series $\sum_{n=0}^{\infty} a_n\,(z - z_0)^n$ which converges in a circle around z_0.

Exponential function

Definition 5

$$\exp(z) = 1 + z + \frac{z^2}{2\,!} + ... = \sum_{n=0}^{\infty} \frac{z^n}{n\,!}.$$

The radius of convergence is infinitely large. So $\exp(z)$ is holomorphic in the whole finite z-plane.

Properties:

1. $(\exp(z))' = \exp(z)$.
2. additon theorem: $(\exp(z_1 + z_2))' = \exp(z_1)\,\exp(z_2)$ (follows from (1)).
3. $\exp(z)$ has no zeros (follows from (2)).

The usual notation is e^z in stead of $\exp(z)$.

Theorem 9

$$\lim_{n\to\infty}(1 + \frac{z}{n})^n = \lim_{n\to\infty}(\sum_{k=0}^{\infty}\binom{n}{k}\,(\frac{1}{n})^k) = \sum_{k=0}^{\infty}\frac{z^k}{k\,!} = e^z.$$

Proof

At k fixed, n increasing: $\binom{n}{k}\,(\frac{1}{n})^k \uparrow \frac{1}{k\,!}$. Take $|z| < R$ (arbitrary). Then K to determine so that $|\sum_{k=K}^{\infty}(\binom{n}{k}\,(\frac{z}{n})^k - \frac{z^k}{k\,!})| < 2\sum_{k=K}^{\infty}\frac{R^k}{k\,!} < \epsilon$. Subsequently

$$\sum_{k=0}^{(K-1)} \left(\binom{n}{k} \left(\frac{z}{n}\right)^k \rightarrow \sum_{k=0}^{(K-1)} \frac{z^k}{k!} \text{ as } n \rightarrow \infty. \right.$$

□

The trigoniometric functions are defined by

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n+1)}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n)}}{(2n)!}$$

They are everywhere analytic. The formulas apply:

$$\sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)), \quad \cos(z) = \frac{1}{2}(\exp(iz) + \exp(-iz)),$$

$$(\sin(z))' = \cos(z), \quad (\cos(z))' = -\sin(z),$$

$$\exp(iz) = \cos(z) + i\sin(z) \quad (\text{note: } |\exp(iz))| = \exp(-\text{Im}(z)) \neq 1 \text{ unless } z \text{ real}),$$

everywhere in the finite z-plane. This is completely in line with the elementary theory. We say: the exponential and the trigoniometric functions of a real variable we continued analytically in the complex plane. (See chapter 6.)

The connection with the hyperbolic functions.
For all z : $\cos(iz) = \cosh(z)$, $\sin(iz) = i\sinh(z)$, $(\cosh(z))^2 - (\sinh(z))^2 = 1$, etc.. See Wiskunde 10.

Exercise: Solve $\cosh(z) = 0$; $\sinh(z) = 0$.
Has $\cosh(z) = \sin(z)$ a solution?

Caution: For many values of z holds $|\cos(z)| > 1$ and $|\sin(z)| > 1$!
At once is $\cos(i) = \frac{1}{2}(\exp(-1) + \exp(1)) > 1$ because $e > 2$.
Out of

$$\sin(z) = \sin(x + iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$

$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

follows

$$|\sin(z)| = (\sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y))^{\left(\frac{1}{2}\right)} = (\sin^2(x) + \sinh^2(y))^{\left(\frac{1}{2}\right)}.$$

So $|\sin(z)| = 1$ if $\sinh(y) = \cos(x)$ or $\sinh(y) = -\cos(x)$.
This leads to two curves in the z-plane

$y_1 = \log(\sqrt{1 + \cos^2(x)} + \cos(x))$ and $y_2 = \log(\sqrt{1 + \cos^2(x)} - \cos(x)) = -y_1$.

On it is $|\sin(z)| = 1$; between $|\sin(z)| < 1$ and everywhere else $|\sin(z)| > 1$:

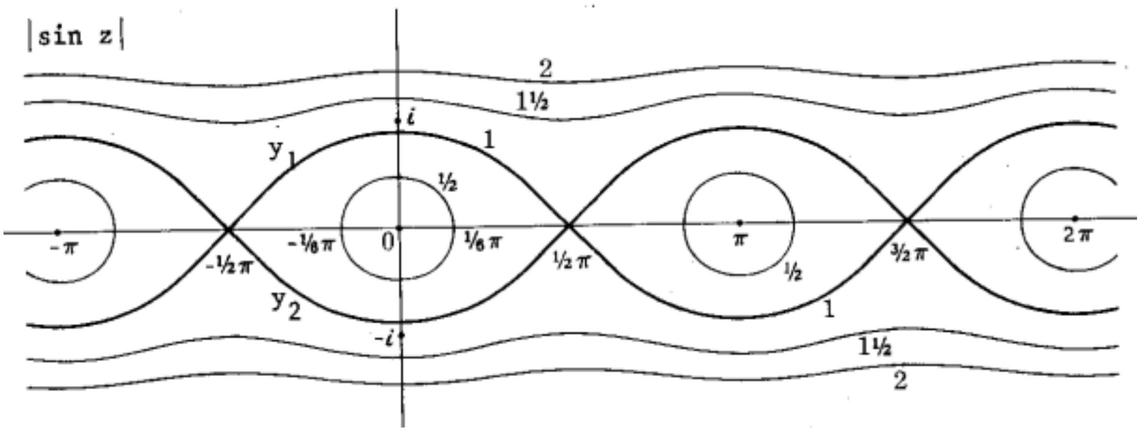

Figure 3

Analoguous for $|\cos(z)|$ (curves shifted over $\frac{\pi}{2}$):

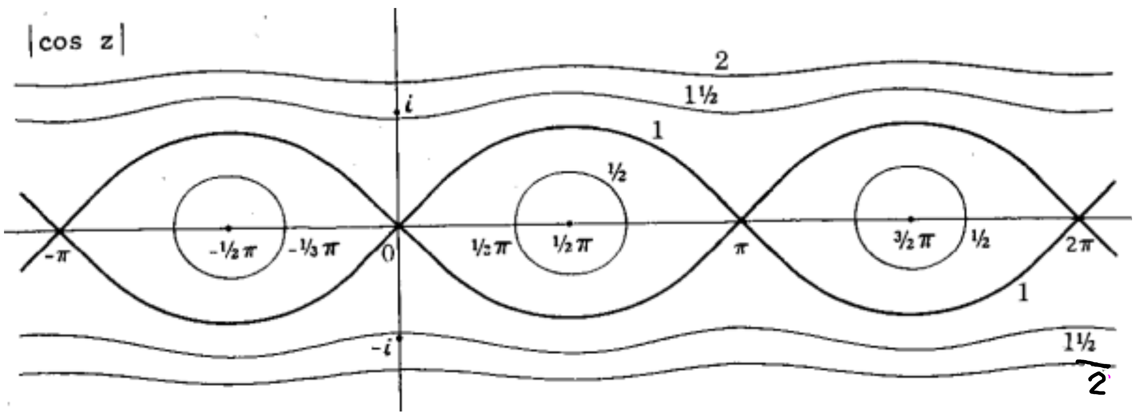

Figure 4

3 Integration in the complex plane

3.1 Complex integration

We start from the definition of a definite integral of a real function, which we assume to be known out of the lecture Wiskunde 10, even as the (then not proven) theorem:

If $g(x)$ continuous for $a \leq x \leq b$ then exists $\int_a^b g(x)\,dx$.

Let now K be a smooth arc with parameterisation $z = \phi(t) \quad (a \leq t \leq b)$. We remind to the fact that in section 1.3 is required that $\phi'(t)$ is continuous. Let further $f(z)$ be a continuous function of z at the arc K. Then we define

$$\boxed{\int\limits_K f(z)\,dz = \int\limits_a^b f(\phi(t))\,\phi'(t)\,dt.}$$

(Notice: the integrand is a complex-valued function of the real variable t.)

Because of the above-mentioned theorem out of Wiskunde 10 exists the integral at the righthand side (You see that it had a reason to demand that arcs are always continuous differentiable).
We define the integral of $f(z)$ over a curve K as the sum of the integrals over the arcs from which K is built. (Verify that above-mentioned definition is independent of the choice of the parametrisation.)

Some examples:

a. $f(z) = 1$, K is given by $\phi(t)$, $a \leq t \leq b$ with starting point $A = \phi(a)$, and end point $B = \phi(b)$.

$$\int\limits_K f(z)\,dz = \int\limits_a^b 1 \cdot \phi'(t)\,dt = \phi(b) - \phi(a) = B - A.$$

b. $f(z) = z$, K as in a).

$$\int\limits_K f(z)\,dz = \int\limits_a^b \phi(t) \cdot \phi'(t)\,dt = [\frac{1}{2}\phi^2(t)]_{t=a}^{t=b} = \frac{B^2 - A^2}{2}.$$

In a) and b) is the result of the integral dependent of the starting and end point of K but not of the chosen way between A and B. This is not always the case (and that will become one of the main points of this lecture).
Check this with the examples c) and d)

c. $f(z) = |z|$ is nowhere analytic. Let K be the curve consisting of the line segment $[0, 1]$ and the circular arc of the unit circle from 1 to i. Let K' be the line segment from 0 to i. Check that

$$\int_K f(z)\,dz = \frac{1}{2} + (i - i) \text{ and } \int_{K'} f(z)\,dz = \frac{i}{2}.$$

d. Choose $f(z) = \frac{1}{z}$, $A = 1$, $B = -1$ and as curves K resp. K' we take

$|z| = 1$, $\mathrm{Im}(z) \geq 0$ resp. $|z| = 1$, $\mathrm{Im}(z) \leq 0$.

Show that the integrals over these curves are different.

The following result is of great importance (memorize!):
Let K be some circle with center z_0 and radius a. For K we can choose the parametrisation $z = z_0 + a \exp(i\,t)$ $(0 \leq t \leq 2\,\pi)$. Let m be an integer. Then is

$$\frac{1}{2\,\pi} \int_K (z - z_0)^m\,dz = \begin{cases} 0 & \text{if} \quad \text{m} \neq \text{-1}, \\ 1 & \text{if} \quad \text{m} = \text{-1}. \end{cases}$$

Prove: apply the definition.

Remark: The result of the integral does not depend on the radius of the circle.
The known theorems about integrals of real functions are directly to convey to complex integrals. We mention only one example

$$\int_K \{\lambda\,f(z) + \mu\,g(z)\}\,dz = \lambda \int_K f(z)\,dz + \mu \int_K g(z)\,dz.$$

In the sequel it will be necessary to get a rough estimate of the result of the integral. Therefore we use

Theorem 10

If $|f(z)| \leq M$ for $z \in K$ and L the length of the curve K
then holds
(assumed that the integral exists)

$$\boxed{|\int_K f(z)\,dz| \leq M \cdot L.}$$

Proof

Let $z = \phi(t)$, $(a \leq t \leq b)$ a parametrisation of K. Then is

$$|\int_K f(z)\,dz| = |\int_a^b f(\phi(t))\,\phi'(t)\,dt| \leq$$

$$\int_a^b |f(\phi(t))\,\phi'(t)|\,dt \leq M \int_a^b |\phi'(t)|\,dt = M \cdot L$$

(compare section 1.3).

□

Remark: In this proof we use the inequality

$$|\int_a^b z(t)\,dt| \leq \int_a^b |z(t)|\,dt$$

where $z(t)$ nis a complex function of the real variable t (and $a \leq b$). This is simple to prove with the definition of integral (Wiskunde 10) and the inequality $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$.

An analog of which to the calculation of integrals of real functions was called the main theorem can also be defined for complex functions, namely

Theorem 11

Let G be a region and f continuous inside G. Let F be a function, defined in G, such that holds $F' = f$. If the curve K (starting point A, end point B) lies inside G then holds

$$\boxed{\int\limits_K f(z)\,dz = F(B) - F(A).}$$

Proof

$$\int\limits_K f(z)\,dz = \int\limits_a^b f(\phi(t))\,\phi'(t)\,dt = \int\limits_a^b F'(\phi(t))\,\phi'(t)\,dt =$$

$F(\phi(b)) - F(\phi(a)) = F(B) - F(A)$.

(So we use the chain rule for integrals of a complex function of a real variable).

$\square$

Examples

1. $f(z) = z^2$, $F(z) = \dfrac{z^3}{3}$ and $\int_K z^2\,dz = \dfrac{B^3 - A^3}{3}$ if A and B are the starting point resp. the end point of K.

2. $f(z) = z^m$, m an integer, $m \neq -1$. Then is $F(z) = \dfrac{z^{(m+1)}}{(m+1)}$ The functions $f(z)$ and $F(z)$ are for all z defined for non-negative exponents, otherwise only for $z \neq 0$. For a closed curve (eventual not through 0) is then $\int_K f(z)\,dz = 0$, such we above already proved with the help of the definition. That this is not the case for $f(z) = \dfrac{1}{z}$ becomes later on an important case!

In the last theorem depends $F(B) - F(A)$ not of the curve K. Every way of integration in G from A to B gives the same solution. Condition is that there exists a F with $F' = f$. G may be multiple connected (for instance $f(z) = \dfrac{1}{z^2}$, $F(z) = \dfrac{-1}{z}$).

Theorem 12

Let f be continuous in G and p some fixed point in G. If for every $z \in G$ and for every way in G from p to z the integral

$$\int_p^z f(\zeta)\,d\zeta$$

only depends on z (but not of the chosen way) then is $F(z) := \int_p^z f(\zeta)\,d\zeta$ in G a holomorphic function and $F'(z) = f(z)$.

Proof

F is uniquely defined. Let $a \in G$. Because f is continuous, there exists for every $\epsilon > 0$ a surrounding of a such that inside holds: $|f(z) - f(z)| < \epsilon$. Look to a way in G from p to a, lengthen with the line segment for a to z. Then holds:

$$F(z) - F(a) = \int_a^z f(\zeta)\,d\zeta = (z-a)\,f(a) + \int_a^z (f(\zeta) - f(a))\,d\zeta.$$

In here is the last term in absolute value $< \epsilon\,|z-a|$ if z lies densily enough to a. This means that F is differentiable in a with derivative $f(a)$. This holds for every $a \in G$ so the supposition is proved.

$\square$

Exercise: Let $f(z) = \sum_{n=0}^{\infty} a_n\,z^n$ a power series with domain of convergence $G : |z| < R$. Show that

$$\int_0^{z_0} f(\zeta)\,d\zeta \quad |z_0| < R$$

by two different ways of integration (inside G) give the same result, namely $\sum_{n=0}^{\infty} a_n \dfrac{z_0^{(n+1)}}{(n+1)} = F(z_0)$. (Hint: use theorem 11 on page 28).

3.2 Main theorem of complex integration

We have seen that in many cases the choice of the way of integration has no influence on the result of the integral, as long as the end points of the way of integration keep fixed. This holds for a great class of integrable functions and in particular for holomorphic functions.

Theorem 13

Let $f(z)$ be holomorphic in a region G; α and β starting and end point of a curve K which lies completely in G. Let K' be a curve with the same starting and end point, also completly lying in G, and continuously to transform to K without passing a point which does not belong to G. Then holds

$$\int\limits_K f(z)\,dz = \int\limits_{K'} f(z)\,dz.$$

The integral is not dependent of the choice of the connecting curves, but only of the end points α and β.

Theorem 14

Let $f(z)$ be holomorphic in a region G. Let K be a Jordan curve, of which the inner region, lies completely in G. Then exists $\int_K f(z)\,dz$ and is equal to zero.

The above mentioned theorems are equivalent formulations of the main theorem of complex integration. (Not to be confused with what in real integration is called the main theorem.)
Before we prove these theorems first a remark in the form of and
Exercise: Check that the theorem for power series is easy to prove along the lines of the last exercise of the previous section. (The remarks after the second theorem in section 2.3 make this assignment even more interesting!)

We now give the proof of the theorem in the second formulation and in three steps:

1. K is the boundary of a triangle,
2. K is the boundary of a polygon,
3. K is arbitrary.

Proof

(of theorem 14)

First step:
Consider in G a triangle D with boundary K and circumference L. We connect the midpoints of the sides. So arise 4 (congruent) triangles of which we call the boundaries C_1 upto C_4.

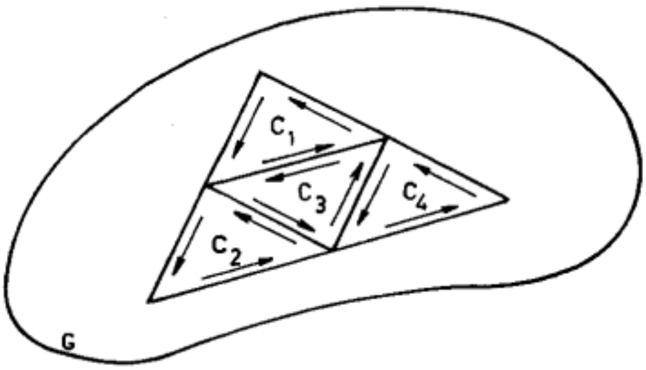

Figure 5

If we integrate over the four triangles in positive sense, the pieces at the inner side are run twice and in the opposite sense. So is

$$I := \int_K f(z)\, dz = \sum_{i=1}^{4} \int_{c_i} f(z)\, dz.$$

There then

$$|I| \leq \sum_{i=1}^{4} | \int_{c_i} f(z)\, dz|$$

gives that at least at on of the ways of integration $c_1, ..., C_4$

$$| \int_{c_i} f(z)\, dz| \geq \frac{1}{4}\, |I|.$$

This triangle we call D_1 (boundary $=$ K_1). Its circumference is $L_1 = \frac{1}{2}L$. Now D_1 is quartered, etc, etc. After n steps we have a triangle D_n with boundary K_n, circumference $L_n = 2^{-n}\,L$, for which

$$|\int_{K_n} f(z)\,dz| \;\geq\; 4^{-n}\,|I|.$$

There is exactly one point $a \in G$ which lies inside or on all triangles D_i. Let $g(z) := f(z) - f(a) - (z-a)\,f'(a)$. Because f is differentiable, there is for every $\epsilon > 0$ a surrounding of a such that:

$$|g(z)| \;<\; \epsilon\,|z - a|.$$

If n is great enough lies the boundary of K_n of the triangle D_n completely inside the surrounding of a. Further holds

$$\int_{K_n} f(z)\,dz = \int_{K_n} g(z)\,dz.$$

On K_n holds:

$$\max|z - a| \;\leq\; \frac{1}{2}\,L_n.$$

If we combine the founded inequalities, we find

$$|I| \;\leq\; 4^n\,|\int_{K_n} f(z)\,dz| = 4^n\,|\int_{K_n} g(z)\,dz| \;\leq\; 4^n\,\frac{1}{2}\,\epsilon\,(L_n)^2 = \frac{1}{2}\,\epsilon\,L^2.$$

Because this holds for every $\epsilon > 0$ is $I = 0$.

Second step:
Let now K be the boundary of a polygon without double points, with its inner region completely lying in G. The inner regeion of K we can built up out of a finite number of triangles witout overlap. We can orient all the boundaries in positive sense. (as that of K). Let K_ν be the oriented boundary of any triangle D_ν. Then holds

$$\int_K f(z)\,dz = \sum_\nu \int_{k_\nu} f(z)\,dz,$$

because the common sides of two neighbouring triangles is going through in opposite direction, and only the contributions of the boundary K remain. Every integral in the righthand side is zero, because of the first step. So the righthand side is zero, with which the second step is proved.

Third step:
An arbitrary curve we can approximate arbitrary close by a polygon draw. Lies that curve K and its inner region completely in region G, then we can make it on such a way that the polygon draw T and its inner region lies also completely in G. And we can set it up in such a way that

$$|\int\limits_K f(z)\,dz - \int\limits_T f(z)\,dz|$$

becomes arbitrary small. We ignore the real proofs of these assertions, but the result is quite plausible. According to step 2 is, if K is closed and so also T is closed, the integral along T is equal to zero. So the integral along K is arbitrary small. Then has to be $\int_K f(z)\,dz = 0$.

With this the fundamental theorem of integration is proved.

□

Theorem 15

Let $f(z)$ be holomorphic in a region G. Let K and K' be closed curves which are completely located in G and which can continuously be transformed in each other, without passing a point which does not belong to G, then is, provided that K and K' are run through in the same sense,

$$\int\limits_K f(z)\,dz = \int\limits_{K'} f(z)\,dz.$$

Proof

We use a tool that sometimes is called the "channel method". K and K' were connected to each other by line segments C_1 and C_2, such that the by K and K' formed annular region split into 2 pieces. According the main theorem is the integral of f on those pieces 0.

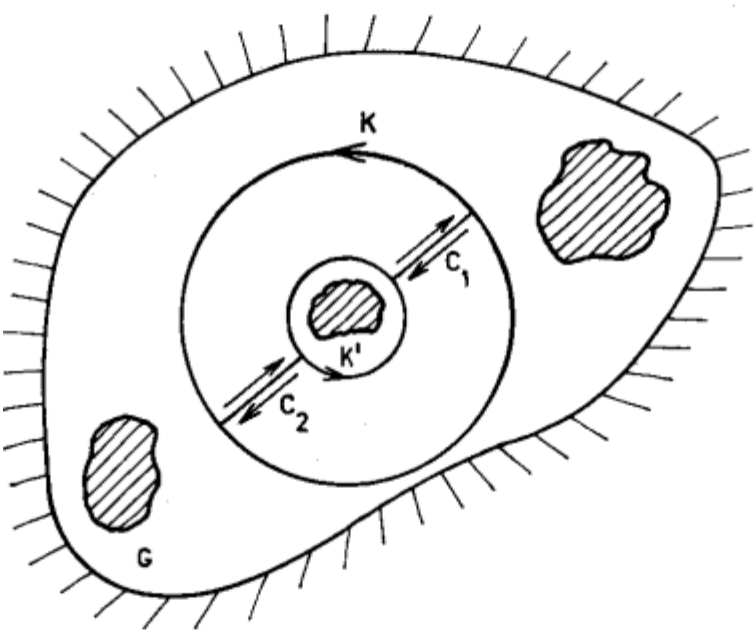

Figure 6

If we add those integrals, the integrals over the channels C_1 and C_2 fall away, because these two times and in opposite direction as part of the integration way appear. In this sum the integration way K' appears in negative sense.
From this follows the statement.

□

The main theorem is often used in the following form

Theorem 16

Let K be a simple closed curve (Jordan curve), $f(z)$ holomorphic on and inside K,
then holds $\int_K f(z)\,dz = 0$.

Remark: Holomorphic on K means: holomorphic in a full surrounding of K. So we can embed K into a region G consisting of the inner region of K, K itself, and a strip in the outer region of K. In G is $f(z)$ then holomorphic.

3.3 Residue; integral formula of Cauchy

Let $f(z)$ be holomorphic in a reduced surrounding of the point a. Then there is a circle C_a (radius ρ_a, midpoint a such that $f(z)$ is differentiable for every point

$z \neq a$ inside C_a. The set $0 < |z-a| < \rho_a$ is a region which is double connected. Consider two Jordan curves, K_1 and K_2, which have the point a in their inner region and lie completely inside C_a. These curves can be continously be transformed in each other without leaving G. Let both curves be positive oriented. then we know

$$\int_{K_1} f(z)\,dz = \int_{K_2} f(z)\,dz.$$

We consider now the expression

$$\frac{1}{2\,\pi\,i}\int_{K} f(z)\,dz,$$

where K is an arbitrary Jordan curve, which the point a encloses in positive sense and lies completely inside C_a. Then it is clear that this expression through a and f uniquely is determined and not depends on K. We call this expression the residue of f at the point a. Is $f(z)$ also in a itself holomorphic, then is the residue apparently zero.

Definition 6

A point where $f(z)$ is analytic, is called a regular point of $f(z)$. Otherwise z is called a singular point.

So there holds

Theorem 17

Is $f(z)$ in a reduced surrounding of a holomorphic, then has $f(z)$ a residue in a. This residue is zero if a is a regular point of f (a.o.!).

Theorem 18

> Is $f(z)$ on or inside a Jordan curve K holomorphic, with the exception of a finite number singular points a_1, a_2, ..., a_n which lie within K, then holds
>
> $$\frac{1}{2\,\pi\,i}\int\limits_K f(z)\,dz = \sum_{\nu=1}^{\nu=n}\{\text{Res.}\,f(z)\}_{z=a_\nu}.$$

Proof

With the "channel method".

With the help of the residue theorem we can calculate the integrals if we have a method to determine the residues. How do we do that?

Let $f(z)$ be holomorphic in a reduced surrounding of a, and there holds $f(z) = g(z) + h(z)$, where $h(z)$ is holomorphic in the point a. Then it is clear that the residue of f in a is equal to the residue of g in a, because the residue of h in a is zero.

In section 3.1 was found that $\frac{1}{2\,\pi\,i}\int_K \frac{dz}{(z-a)^m} = 0$ for m integer ($\geq$ 2), with K a circle around $z = a$. Now we know that this results holds for every arbitrary K which a surrounds. The function $f(z) = (z - z)^{(-m)}$, $m \geq 2$, is everywhere analytic except for $z = a$, where f is not defined. The point a is so a singular point of f. In a has f a residue. But the residue is zero. So there holds not: if the residue is zero we have a regular point.

The function $f(z) = (z - a)^{(-1)}$ has apperently in a a residue equal to 1. In order to get this standard was entered the factor $\frac{1}{2\,\pi\,i}$. Consider now

$$f(z) = \sum_{m=1}^{k}\frac{A_m}{(z-a)^m} + h(z),$$

with k a positive whole number, A_m independent of z, and $h(z)$ holomorphic in a; $f(z)$ is not defined for $z = a$. Well is $f(z)$ in a reduced surrounding of a holomorphic. The residue of f in a is the sum of the residues of the individual terms. So

$$\{\text{Res.}\,f(z)\}_{z=a} = A_1,$$

the coefficient of $(z - a)^{(-1)}$.

Definition 7

(see also section 4.4): If $f(z)$ is regular in a reduced surrounding of a and $\lim_{z \to a}(z - a)^k f(z)$ is finite and $\neq 0$, then we say that $f(z)$ in $z = a$ has a pole of the order k (k whole, > 0).

An example is the above mentioned function $f(z)$ if $A_k \neq 0$.

We can now of function calculate the residue in a pole, if we succeed to present the function in the above form. The residue is then A_1. See further the sections 4.4 and 4.5.

Examples

a. To determine $I = \int_{|z|=2} f(z)\, dz$ if $f(z) = \dfrac{z^2 + 1}{z^2\,(z + 1)}$.
   Partial fraction gives $f(z) = \dfrac{-1}{z} + \dfrac{1}{z^2} + \dfrac{2}{(z + 1)}$.
   We see that f has in (-1) has a pole of the first order. In the notation of above is

$$f(z) = \frac{2}{(z + 1)} + h(z)$$

   with $h(z) = \dfrac{-1}{z} + \dfrac{1}{z^2}$, a function which is holomorphic in a surrounding of (-1). The residu of f in (-1) is 2. In the same way we see that the residue of f in 0 is equal to (-1). In 0 has f a pole of order 2.

b. The same as a) but with $f(z) = \dfrac{z^2 + 1}{z^3\,(z + 1)}$.

c. Determine $\int_{|z|=1} \dfrac{\sin(z)}{z^4}\, dz$. We do this as follows:
   $\sin(z)$ is defined as $z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + ...)$, which series represents a holomorphic function in the whole plane. We may also write

$\sin(z) = z - \frac{z^3}{3!} + z^4\,(\frac{z}{5!} - \frac{z^3}{7!}\,...).$

If we call the holomorphic function between the brackets h we have

$$\frac{\sin(z)}{z^4} = \frac{1}{z^3} - \frac{1}{6\,z} + h(z)$$

where $h(z)$ is holomorphic in the whole plane. We see that the residue of $\frac{\sin(z)}{z^4}$ in $z = 0$ is equal to $\frac{-1}{6}$. De requested integral is $-\frac{1}{3}\,\pi\,i$.

Remark: If $f(z) = \frac{A}{(z - a} + h(z)$ $(A \neq 0)$ where h is analytic in a surrounding of a, then has f in a a pole of the first order with

$$\{\text{Res.}\,f(z)\}_{z=a} = A = \lim_{z \to a}(z - a)\,f(z).$$

This method to determine a residue, which is only possible to poles of the first order, is implied in the following general

Theorem 19

If $\phi(z)$ analytic in $z = a$, then holds

$$\boxed{\{\text{Res.}\,\frac{\phi(z)}{z - a}\}_{z=a} = \phi(a).}$$

Proof

Define f by

$$f(z) = \frac{\phi(z)}{z - a}.$$

Let K be a cricle around a with a small radius ρ (to be determined later). The function $f(z)$ is holomorphic in the surrounding of a, with the possible exception of a itself. So f has a residue in a:

$$\{\text{Res.}\,f(z)\}_{z=a} = \frac{1}{2\,\pi\,i}\int_K \frac{\phi(z)}{z - a}\,dz.$$

For this can be written

$$\{\text{Res.}\, f(z)\}_{z=a} = \frac{1}{2\pi i}\int_K \frac{\phi(a)}{z-a}\,dz + \frac{1}{2\pi i}\int_K \frac{\phi(z)-\phi(a)}{z-a}\,dz$$
$$= \phi(a) + \frac{1}{2\pi i}\int_K \frac{\phi(z)-\phi(a)}{z-a}\,dz.$$

Because $\phi(z)$ is holomorphic, it is continuous in a. So at $\epsilon > 0$ there can be found a $\delta > 0$ such that $|\phi(z) - \phi(a)| < \epsilon$ for all z with $|z - a| < \delta$. Choose $\rho < \delta$, then is

$$0 \leq |\text{Res.}\, f(z)\}_{z=a} - \phi(a)| = \frac{1}{2\pi}\left|\int_K \frac{\phi(z)-\phi(a)}{z-a}\,dz\right| \leq \frac{1}{2\pi}\frac{\epsilon}{\rho}\,2\pi\rho = \epsilon.$$

This holds for every $\epsilon > 0$ so

$|\text{Res.}\, f(z)\}_{z=a} - \phi(a)| = 0$, this means $\{\text{Res.}\, f(z)\}_{z=a} = \phi(a)$.

□

Remark: Instead of $\{\text{Res.}\, f(z)\}_{z=a}$ is often written $\text{Res}_a\, f(z)$.

Theorem 20

Integralformula of Cauchy

Is $\phi(z)$ on and inside the Jordan curve J holomorphic, then holds

$$\boxed{\frac{1}{2\pi i}\int_J \frac{\phi(z)}{(z-a)}\,dz = \begin{cases} \phi(a) & ,\text{a inside J} \\ 0 & ,\text{a outside J} \end{cases}}$$

Proof

Following the residue theorem is the integral equal to the sum of the residues in the singular points inside J. Lies a outside J, then there is no singular point inside J and then is the integral zero. Lies a inside J, the is the residue in a equal to $\phi(a)$.

□

Remark: We may not integrate through a singular point of the integrand. The integral has no sense if a is a point of J.

Remark: The above formula is "curious". The analytic function $\phi(z)$ is inside J fully determined by its values on J.

Theorem 21

Let K be an arbitrary curve; $\phi(z)$ continuous on K, a not on K. Then is

$$f(z) := \frac{1}{2\,\pi\,i}\int\limits_K \frac{\phi(z)}{(z-a)}\,dz$$

a holomorphic function of a. (Important note: K has not to be closed.)

Proof

Let

$$g(a) := \frac{1}{2\,\pi\,i}\int\limits_K \frac{\phi(z)}{(z-a)^2}\,dz.$$

This integral exists for a not on K; we expect dat $g(a) = f'(a)$, because the second integral can be obtained out of the first integral by formal differentiation under the integral sign.
Because of the fact that a lies not on K, we can specify a circle C with midpoint a such that K and C don't cut each other. Call ρ the radius of that circle. Let $|h| < \rho$. Then lies $a + h$ inside C. For $h \neq 0$ holds

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} - g(a) &= \frac{1}{2\,\pi\,i}\int\limits_K \phi(z)\,\{\frac{1}{h}\,[\frac{1}{z-a-h} - \frac{1}{z,\,a}] - \frac{1}{(z-a)^2}\}\,dz\\ &= \frac{1}{2\,\pi\,i}\int\limits_K \phi(z)\{\frac{h}{(z-a)^2\,(z-a-h)}\}\,dz.\end{aligned}$$

Therefore:
Absolute value left-hand side $\leq \frac{|h|}{2\,\pi}\mathrm{Max}_{z\in K}|\frac{\phi(z)}{(z-a)^2\,(z-a-h)}|\cdot$ (length of K).
Now has C a positive distance d to K. So $|z-a| > d$, $|z-a-h| > d$. $\phi(z)$ is continuous, so bounded on K : $|\phi(z)| \leq p$, with p some constant. Therefore

$\text{Max}_{z\in K}|\dfrac{\phi(z)}{(z-a)^2\,(z-a-h)}| < \dfrac{p}{d^3} =$ constant independent of h.

The absolute value of the "left-hand side" is smaller than a suitable constant times $|h|$, where the constant doesn't dependent on h. Therefore: $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists and is equal to $g(a)$. This means: $f(a)$ is differentiable in a and in addition

$$f'(a) = \frac{1}{2\,\pi\, i}\int\limits_K \frac{\phi(z)}{(z-a)^2}\,dz.$$

The proof holds for an arbitrary a which lies not on K. So $f(a)$ is a holomorphic function of a in every region that has no points in common with K.

$\square$

Remark: If K is a curve without double points, then is the region: the whole z-plane minus K. Is K a Jordan curve, then is in both regions determined by K (inner and outer region) $f(a)$ a holomorphic function.

The previous statement is just a special case of a more general statement about differentiation under the sign of the integral.

Theorem 22

Let K be a curve, G a region. Let $\phi(z,a)$, $z\in K$, $a\in G$, be bounded ($|\phi(z,a)|\le M$), in addition holomorphic in G (as function of a with fixed z), and just as $\dfrac{\partial\phi(z,a)}{\partial a}$, continuous on K (as function of z with fixed a). Then holds

$$f(a) := \int\limits_K \phi(z,a)\,dz \,\text{is holomorphic in}\, G,$$

with derivative

$$f'(a) = \int\limits_K \frac{\partial\phi(z,a)}{\partial a}\,dz.$$

Proof

Take a $z \in K$. Because ϕ is a holomorphic function of a in G holds (because of the integral formula of Cauchy and the previous theorem):

$$\phi(z, a) = \frac{1}{2\,\pi\,i} \int\limits_J \frac{\phi(z, \alpha)}{(\alpha - a)}\, d\alpha \quad \text{and} \quad \frac{\partial \phi}{\partial a} = \frac{1}{2\,\pi\,i} \int\limits_J \frac{\phi(z, \alpha)}{(\alpha - a)^2}\, d\alpha,$$

where J is a circle around a, which lies in G. If h is so small that also $a + h$ lies inside J holds:

$$\frac{\phi(z, a + h) - \phi(z, a)}{h} - \frac{\partial \phi}{\partial a} = \frac{1}{2\,\pi\,i} \int\limits_J \frac{h\,\phi(z, \alpha)}{(\alpha - a)^2\,(\alpha - a - h)}\, d\alpha.$$

With $|\phi(z, a)| \leq M$ for $z \in K$, $a \in G$, ρ the radius of J and $|h| < \frac{1}{2}\rho$, one finds

$$|\int\limits_J \frac{h\,\phi(z, \alpha)}{(\alpha - a)^2\,(\alpha - a - h)}\, d\alpha| \leq \frac{|h|\,M}{\rho^2 \cdot \frac{1}{2}\rho}\, 2\,\pi\,\rho = |h| \frac{4\,\pi\,M}{\rho^2}.$$

This holds for all $z \in K$. By integration over the curve K, with length L, we find

$$|\frac{f(a + h) - f(a)}{h} - \int\limits_K \frac{\partial \phi}{\partial a}\, dz\,| \leq |h| \frac{2\,M\,L}{\rho^2}.$$

If we in this let go $h \to 0$, then the stated follows by the definition of derivative.

□

Applications

1. $\phi(z,\, a) = \exp\,(a\,z)$ is bounded if we take for K the line segment from $-i$ to $+i$ and for G the circle $|a| < R$. This gives

   $$f(a) := \int\limits_{-i}^{i} \exp\,(a\,z)\, dz,\ |a| < R.$$

   Because R was arbitrary, f is everywhere holomorphic! We already knew that. Because $f(a) = 2\,i\, \frac{\sin\,(a)}{a}$.

2. With the help of integrals there can be defined new functions. Example:

$$f(a) := \int\limits_0^1 \cos\left(a \sin()\pi\, z\right) dz,$$

a Bessel-function (namely $J_0(a)$, see Whittaker and Watson, CH. XVII).

3. If ϕ is on the curve K and G a region such that all $a \in G$ have a distance $> \delta > 0$ to K then is $\phi(z, a) = \dfrac{\phi(z)}{(z - a)}$ bounded. Application of the theorem to this case gives the theorem of page 40. Now we can continue differentiating:

$$\boxed{f^{(n)}(a) = \frac{n!}{2\,\pi\, i} \int\limits_K \frac{\phi(z)}{(z - a)^{(n+1)}}\, dz}$$

if

$$f(a) = \frac{1}{2\,\pi\, i} \int\limits_K \frac{\phi(z)}{(z - a)}\, dz.$$

In particular follows from this the first mentioned:

Theorem 23

(property*)[3].

> If f is holomorphic in G then also f'.

Proof

Let z be point of G and J a Jordancurve (positive oriented) around z, with its inner region completely in G. That gives the integral formula of Cauchy

$$f(z) = \frac{1}{2\,\pi\, i} \int\limits_J \frac{f(\zeta)}{(\zeta - z)}\, d\zeta.$$

We have just seen that the right-hand side can be differentiated arbitry often:

$$f^{(n)}(z) = \frac{n!}{2\,\pi\, i} \int\limits_J \frac{f(\zeta)}{(\zeta - z)^{(n+1)}}\, dz.$$

In particular has f' a derivative in z. This holds for every $z \in G$, which means: f' is holomorphic in G.

Another pleasant consequence of our propositions is the following

Theorem 24

Let ϕ be holomorphic in a reduced surrounding of a and let $\lim_{z\to a}\phi(z) = l$. If we define $\phi(a) = l$ then is ϕ also in a analytic.

Proof

Let K be the circle $\{\zeta \mid |\zeta - a| = \rho\}$, ρ so small that $|\phi(z) - l| < 1$. Inside K is $\int_K \dfrac{\phi(\zeta)}{(\zeta - z)}\, d\zeta \;=:\; \phi^*(z)$ holomorphic. For $z \neq a$ holds (channel method, integralformule) $\phi^*(z) = \phi(z) + \mathrm{Res}_a \dfrac{\phi(\zeta)}{(\zeta - z)}$. This residue is 0, because there is $|\mathrm{Res}_a| < \dfrac{\sigma\,(1 + |l|)}{(|z-a| - \sigma)}$ if $\sigma < |z - a|$. So $\phi(z) = \phi^*(z)$ for all $z \neq a$ and (let $z \to a$) $l = \phi^*(a)$. So with $\phi(a) = l$ becomes ϕ analytic in a.

□

More or less a reversal of the main theorem

Theorem 25

Theorem of Morera

Is f continuous in a region G and
is for every closed path in G the integral of f over this path zero,
then is f holomorphic.

Proof

Let p some fixed point in G. Define

$$F(z) := \int_p^z f(\zeta)\, d\zeta$$

where de path runs through G. The definition is unambigious because the integral depends not of the path. At page 29 is proved that F is holomorphic and $F'(z) = f(z)$. Then is f also holomorphic.

□

We will apply this theorem to the series in a beautiful way.

Exercise:

Inequality of Cauchy: Let f be on and inside the circle $C := \{z \mid |z - a| = r\}$ analytic and $|f(z)| \leq M$ for z on C. Then holds

$$\boxed{|f^{(n)}(a)| \leq \frac{M \cdot (n!)}{r^n}.}$$

Exercise: Let $\phi(z)$ be holomorphic in a surrounding of a curve K and

$$f(a) := \frac{1}{2\,\pi\, i} \int_K \frac{\phi(z)}{(z - a)}\, dz,\ a \text{ not on } K.$$

Take b (not end point) on K, b_1 and b_2 close to b on either side of (left resp. right) K; then holds ("jump of f at b"):

$$\lim_{b_1 \to b} f(b_1) - \lim_{b_2 \to b} f(b_2) = \phi(b).$$

(Draw a circle C around b, with the arc C_1 around b_1, with the arc C_2 around b_2; check that

$$f(b_1) := \frac{1}{2\,\pi\, i} \int_{K_2} \frac{\phi(z)}{(z - b_1)}\, dz,\ \ f(b_2) := \frac{1}{2\,\pi\, i} \int_{K_1} \frac{\phi(z)}{(z - b_2)}\, dz,$$

where K_2 and K_1 propose diversions of K through C_2 and C_1;
let now go b_1 and $b_2 \to b$.)
Check result for the case $K =$ Jordancurve!

3.4 Construction of analytic functions by series; theorem of Weierstrass

The concept of uniform convergence is known out of the lectures of Wiskunde 30.

We give here an analogous definition for complex functions. A series of functions $\sum_{n=1}^{\infty} f_n(z)$, defined at some region G we call convergent with sum $F(z)$ if for every $\epsilon > 0$ there can be found some N_0 sucht that

$$|F(z) - \sum_{n=1}^{N} f_n(z)| < \epsilon \text{ if } N > N_0.$$

This have to be possible for every $z \in G$. In general the N_0 will not only depend on ϵ, but also on z. If we however can find to every $\epsilon > 0$ a N_0 such that for every $z \in G$ and $N > N_0$ the above given inequality is satisfied then the series is called uniform convergent on G. (That means that N_0 only depends on ϵ.)

Note: Uniform convergent means much more then only
"convergent for every z". Example: inside the unit circle is $\sum_{n=1}^{\infty} z^n$ convergent for every z, but not uniform convergent. The series is uniform convergent on $\{z \mid |z| < \rho < 1\}$, even if ρ lies close to 1.

Theorem 26

Characteristic of Weierstrass

(compare Wiskude 30, chapter 3).

If for all z in a region G holds $|f_n(z)| < a_n$ and the majorant $\sum_{n=1}^{\infty} a_n$ converges, then is $\sum_{n=1}^{\infty} f_n(z)$ uniform convergent on G.

Theorem 27

(proof as in Wiskunde 30).

The sum of an uniform convergent series of continuous functions is a continuous function.

Theorem 28

An uniform convergent series of continuous functions may term wise be integrated about every every curve lying in the region of uniform convergence, which means

$$\int_K \{\sum_{n=1}^{\infty} f_n(z)\}\,dz = \sum_{n=1}^{\infty}\{\int_K f_n(z)\,dz\}.$$

Proof

Let $F(z)$ be the sum of the series and L the length of the curve K. To every $\epsilon >)$ there is a N_0 such that for $N > N_0$ and all z on K holds

$$|F(z) - \sum_{n=1}^{N} f_n(z)| < \epsilon.$$

Then according to the known estimates for integrals:

$$|\int_K F(z)\,dz - \sum_{n=1}^{N}\{\int_K f_n(z)\,dz\}| = |\int_K \{F(z)\,dz - \sum_{n=1}^{N} f_n(z)\}\,dz\}| \leq \epsilon \cdot L.$$

With this the stated is proven.

□

Theorem 29

Theorem of Weierstrass

> The sum of an in a region uniform convergent series of holomorphic functions is holomorphic.

Proof

Let G be the region of uniform convergence, G' an arbitrary simple connected subregion of G and K an arbitrary curve in G'. Then is following the main theorem $\int_K f_n(z)\,dz = 0$ for every n, so according the previous theorem $\int_K F(z)\,dz = 0$. We have already proven that $F(z)$ is continuous and can now conclude that $F(z)$ is holomorphic based on the theorem of Morera.

□

A special case of the theorem of Weierstrass: a power series in a closed and bounded set inside the circle of convergence.

Remark: Check with the example $\sum_{n=1}^{\infty} \dfrac{z^n}{n\,(n\,+\,1)}$ that out of the uniform convergence for $|\,z\,| \,\leq\, 1$ (no region!) not follows that the sum also for $|z| \,\leq\, 1$ is analytic function. (In $z = 1$ is the sumfunction not differentiable, so certainly not analytic!)

In the treatment of real functions has been warned not to make the known mistake: to differentiate an uniform convergent series term by term.

We see from the previous example: the series $\sum_{n=1}^{\infty} \dfrac{x^n}{n\,(n\,+\,1)}$ is for $-1 \,\leq\, x \,\leq\, 1$ uniform convergent but the sum is in $x = 1$ not even differentiable. For complex series, uniform convergent on a region, is termwise differentiation allowed:

Theorem 30

Is at the region G the series $\sum_{n=1}^{\infty} f_n(z)$ uniform convergent with sum $F(z)$ and are all $f_n(z)$ holomorphic, then is

$$F^{(p)}(z) = \sum_{n=1}^{\infty} f_n^{(p)}(z) \quad (p = 1,\, 2\, ...).$$

Proof

In the previous theorem is proved that $F(z)$ is holomorphic and so exists the derivative $F^{(p)}(z)$. We can write for it:

$$\frac{p!}{2\,\pi\,i} \int\limits_C \frac{F(\zeta)}{(\zeta\,-\,z)^{(p+1)}}\, d\zeta = \frac{p!}{2\,\pi\,i} \int\limits_C \sum_{n=1}^{\infty} \frac{f_n(\zeta)}{(\zeta\,-\,z)^{(p+1)}}\, d\zeta = \sum_{n=1}^{\infty} \frac{p!}{2\,\pi\,i} \int\limits_C \frac{f_n(\zeta)}{(\zeta\,-\,z)^{(p+1)}}\, d\zeta,$$

where C is a circle in G: $C = \{\,\zeta\,|\,|\zeta\,-\,z| = \rho\}$. The series, which is integrated, is just as $\sum f_n(z)$ uniform convergent on C, because on C the denominator has a constant value. So we may termwise integrate with which the stated is proven.

$\square$

Exercise: (Besselfunction)

$$\int\limits_0^1 \cos\,(a\,\sin\,(n\,z))\,dz = \sum_{n=0}^{\infty} \frac{(-1)^n\,a^{(2\,n)}}{((2\,n)!!)^2}.$$

Exercise: Show that (compare with section 3.3 where K is finite) that

$$f(a) = \int\limits_0^{\infty} \phi(x,\,a)\,dx = \sum_{n=1}^{\infty} \int\limits_{(n-1)}^{n} \phi(\nu, a)\,d\nu = \sum_{n=1}^{\infty} f_n(a)$$

is a holomorphic function of a in region G, if for every $a \in G$ $\phi(\nu, a)$ and $\dfrac{\partial\phi}{\partial a}$ are continuous on $(n-1) \le \nu \le n$ and

$|\phi(\nu, a)| \le u_n$ = term of convergent sequence.

Exercise: Check based on the foregoing exercise that

1.

$$\int_0^\infty \exp(-a\,x)\,dx = a^{-1} \;\; (\mathrm{Re}(a) > 0) \text{ gives } \int_0^\infty x^n \, \exp(-a\,x)\,dx = n!\,a^{(-n-1)}.$$

2. Check that for $a \in \mathbb{C} \setminus \{z \,|\, z = \mathrm{Re}(z) \leq 0\}$

$$f(a) := \int_0^\infty \frac{dx}{(x^2 + a^2)} = \int_0^\infty \frac{x^2\,dx}{(x^2 + a^2)^2} - a\,f'(a).$$

Deduce from this that: $\frac{1}{2}\,f(a) = -a\,f'(a)$.

3. Show that for all a with $\mathrm{Re}(a) > 0$ holds

$$\int_0^\infty \exp(-a\,x)\,\frac{\sin(x)}{x}\,dx = \frac{\pi}{2} - \int_0^a \frac{d\,\zeta}{(1 + \zeta^2)}.$$

(Differentiate; let $a \to \infty$; the relationship remains valid for $a = 0$, see chapter 5.

4 Holomorphic functions; Series

4.1 Theorem of Taylor

The theorem of Taylor, proven in the lecture of Wiskunde 10 for real functions, we can now also formulate for complex functions:

Theorem 31

Is $f(z)$ holomorphic in the region G and a an arbitrary point of G, then has the power series

$$\sum_{n=0}^{\infty} c_n\,(z-a)^n \quad \text{with} \quad c_n = \frac{1}{n!}\,f^{(n)}(a)$$

a radius of convergence, which has at least the same size as the distance from a to the boundary of G and the sum is $f(z)$ for all $z \in G$ inside the circle of convergence.

Proof

Let r be the distance of a to the boundary of G; choose two numbers ρ and ρ_1 with $0 < \rho_1 < \rho < r$. Then lies the circle C with radius ρ and center a with its inner region completely inside G. Let z satisfy the condition $|z-a| \le \rho_1$. Then holds following the integralformula of Cauchy (section 3.3 , theorem 20)

$$f(z) = \frac{1}{2\,\pi\,i}\int\limits_C \frac{f(w)}{(w-z)}\,dw.$$

Now is $\dfrac{1}{(w-z)} = \dfrac{1}{(w-a-(z-a))} = \dfrac{1}{(w-a)}\,\dfrac{1}{(1-\dfrac{(z-a)}{(w-a)})}$. Further

$|\dfrac{z-a}{w-a}| = \dfrac{1}{\rho}\,|z-a| \le \dfrac{\rho_1}{\rho} < 1,$

and so $\dfrac{1}{(1-(\dfrac{z-a}{w-a}))} = \sum_{n=0}^{\infty}(\dfrac{z-a}{w-a})^n.$

This series converges uniform in w on C because she has the from w independent majorante $\sum_{n=0}^{\infty}(\dfrac{\rho_1}{\rho})^n$. Because $f(w)$ is bounded on C, converges the series

$\dfrac{f(w)}{(w\,-\,z)} = \sum_{n=0}^{\infty} (z\,-\,a)^n\, \dfrac{f(w)}{(w\,-\,a)^{(n+1)}}$ uniform in w on C. This series can be termwise be integrated, and we find

$$f(a) = \sum_{n=0}^{\infty} c_n\,(z\,-\,a)^n, \quad \text{met} \quad c_n = \frac{1}{2\,\pi\,i}\int\limits_C \frac{f(w)}{(w\,-\,a)^{(n+1)}}\,dw.$$

From the past we know that $c_n = \dfrac{f^{(n)}(a)}{n!}$. This is the proof.

$\square$

Exercise: If $f(z)$ and $g(z)$ have convergent power series $\sum_{n=0}^{\infty} a_n\, z^n, \sum_{n=0}^{\infty} b_n\, z^n$ in $|z|\,<\,C$, then evenso $h(z) = f(z)\cdot g(z)$: $\sum_{n=0}^{\infty} c_n\, z^n$ with $c_n = \sum_{k=0}^{n} a_k\, b_{(n-k)}$. (Hint: $\int_J \frac{h(z)}{z^{(n+1)}}\,dz = \int_J \sum_{k=0}^{\infty} a_k\, \frac{g(z)}{z^{(n-k+1)}}\,dz = \sum \int$; the sum cuts off after $k = n$; one can also use the rule of Leibniz for $(f\cdot g)^{(n)}$.

4.2 Behaviour near a regular point

Let f holomorphic in a surrounding of a, so according to Taylor $f(z) = \sum_{n=0}^{\infty} c_n\,(z\,-\,a)^n$ in a surrounding of a. If all the numbers $c_1, c_2, \ldots$ are zero, then is f constant. If not, let c_k be the first one which is not zero. Then we can write f in a surrounding of a as

$$f(z) = c_0\,+\,c_k\,(z\,-\,a)^k\{1\,+\,d_1\,(z\,-\,a)\,+\,d_2\,(z\,-\,a)^2\,+\,\ldots\},$$

where the power series between the braces presents a holomorphic function which in $z = 1$ has the value 1 and therefore in a surrounding of a is not equal to zero. We distinguish two cases:

1. $c_0 = f(a) = 0$ We call a a k-multiple zero of f or also a zero with multiplicity k.

2. $c_0 \neq 0$. Write now c_k in the form $c_k = c_0\cdot r\cdot \exp{(i\,\phi)}$ (r and ϕ real with $r\,>\,0$). Then is

   $$f(z) = c_0\,[1\,+\,r\,\exp{(i\,\phi)}\,(z\,-\,a)^k\{1\,+\,d_1\,(z\,-\,a)\,+\,\ldots\}].$$

   If $z = a\,+\,\rho\,\exp{(-i\,\frac{\phi}{k})}$ and ρ(real) small enough then is

   $$1\,+\,r\,\exp{(i\,\phi)}\,(z\,-\,a)^k\{1\,+\,d_1\,(z\,-\,a)\,+\,\ldots\} = 1\,+\,r\,\rho^k\,\{1\,+\,\delta\}$$

with $|\,\delta\,|$ small. If δ is so small that $\mathrm{Re}(1\,+\,\delta)\,>\,0$, then is $|1\,+\,r\,\rho^k\,(\,1\,+\,\delta\,)\,|\,>\,1$ this means that $|\,f(z)\,|\,>\,|\,c_0\,| = |\,f(a)\,|$, so $|\,f\,|$ takes no maximum in a!

In both cases (1) and (2) we see that here is a surrounding of a where $f(z) \neq 0$ eventual with the exception of a itself (case (1)). With this we have proved two theorems.

Theorem 32

If f is holomorphic in a surrounding of a then is a not a limit point of zeros of f, unless f is equal to 0.

Theorem 33

If f is holomorphic in a surrounding of a then takes $|f|$ not a maximum in the point a, unless f is constant.

Let's consider the last statement first:

Let's D be a bounded domain (region + boundary) located inside the region G where f is holomorphic. The function $|f|$ is on D continuous and so it takes a maximum on D. Based on the above statement we can now conclude that this maximum has te be taken at the boundary of D. Analoguous considerations are valid for the minimum of $|f|$. Is f in a surrounding of a holomorphic and not constant then is $|f|$ minimal in a if $f(a) = 0$ (trivial) and otherwise not.
After all if $f(a) \neq 0$ then there is a surrounding of a where $f(z) \neq 0$ and in that surrounding is $\frac{1}{f}$ holomorphic and $|\frac{1}{f}|$ takes in a not a maximum.

Exercise: The functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ can not take their maximal values other than on the boundary of domain D. This comes in handy in potential theory.

We come now to a very important consequence of the first theorem.

Theorem 34

Identity Theorem

Let f and g be defined and holomorphic in region G. Let z_0 be a point of G. If in every reduced surrounding of z_0 lies a point where $f(z) = g(z)$ then is $f(z) = g(z)$ everywhere in G.

Proof

First step:
The function $(f - g)$ is holomorphic in z_0 and out of what is given follows that z_0 is a limitpoint of zeros of this function. Then should be $(f - g) = 0$ in a surrounding of z_0.
Second step:
If there are points in G where $(f - g)(z) \neq 0$ then there would be a point $p \in G$ such that in every reduced surrounding of p there are points with $(f - g)(z) = 0$ and also points with $(f - g)(z) \neq 0$. We have already seen that this is impossible, which means that $f(z) = g(z)$ everywhere in G.

□

Entire functions

Definition 8

A function $f(z)$ is called entire if $f(z)$ is holomorphic for every z.

Theorem 35

If $f(z)$ is entire, a arbitrary, then is $f(z)$ to expand in a power series around $z = a$ with radius of convergence ∞.

Proof

See above. Note that the property of this theorem can also be taken as the definition of entire.

□

Definition 9

A function whose power series breaks off is called an entire rational function (polynomial). Is this not the case then the functions is called transcedent.

Examples of entire (transcedent) functions are the exponential and trigoniometric functions (2.3).

Theorem 36

Theorem of Liouville

A bounded entire function is necessarily constant.

Proof

Let $f(z)$ be an entire function. Then has $f(z)$ a power series development around the origin, of which the radius of convergence is equal to infinity

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n, \quad \text{with} \quad a_n = \frac{1}{2\,\pi\, i} \int\limits_C \frac{f(w)}{w^{(n+1)}}\, dw.$$

Hereby is C an arbitrary positive oriented Jordan curve around the origin. We can take for C a circle with radius R. Out of the integral representation follows (see also the inequality of Cauchy)

$$|a_n| \leq \frac{M}{R^n} \quad (n = 0,\, 2,\, \ldots)$$

if M is the maximum of $|f(z)|$ for z on C. If $f(z)$ is bounded, then $|f(z)| \leq M'$ (all z)then is $M \leq M'$ and so $|a_n| \leq \frac{M'}{R^n}$, where M' does not depend anymore on R. The coefficients a_n are independent of R. Their upper boundary $\frac{M'}{R^n}$ goes to zero for $R \to \infty$, unless $n = 0$. So $a_n = 0$ except for $n = 0$. So $f(z)$ is indeed constant.

□

We can formulate a more general theorem of this type:

Theorem 37

An entire function f for which holds

$|f(z)| \leq p + q\,|z|^k$

for all z is a polynomial of order $\leq k$.

Proof

Following the inequality of Cauchy holds to every $r > 0$

$$|f^{(n)}(0)| \leq \frac{p + q\,r^k}{r^n}.$$

For $n > k$ approaches the right-hand side to 0 if $r \to \infty$. So $f^{(n)}(0) = 0$ for $n > k$.

□

Remark:
To every polynomial $f(z) = \sum_{i=0}^{k} a_i\, z^i$ can be find constants p and q such that for every z

$|f(z)| \leq p + q\,|z|^k.$

Take $q > |a_k|$, then holds

$$|f(z)| - q\,|z|^k \leq \sum_{i=0}^{(k-1)} |a_i|\,|z|^i - (q - |a_k|)\,|z|^k.$$

Take now for p the maximum of the right-hand side.

Theorem 38

Main Theorem of Algebra

> A n-th degree equation has at least one root if $n \geq 1$.

Proof

Consider the equation

$$a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0, \quad a_n \neq 0, n \geq 1.$$

Call the polynomial at the left-hand side $f(z)$.
Assume that $f(z)$ has no zeros, then should $\frac{1}{f(z)}$ be an entire function (and it would be a whole transcedent function). But

$$f(z) = a_n z^n (1 + \frac{d_1}{z} + ... + \frac{d_n}{z^n}), \quad \text{with} \quad d_k = \frac{a_{(n-k)}}{a_n}.$$

For great values of $|z|$ approaches the expression between the brackets to one. So $|\frac{1}{f(z)}|$ approaches to zero for $|z| \to \infty$ (see page 60). Then is $\frac{1}{f(z)}$ in the neighbourhood of $z = \infty$ bounded. So $\frac{1}{f(z)}$ is everywhere bounded. With Liouville follows that $\frac{1}{f(z)}$ is constant and that is impossible. So $f(z)$ does have a zero.

□

Remark:
A whole transcedental function need not to have zeros; example: $\exp(z)$.

4.3 Laurent series

We know that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all z, so $\exp(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{z^{(-n)}}{n!}$. Analogu-ous:

$$\sum_{n=0}^{\infty} z^{(-n)} = \frac{z}{z-1} \quad \text{for} \quad |z| > 1 \quad \text{and further}$$

$$\frac{\sin(z)}{z^6} = z^{(-5)} - \frac{z^{(-3)}}{3!} + \frac{z^{(-1)}}{5!} + \sum_{k=1}^{\infty}(-1)^k \frac{z^{(2k-1)}}{(2k+5)!}.$$

This are three examples of series which look like power series but also contain negative powers. This are generalisations of power series which we will call Laurent series. We prove an expansion theorem which is an extension of the theorem of Taylor.
Let G be the region between two concentric circles c and C (the annular region G is a doubly connected domain) with radii r and R. Let $f(z)$ be defined and holomorphic in G. Consruct two other concentric circles γ and Γ inside G, and take z inside the ring between γ and Γ. The radii of γ and Γ are ρ and P.
The common center point of all the d=circles is a. Then applies by appointment:

$0 < r < \rho < |z - a| < \mathrm{P} < R.$

The integrals

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{(w - z)}\, dw \quad \text{and} \quad \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w - z)}\, dw$$

both have meaning. Allcircles are positive oriented. With the channel method and the integral formula of Cauchy we find

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w - z)}\, dw - \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{(w - z)}\, dw.$$

Now is

$$\frac{1}{(w - z)} = \frac{1}{(w - a - (z - a))} = \frac{1}{(w - a)}\,\frac{1}{(1 - \dfrac{(z - a)}{(w - a)})} = \frac{-1}{(z - a)}\,\frac{1}{(1 - \dfrac{(w - a)}{(z - a)})}.$$

Lies w on γ then holds that $|\frac{(w - a)}{(z - a)}| < 1$; lies w on Γ then holds that $|\frac{(z - a)}{(w - a)}| < 1$. So if:

$$w \quad \text{on} \quad \gamma \quad \text{then} \quad \frac{1}{(w - z)} = -\sum_{n=0}^{\infty} \frac{(w - a)^n}{(z - a)^{(n+1)}},$$

and this series is uniform convergent in w on γ,

$$w \quad \text{on} \quad \Gamma \quad \text{then} \quad \frac{1}{(w - z)} = -\sum_{n=0}^{\infty} \frac{(z - a)^n}{(w - a)^{(n+1)}},$$

and this series is uniform convergent in w on Γ.
The function $f(w)$ is bounded on γ en also on Γ, so $|f(w)| \leq M$. So

$$w \quad \text{on} \quad \gamma: \quad \frac{f(w)}{(w-z)} = -\sum_{n=0}^{\infty}(z-a)^{(-n-1)}\, f(w)\,(w-a)^n,$$

and this series is uniform convergent in w on γ,

$$w \quad \text{on} \quad \Gamma: \quad \frac{f(w)}{(w-z)} = -\sum_{n=0}^{\infty}(z-a)^{(n)}\, f(w)\,(w-a)^{(-n-1)},$$

and this series is uniform convergent in w on Γ.
Uniform convergent series may be termwise integrated. Therefore

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{1}{2\,\pi\,i} \int_{\Gamma} \frac{f(w)}{(w-a)^{(n+1)}}\, dw \, +$$

$$+ \sum_{n=0}^{\infty} \frac{1}{(z-a)^{(n+1)}} \frac{1}{2\,\pi\,i} \int_{\gamma} f(w)\,(w-a)^n\, dw.$$

With this is $f(z)$ written as the sum of two series; the one as a power series in $(z-a)$, the other one as a power series in $\dfrac{1}{(z-a)}$. They converge respectively inside Γ and outside γ.
Instead of Γ or γ in the integrals we can take an arbitrary Jordan curve J which runs entirely between c and C and is oriented in positive sense, because we can deform Γ and γ to J without passing the singularties of the respective integrals. So we can write

$$f(z) = \sum_{-\infty}^{\infty} c_n\,(z-a)^n, \quad c_n = \frac{1}{2\,\pi\,i} \int_{J} \frac{f(w)}{(w-a)^{(n+1)}}\, dw \quad \left(n \begin{array}{c} \geq \\ = \\ \leq \end{array} 0 \right).$$

The coeffients c_n are independent of z and uniquely determined by f. We can $f(z)$ slpit up as $f(z) = f_1(z) + f_2(z)$, with

$$f_1(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^n, \text{a power series in} (z-a),$$

$$f_2(z) = \sum_{n=-1}^{-\infty} c_n\,(z-a)^n = \sum_{n=1}^{\infty} c_{-n}\,(z-a)^{(-n)}, \text{a power series in} \frac{1}{(z-a)}.$$

The series of $f_1(z)$ converges for $|(z-a)| < R$, also there where $f(z)$ is not defined (inside c); that for $f_2(z)$ converges for $|(z-a)| > r$, also where $f(z)$

is not defined (outside C). In the annular ring between c and C converges both series, and their sum suggest $f(z)$.
This series epansion is called Laurent-expansion of $f(z)$. This Laurent-expansion is uniform convergent in z on every domain lying inside G.

Remark: A special case occurs when $f(z)$ is holomorphic inside C. Then only remains the Taylor series: $c_n = 0$ for $n = -1, -2, \ldots$.

Definition 10

The series for $f_1(z)$ we call the positive part of the Laurent expansion. The series for $f_2(z)$ we call the negative part of the Laurent expansion either the main part of the Laurent expansion.

Another special case occurs if the circle c contracts to a point a.
Then apparently holds

Theorem 39

Is $f(z)$ holomorphic in the surrounding of a, the point a itself excluded, the is in that reduced surrounding of a the function $f(z)$ to expand to a Laurent-series. The positive part if the Laurent expansion converges in a full surrounding of a (so a itself included); the negative part converges for all $z \neq a$.

Exercise: If $f(z)$ and $g(z)$ have convergent Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$, $\sum_{n=-\infty}^{\infty} b_n z^n$ in $C_1 < |z| < C_2$ then has $h(z) = f(z)\,g(z)$ likewise: $\sum_{n=-\infty}^{\infty} c_n z^n$ with $c_n = \sum_{k=-\infty}^{\infty} a_k\, b_{(n-k)}$. (See analoguous exercise power series.)

4.4 Singular points; addition of the point $z = \infty$

We will now consider the different cases of so called isolated points. We look here to functions $f(z)$ which are regular in a reduced surrounding of a point $z = a$ and find out what could be going on in $z = a$.

Remark: In chapter 6 we will meet another kind of sungular point, namely branch points.

To the Laurent expansion of a function, which is regular in a reduced surrounding of a, we can distinguish three cases:

- First case
  The lack of the main part. Then holds in a reduced surrounding of a

  $$f(z) = \sum_{n=0}^{\infty} c_n \, (z - a)^n.$$

  In this case we speak about removable singularity. If we define $f(a) = c_0$ then is $f(z)$ also regular in $z = a$.

- Second case
  In the laurent series are pnly a finite number of terms with a negative exponent:

  $$f(z) = \sum_{n=0}^{\infty} c_n \, (z - a)^n + \frac{c_{-1}}{(z - a)} + \frac{c_{-2}}{(z - a)^2} + \ldots \frac{c_{-k}}{(z - a)^k}$$

  with $k \geq 1$ and $c_{-k} \neq 0$. In this case is $\lim_{z \to a}(z - a)^k \, f(z) = c_{-k} \neq 0$. According the definition on page 37 has $f(z)$ in $z = a$ a pole of order k, also called a k-fold pole. The function $(z - a)^k \, f(z)$ is holomorphic and $\neq 0$ in a surrounding of a.

- Third case
  The main part contains infinitely many terms. The main part is then an entire transcedental function of $\frac{1}{(z - a)}$. The point $z = a$ is natural singular, much "stronger" singular then the pole in the second case. Therefore $z = a$ is called an essential singular point.

About the behaviour of the function $f(z)$ in the neighbourhood of the point a we can notice that

1. Is a a regular point, then is $f(z)$ in the surrounding of a bounded and conversely. (Check that!)

2. Is a a pole, then is $|f(z)|$ big for all z in the surrounding of a. More clearly: is G an arbitrary big number, then one can find some $\delta > 0$, such that $|f(z)| > G$ for $0 < |z - a| < \delta$.

3. In the surrounding of an essential singular point any number α is arbitrary closed approximated by $f(z)$. This means: to every $\epsilon > 0$, $\delta > 0$, α, can be found at least one point z such that: $|f(z) - \alpha| < \epsilon$ and $0 < |z - a| < \delta$.

1) The proofs of 1) and 2) are trivial. The proof of 3) is difficult. We ommit it.

Remark: Out of 1), 3) and 3) follows that only in the second case there exists a whole number $k \geq 1$ such that $\lim_{z \to a}(z-a)^k f(z)$ is bounded and $\neq 0$.

Addition of the point $z = \infty$

To simplify speaking we introduce formal "the point ∞". What we mean becomes evident from the following.
In the discussion of above, and also before, was a assumed to be finite. Now we will define the concepts holomorphic, pole, etc. for that one improper point, $z = \infty$. The common rule is: the behaviour of a function $f(z)$ in $z = \infty$ is defined based on the behaviour of the function $g(z) = f(\frac{1}{z})$ in the point $z = 0$.

A model of the so called extended complex plane we can get by viewing a sphere which is tangent to the complex plane in the origin, and then to project the complex plane to this ball out of the highest point (the "North Pole" N). The point N we consider as the image of ∞. To this model (stereographic projection) we see that it is useful to call for instance the outer region of a Jordan curve a "surrounding of ∞". After all the projection at the sphere is a surrounding of N. We will not go into this further.

In accordance with the general rule (with $g(z) = f(\frac{1}{z})$) we can also for $z = \infty$ distinguish the three cases of above:

1. $f(z)$ is holomorphic in $z = \infty$, if $g(z)$ is holomorphic in $z = 0$.
2. $f(z)$ has in $z = \infty$ a pole of order k, if $g(z)$ in $z = 0$ has a pole of order k.
3. $f(z)$ has in $z = \infty$ an essential singular point, if $g(z)$ has an essential singular point in $z = 0$.

Examples

An entire not constant rational function has a pole in infinity; an entire transcedental function has an essential singular in infinity. So has $f(z) = z^2$ a second order pole in $z = \infty$; $f(z) = z^{(-3)} + z^{(-2)}$ has a second order zero in $z = \infty$. And $\sin(z)$ has an essential singular point in $z = \infty$, $\sin(\frac{1}{z})$ is regular in $z = \infty$, etc..

Exercise: Has $f(z)$ just regular points and a finite number of poles ($z = \infty$ included), then is $f(z)$ rational. A rational function kan be splitted up into partial fractions.
Hint: if a is a pole consider then $h(z) = f(z) -$ main part; etc..

4.5 Further details about residues

Residue in $z = \infty$

We define not: the residue of $f(z)$ in ∞ is the residue of $f(\frac{1}{z})$ in $z = 0$. We can define this concept in the usual way. Is $f(z)$ regular on and outside the Jordan curve K then we define

$$\mathrm{Res}_\infty f(z) = -\frac{1}{2\,\pi\,i} \int\limits_K f(z)\,dz.$$

The minus sign is explained by the fact that if we consider K as the boundary of the outer region, the orientation with respect to that region is negative.
So: the function $\frac{1}{z}$ is regular in $z = \infty$, but has in ∞ the residue -1; likewise $\sin(\frac{1}{z})$ [note: regular and still residue $\neq 0$].

A consequence of the given definition is, that always holds

$$\sum \text{ all residues (inclusive at } z = \infty) \;=\; 0$$

for functions with a finite number of singularities.

There holds $\mathrm{Res}_\infty f(z) = -\mathrm{Res}_0 (\dfrac{f(\frac{1}{z})}{z^2})$.

Proof

Let R be great such that $|z| < R$ contains all singularities; $\rho = R^{(-1)}$.

$$\mathrm{Res}_{\infty} f(z) = -\frac{1}{2\,\pi\,i} \int\limits_{0}^{2\,\pi} f(R\,\exp{(i\,\phi)})\, d(R\,\exp{(i\,\phi)}) =$$

$$\frac{1}{2\,\pi\,i} \int\limits_{0}^{2\,\pi} f(\frac{1}{(\rho\,\exp{(-i\,\phi)})})\,\frac{d\,(\rho\,\exp{(-i\,\phi)})}{\rho^2\,\exp{(-2\,i\,\phi)}} = -\mathrm{Res}_0(\frac{f(\frac{1}{z})}{z^2}),$$

because the second integral $|z| = \rho$ is passed in negative sense.

$\square$

Residues in essential singular points

In section 3.3 we did discuss the resiudue theorem of Cauchy, and illustrated to functions which have singularities as poles. The residue theorem holds for arbitrary isolated singuar points. So we can also allow essential singular points. Has $f(z)$ an essential singular point in $z = a$, then

$$f(z) = \sum_{-\infty}^{\infty} c_n\,(z\,-\,a)^n \quad \text{in surrounding of } a$$

so $\{\mathrm{Res} f(z)\}_{z=a} = c_{-1}$ following the in the proof given formula of the Laurent expansion for c_n.
Also to an essential singular point is so the residue equal to the coefficient of $\dfrac{1}{(z\,-\,a)}$ in the Laurent expansion.

Exercise: Let J be a Jordan curve around 0, then holds

$$\frac{1}{2\,\pi\,i} \int\limits_{J} \exp{(\frac{1}{z})}\, z^{(n-1)}\, dz = \frac{1}{n!}; \quad \text{also for } n = 0\,?$$

$$\frac{1}{2\,\pi\,i} \int\limits_{J} \exp{(\frac{1}{z})}\, \frac{dz}{(z\,-\,a\,z^2)} = \exp{(a)}; \quad a^{-1} \text{ inside or outside } J\,?$$

Residues of $\dfrac{f'(z)}{f(z)}$; number of zeros inside a curve

Let $f(z)$ be holomorphic on and inside the Jordan curve J, and $f \neq 0$ on J. Then is the number of zeros of f inside J, if we count a zero with multiplicity k, also k times:

$$N = \frac{1}{2\,\pi\,i} \int\limits_{J} \frac{f'(z)}{f(z)}\, dz.$$

Proof

Define the function g by $g(z) = \frac{f'(z)}{f(z)}$. This function is continuous on J, and so exists the integral. Inside J lie a finite number of zeros of f. Let a be such a zero, and let it be zero with multiplicity k. Then holds in a surrounding of a:

$$f(z) = a_k\,(z - a)^k\,(1 + ...) \quad (a_k \neq 0),$$
$$f'(z) = k\,a_k\,(z - a)^{(k-1)}\,(1 + ...)$$

and so

$$g(z) = \frac{f'(z)}{f(z)} = \frac{k}{(z - a)}\,(1 + c_1\,(z - a) + c_2\,(z - a)^2 + ...).$$

The residue of $g(z)$ in a zero of $f()$ is therefore equal to the multiplicity of that zero. Apply the residue theorem and the deired result is obtained.

$\square$

Application: A polynomial of the degree n has exactly n zeros.

Proof

Is $f(z) = a_0 + a_1\,z + ... + a_n\,z^n$ $(n \geq 1,\, a_n \neq 0)$. For $|z|$ large enough is $|f(z)| > 1$. The possible zeros lie all inside some circle J with the center in the origin. This number is

$$N = \frac{1}{2\,\pi\,i}\int_J g(z)\,dz, \text{ with } g(z) = \frac{f'(z)}{f(z)} =$$
$$\frac{n\,a_n\,z^{(n-1)} + ...}{a_n\,z^n + ...} = \frac{n}{z}\,(1 + \frac{b_1}{z} + \frac{b_2}{z^2}\,...).$$

This series is a Laurent expansion of $g(z)$ in the surrounding of the point $z = \infty$, and one can choose J such that it on and outside J uniformly converges. The series may be integrated termwise, and the result is n, the coefficient of $\frac{1}{z}$.

$\square$

Remark: $g(z)$ has in $z = \infty$ a simple zero.

Exercise: Let $f(z)$ be on and inside J holomorphic with the exception of a finite number of poles, not lying on J, and $f \neq 0$ on J, then holds

$$N - P = \frac{1}{2\pi i} \int\limits_J \frac{f'(z)}{f(z)}\, dz.$$

Here is N the number of zeros of f inside J, P is the number of poles of f inside J (multiple also counted multiple). A pole of order k is in certain sense a zero with multiplicity $(-k)$, and the other way around.

Exercise: Let $\phi(z)$ be holomorphic on and inside J, $f(z)$ be holomorphic on and inside J, and $f \neq 0$ on J. Let further z_k be a zero of f inside J with multiplicity m_k. Then holds

$$\frac{1}{2\pi i} \int\limits_J \phi(z)\, \frac{f'(z)}{f(z)}\, dz = \sum_k \phi(z_k)\, m_k.$$

5 Application of the residues in the integral calculus

5.1 Definite integrals

In this chapter we are going to use complex function theory to the calculation of definite integrals. In the elemtary analysis we can actually only calculate definite integrals if we know the indefinite integrals, if we know the primitive function. In that sense is integration then the opposite of differentiation. The complex function theory is much powerful. There we can calculate certain integrals with the theory of the residues.

A. Integrals of the form: $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$,
where R ia a rational function in each of the arguments.
In the past we learned how we could find the result by using the substitution $t = \tan(\frac{\theta}{2})$ and partial fractions. Now we work differently:
we equate $z = \exp(i\,\theta)$, $dz = i\,z\,d\theta$, and the integral becomes

$$\int_C R(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{(2\,i)}(z - \frac{1}{z}))\,\frac{dz}{i\,z} = \int_C R_1(z)\,dz$$

where C is the unit circle and R_1 another rational function, now of z. We search the sinular points of R_1 inside C and apply the residue theorem.

Example: $I(a) = \int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)} \quad (0 < a < 1).$
Put $z = \exp(i\,\theta)$; z walks around the unit circle C in poisitive sense; $dz = i\,\exp(i\,\theta)\,d\theta = i\,z\,d\theta$, so $d\theta = -\frac{i}{z}\,dz$. Further

$$(1 + a\cos\theta) = 1 + \frac{1}{2}a\,(\exp(i\,\theta) + \exp(-i\,\theta)) = 1 + \frac{1}{2}a\,(z + \frac{1}{z}).$$

With this we find

$$\frac{d\theta}{(1 + a\cos\theta)} = -\frac{2\,i}{a}\,\frac{dz}{(z^2 + \frac{2}{a}z + 1)} = -\frac{2\,i}{a}\,\frac{dz}{(z - z_1)\,(z - z_2)},$$

where z_1 and z_2 are the roots of $z^2 + \frac{2}{a}z + 1 = 0$:

$$z_1 = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1} = \frac{\sqrt{1 - a^2} - 1}{a},$$

$$z_2 = -\frac{1}{a} - \sqrt{\frac{1}{a^2} - 1} = \frac{-\sqrt{1 - a^2} - 1}{a} \quad (< -\frac{1}{a}).$$

The number z_1 lies inside C, the number $z_2 = \frac{1}{z_1}$ lies outside C.
We have only to do with the singular point inside C, so with z_1. There is the residue of the integrand

$$-\frac{2\,i}{a}\,\frac{1}{z_1 - z_2} = -\frac{2\,i}{a}\,\frac{a}{2\,\sqrt{(1 - a^2)}}.$$

Multiply this with $2\,\pi\,i$, and we obtain the asked result

$$I(a) \;=\; \frac{2\,\pi}{\sqrt{(1 - a^2)}}.$$

This formula holds also for $-1 \,<\, a \,<\, 1$.
(After reading chapter 6 this is extendable to complex a. Check that!)

Exercise:

$$I(a) \;=\; \int_0^{2\,\pi} (1 \,-\, 2\,a\,\cos(t) \,+\, a^2)^{-1}\,dt \;\; (a\,\text{complex},\; |a| \,\neq\, 1).$$

Then holds:

$$\begin{aligned} I(a) \;&=\; 2\,\pi\,/\,(1 \,-\, a^2) \text{ for } |a| \,<\, 1,\\ &=\; 2\,\pi\,/\,(a^2 \,-\, 1) \text{ for } |a| \,>\, 1. \end{aligned}$$

(Integral is divergent for $|a| \,=\, 1$.)

Exercise:

$$\int_0^{2\,\pi} \frac{\cos(m\,t)}{(1 \,+\, a\,\cos(t))}\,dt \;=\; \frac{2\,\pi}{\sqrt{(1 - a^2)}}\,(\frac{-1 \,+\, \sqrt{(1 \,-\, a^2)}}{a})^m$$

with m whole ≥ 0, $0 \,<\, a \,<\, 1$.

Exercise:

$$\int_0^{2\pi} \frac{\cos(m\,t)}{(1 - 2\,a\,\cos(t) + a^2)}\,dt = 2\,\pi\,a^m\,(1 - a^2)^{-1} \text{ for } |a| < 1,$$

$$= 2\,\pi\,a^{-m}\,(a^2 - 1)^{-1} \text{ for } |a| > 1.$$

(Integral diverges for $|a| = 1$.)

Exercises:

$$\int_0^{2\pi} \frac{dt}{(a + b\,\sin(t))} = \frac{2\,\pi}{\sqrt{(a^2 - b^2)}},\ 0 \leq b < a,$$

$$\int_0^{2\pi} \frac{dt}{(a + i\,b\,\cos(t))} = \int_0^{2\pi} \frac{dt}{(a + i\,b\,\sin(t))} = \frac{2\,\pi}{\sqrt{(a^2 + b^2)}},\ 0 < a.$$

B. Integrals of the form: $\int_{-\infty}^{\infty} f(x)\,dx$.
Suppose of the function $f(z)$:

1. $f(z)$ is holomorphic on and above the real axis, with the eventual exception of a finite number of poles, none of which lie on the real axis;
2. If $M(\rho)$ is the maximum of $|f(z)|$ on the semicircle $\{z\,|\,|z| = \rho,\ \mathrm{Im}z \geq 0\}$, then holds

   $\rho\,M(\rho) \to 0$ if $\rho \to \infty$.
3. $\int_{-\infty}^{\infty} f(x)\,dx$ exists.

Remarks:
Condition (2) does not guarantee the existence of the integral. Just take a function such that $f(x) \sim \dfrac{1}{(x\,\ln(x))}$ $(x \to \infty)$. Requirement (3) means that $\int^{\infty}$ and $\int_{-\infty}$ both exist. Out of (1) follows that for finite a and b the integral $\int_a^b f(x)\,dx$ exists. (3) demands that the last integral has a limit for $a \to -\infty$ and $b \to \infty$, independent of each other.

Calculation:
Draw a half circle Γ with midpoint in the origin and radius ρ. Take ρ as big such that outside Γ there are no singularities of $f(z)$. Then

$$\int_{-\rho}^{\rho} f(x)\,dx \,+\, \int_{\Gamma} f(z)\,dz \,=\, 2\,\pi\,i \sum \text{Res.}\, f(z) \,\text{in upper half-plane.}$$

Neither member depends on ρ. Let $\rho \to \infty$. The limit of the first integral exists and is equal to the requested. The second integral must therefore also have a limit for $\rho \to \infty$. That the limit is equal to zero follows from

$$|\int_{\Gamma} f(z)\,dz| \,\leq\, \pi\,\rho\,M(\rho) \to 0 \ (\rho \to \infty).$$

We find thus

$$\int_{-\infty}^{\infty} f(x)\,dx \,=\, 2\,\pi\,i \sum \text{Res.}\, f(z) \ (\text{Im} > 0),$$

where the sum extends over the poles of $f(z)$ in the upper half-plane.

Examples:

1. $\int_{-\infty}^{\infty} \dfrac{1}{1\,+\,x^2}\,dx$; all conditions are fullfilled.
   $f(z) = \dfrac{1}{(1\,+\,z^2)}$ has a single pole in the upper half-plane for $z = i$. The residue is there $\lim_{z\to i} \dfrac{(z\,-\,i)}{(z^2\,+\,1)} \,=\, \dfrac{1}{(2\,i)}$. So the integral is equal to π.

2. $\int_{-\infty}^{\infty} (1\,+\,x^2)^{-3}\,dx$; $f(z) \,=\, (1\,+\,x^2)^{-3}$ has in $z = i$ a triple pole. The Laurent expansion around the point $z = i$ is: (let $(z - i = u)$)

$$\frac{1}{(z\,+\,i)^3}\,\frac{1}{(z\,-\,i)^3} \,=\, \frac{1}{(u\,+\,2\,i)^3}\,\frac{1}{u^3} \,=\, \frac{1}{(8\,i^3)}\,\frac{1}{u^3}\,\frac{1}{(1\,+\,(\frac{u}{2\,i}))^3}$$

$$=\, \frac{i}{8}\,u^{(-3)}\,(1\,-\,\frac{1}{2}\,i\,u)^{(-3)} \,=\, \frac{i}{8}\,u^{(-3)}\,(1\,+\,\frac{3}{2}\,i\,u - \frac{3}{2}\,u^2\,+\,...)$$

$$=\, ...\,-\,\frac{3}{16}\,i\,\frac{1}{u}\,+....$$

The residue in this pole is $-\frac{3}{16}\, i$. Multiply this with $2\,\pi\, i$, and the value of the integral becomes $\frac{3\,\pi}{8}$.

Exercise:

$$\int_{-\infty}^{\infty} \frac{x^4}{(1\,+\,x^2)^4}\, dx \,=\, \frac{\pi}{16}.$$

C. Integrals of the form: $\int_{-\infty}^{\infty} f(x)\, \exp\,(i\,\alpha\, x)\, dx$.
For $\alpha = 0$ already discussed. Let $\alpha\, >\, 0$ (no essential restriction, as long as we look to real α). Assume that:

1. $f(z)$ up and above the real axis as holomorphic, with the exception of a finite number of singular points, none of which at the real axis;
2. If $M(\rho)$ is the maximum of $|f(z)|$ at the semicircle $\{z\,|\,|z|\,=\,\rho,\, \mathrm{Im}\,\geq\, 0\,\}$ then holds

   $M(\rho)\, \to\, 0\, \text{if}\, \rho \to \infty.$
3. The integral exists.

Take Γ just as in the case $\alpha = 0$. Then

$$\int_{-\rho}^{\rho} f(x)\, \exp\,(i\,\alpha\, x)\, dx\, +\, \int_{\Gamma} f(z)\, \exp\,(i\,\alpha\, z)\, dz\, =$$

$$2\,\pi\, i \sum \mathrm{Res.}\,\{f(z)\, \exp\,(i\,\alpha\, z)\},$$

whereby the sum is stretched out over the singular points of $f(z)$ in the upper half-plane. We are going to estimate the integral along Γ:

$$J = \int_{\Gamma} f(z)\, \exp(i\,\alpha\,z)\, dz = \int_0^{\pi} f(\rho\, \exp(i\,t))\, \exp(i\,\alpha\,\rho\, \exp(i\,t))\, i\,\rho\, \exp(i\,t)\, dt;$$

$$|J| = |\int_{\Gamma} f(z)\, \exp(i\,\alpha\,z)\, dz\,| =$$

$$|\int_0^{\pi} f(\rho\, \exp(i\,t))\, \exp(i\,\alpha\,\rho\, \exp(i\,t))\, i\,\rho\, \exp(i\,t)\, dt\,| \leq$$

$$\leq \rho \int_0^{\pi} |\,f(\rho\, \exp(i\,t))\,|\, \exp(-\alpha\, \sin(t))\, dt \leq \rho\, M(\rho) \int_0^{\pi} \exp(-\alpha\, \sin(t))\, dt =$$

$$= 2\,\rho\, M(\rho) \int_0^{\frac{\pi}{2}} \exp(-\alpha\, \sin(t))\, dt < 2\,\rho\, M(\rho) \int_0^{\frac{\pi}{2}} \exp(-\alpha\,\rho\,(\frac{2}{\pi})\,t)\, dt <$$

$$< 2\,\rho\, M(\rho) \int_0^{\frac{\pi}{2}} \exp(-2\,\alpha\,\rho\,\frac{t}{\pi})\, dt = 2\,\rho\, M(\rho)\, \frac{\pi}{2\,\alpha\,\rho} = \frac{\pi\, M(\rho)}{\alpha}.$$

So $\lim_{\rho\to\infty} \int_{\Gamma} f(z)\, \exp(i\,\alpha\,z)\, dz = 0$. There follows easily

$$\int_{-\infty}^{\infty} f(x)\, \exp(i\,\alpha\,x)\, dx = 2\,\pi\,i \sum \text{Res.}\,\{f(z)\, \exp(i\,\alpha\,z)\}$$

in the upper half-plane.

Examples:

1.

$$I = \int_0^{\infty} \frac{\cos(x)}{(a^2 + x^2)}\, dx \quad (a > 0).$$

The integrand is an even function of x, so $I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(x)}{(a^2 + x^2)}\, dx$. The same integral with $\sin(x)$ instead of $\cos(x)$ is zero, because the integrand is then an odd function in x. So it also holds that

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\exp(i\,x)}{(a^2 + x^2)}\, dx.$$

We must remember these pieces of art well!

We apply the general case for $f(z) = \frac{1}{2}(a^2 + z^2)^{-1}$. All the conditions are fullfilled, and the only pole of $f(z)$ in the upper half-plane lies by $z = i\,a \quad (a > 0)$. Further is

$$\{\text{Res.}\, f(z)\, \exp(i\,z)\}_{z = i\,a} = \lim_{z \to i\,a} \exp(i\,z)\, \frac{1}{2}\, \frac{(z - i\,a)}{(z - i\,a)\,(z + i\,a)} = \frac{\exp(-a)}{(4\,i\,a)}.$$

So

$$I = 2\,\pi\,i\, \frac{\exp(-a)}{(4\,i\,a)} = \frac{\pi}{(2\,a)} \exp(-a).$$

Exercise:

$$\int_0^{\infty} \frac{x\, \sin(a\,x)}{(x^2 + k^2)} \quad (a > 0,\, k > 0).$$

(What is the solution if $(a < 0)$?)

2.

$$J = \int_0^{\infty} \frac{1 - \cos(x)}{x^2}\, dx.$$

The function $\frac{\sin(x)}{x^2}$ is odd, but not integrable in the neighbourhood of $x = 0$. Therefore some new piece of art. We take a path of integration as follows: from $-\rho$ to $-\epsilon$, from $-\epsilon$ to ϵ a circular arc with radius ϵ, to avoid the origin, then from ϵ to ρ, and further with Γ back to $-\rho$. This road we call W. And we integrate along W the function $f(z) = \frac{1 - \exp(i\,z)}{z^2}$. If we take the circular arc around de origin in the upper half-plane, then has the integrand no singular points inside W, and $\int_W f(z)\, dz$ is equal to zero. The contribution of Γ goes to zero if $\rho \to \infty$. Then holds

$$\int_{-\infty}^{-\epsilon} f(z)\,dz + \int_{-C_\epsilon} f(z)\,dz + \int_{\epsilon}^{\infty} f(z)\,dz = 0,$$

where the integral in the middle goes about the little circular arc (in negative sense around the origin). In the limit $\epsilon \to 0$ approximates this integral to $2\,\pi\,i$ times minus the half residue of $f(z)$ in $z = 0$, so to $(2\,\pi\,i)\,(-\frac{1}{2})\,(-i) = -\pi$.The two remaining integrals we can contract to

$$2\int_{\epsilon}^{\infty} \frac{1-\cos(x)}{x^2}\,dx,$$

and we find for $\epsilon \to 0$ that $J = \frac{\pi}{2}$.
In the foregoing example is used a principle that we can more often utilize. We formulate it as

Theorem 40

Is a a simple pole of the function $f(z)$ and C_ϵ the semi-circle: $|z-a| = \epsilon$, $0 \le arg(z-a) \le \pi$, then is

$$\lim_{\epsilon \to 0} \frac{1}{2\,\pi\,i} \int_{C_\epsilon} f(z)\,dz = \frac{1}{2}\,\mathrm{Res}_a\{f(z)\}.$$

(Think to the orientation of the semi-circle.)

Exercise: Proof this theorem.

6 Analytic continuation

6.1 Analytic continuation

Let be G_1 and G_2 be two regions of which the intersection D is also a region (for instance two circles). Suppose that in G_1 a holomorphic function f_1 is defined and in G_2 a holomorphic function f_2. If a is point of D such that in every reduced surrounding of a are points scuch that $f_1(z) = f_2(z)$ then is according the identity theorem (subsection 4.2) $f_1(z) = f_2(z)$ everywhere in D. We now define a function f on $G_1 \cup G_2$ by

$$f(x) = \begin{cases} f_1(z) & \text{if } x \in G_1, \\ \\ f_2(z) & \text{if } x \in G_2. \end{cases}$$

That is allowed now because $f_1 = f_2$ at $G_1 \cap G_2$. We see that f is a holomorphic function on a region $G = G_1 \cup G_2$, that G_1 includes such that $f(z) = f_1(z)$ on G_1. We call f an analytic continuation of f_1. Because of the identity theorem we know that for the region $G_1 \cup G_2$ there exists no other analytic continuation. Naturally is f as well an analytic continuation of f_2. f_1 and f_2 are called each other analytic continuation (in G_1, resp. G_2). Another possibility can be found at the end of this section.
We give an example:

Example:

$$f_1(z) = \sum_{n=0}^{\infty} (-1)^n z^n \qquad \text{in } G_1 : |z| < 1$$

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{(n+1)}} (z-i)^n \quad \text{in } G_2 : |z - i| < \sqrt{2}$$

$f(z) = \dfrac{1}{(1+z)}$, not only in $G_1 \cup G_2$, but everywhere (except $z = -1$).
We have so f_1 continued, first using f to the region $G_1 \cup G_2$ and then with $f(z) = \dfrac{1}{(1+z)}$ to the region $\mathbb{C} \setminus \{-1\}$.

f is defined in a region G. Choose a point $a \in G$. On basis of the theorem of Taylor there is a circle around a within $f(z)$ is the sum of a power series. If the circle of convergence C of this power series becomes outside G we can continue f to $G \cup C$, as long as $G \cap C$ is a region.

In the next section we shall, first fast at a special case, a quick way to get to know.

6.2 The logarithm

We let away out of $\mathbb{C}$ the negative real numbers and 0 away. It is also said: we make a cut along the negative real axis. We are left with a region G. In it, we want to define a holomorphic function f such that $f(x) = \log(x)$ if x is real and positive. If this is possible then only in one way (again: the identity theorem!).
If we consider two paths from 1 to z in the region G, then then is value of $\int_1^z \frac{1}{t}\,dt$ equal on these paths, see theorem 13 (page 30). Because of the continuity of $\frac{1}{t}$ in G, is on basis of theorem 12 (page 29): $\int_1^z \frac{dt}{t}$ a holomorphic functions of z in G, with derivative $\frac{1}{z}$. If z is real and positive, is the value of the integral $\log(z)$. This has to be the sought analytical continuation. We define:

$$\log(z) =: \int_1^z \frac{dt}{t},$$

where we require that the integration path does not intersect the negative real axis.
In G is $f(z)$ independent of the course of the integration path. To calculate $f(z)$ we choose our way rectilinearly from 1 to $|z|$, and then from $|z|$ to z along a circular arc centered at the origin. Suppose $z = |z|\exp(i\,\phi)$, wherein $-\pi\,, < \phi < \pi$. Then is

$$f(z) = \int_1^{|z|} \frac{dt}{t} + \int_{|z|}^{|z|\exp(i\,\phi)} \frac{du}{u}.$$

The first integral is simply $\log|z|$ (the ordinary logarithm from the real analysis). In the second integral we substitute $u = |z| \exp(i\,\theta)$, $du = i\,|z| \exp(i\,\theta)\,d\theta = i\,u\,d\theta$, such that the second integral is equal to $i\,\phi$. Therefore $f(z) = \log|z| + i\,\phi$, so

$$\log(z) = \log|z| + i\arg(z) \qquad -\pi < \arg(z) < \pi.$$

We now also define $\log(z)$ for negative real z:

Principal value of the logarithm

This is defined by $\log(z) = \log|z| + i\arg(z)$, where $\arg(z)$ is the principal value of argument of z so $(-\pi < \arg(z) \leq \pi)$. Now the log of a negative number is well defined.

Exercise: For x negative-real holds:

$$\log(x) \stackrel{\text{def}}{=} \log(x + i0) = \log(-x) + i\,\pi \quad \text{and}$$
$$\log(x - i0) = \log(-x) - i\,\pi,$$

such that $\log(z)$ has a jump of $2\,\pi\,i$ at the cut.

Exercise: Show that for $|z| < 1$ holds

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots = \log(1 + z) \quad (\stackrel{\text{def}}{=} \int_1^{(1+z)} \frac{dt}{t}).$$

From now on, unless stated otherwise, we mean by $\log(z)$ the principal value of the logarithm, which is bounded by the principal value of $\arg(z)$.

One can take a broader point of view, and consider all possible analytic continuations of $\log(x)$. This then leads to a function $\log(z)$ which is infinitely many-valued; the various branches of the function differ then a integer multiple of $2\,\pi\,i$. The point $z = 0$ is a singular point of the function. This is called a logarithmic branch point. For applications, it is better, however, to make this infinitely many-valued function monovalent by applying a cut (also called denomination) and in so doing cut open z-plane considers the main branch of the function. If it is necessary we can easily continue this main branch analytically beyond the cut. Apart from this, the cut can also be an arbitrary curve which connects $z = 0$ with $z = \infty$. And $\log(x)$, originally defined at a small interval of the positive-real axis, can then always be analytic continued to the edges of the cut. Along a prescribed path continuation is always unambiguous. This is clearly the same for $\arg(z)$, which we can continue continuously.

Using the logarithm we can now also define arbitrary powers.
Let α be a complex number, then by definition, for $z \neq 0$,

$$z^{\alpha} \stackrel{\text{def}}{=} \exp\left(\alpha \, \log\left(z\right)\right).$$

One gets the principal value of z^{α} if one chooses the principal value of $\log\left(z\right)$.
So is

$$z^{\alpha} \;=\; \exp\left(\alpha\left(\log|z| \,+\, i\arg(z)\right)\right) \;=\; |z|^{\alpha}\,\exp\left(i\,\alpha\arg(z)\right)$$

with $(-\pi \,<\, \arg(z) \,\leq\, \pi)$. If α is whole then we are in familiar territory: z^m, m is whole, is monotonous. If α is rational: $\alpha \,=\, \dfrac{p}{q}$, then is z^{α} q-valued.

Exercise: What is the principal value of $\sqrt{(i)} \,=\, (i)^{\dfrac{1}{2}}$, of $(i)^i$, of $(-i)^i$?

Exercise:

$$\frac{d}{dz}z^{\alpha} \;=\; \alpha\, z^{(\alpha-1)},$$

$$z^{\alpha}\, z^{\beta} \;=\; z^{(\alpha+\beta)},$$

$$z^{(-\alpha)} \;=\; \frac{1}{z^{\alpha}}.$$

We have to be careful with some operations:

$$(z_1)^{\alpha}\,(z_2)^{\alpha} \;=\; \exp\left(\alpha\left(\log\left(z_1\right) \,+\, \log\left(z_2\right)\right)\right)$$
$$=\; \exp\left(\alpha\left(\log|z_1| \,+\, \log|z_2| \,+\, i\arg(z_1) \,+\, i\arg(z_2)\right)\right)$$
$$=\; |z_1\, z_2|^{\alpha}\,\exp\left(i\,\alpha\left(\arg(z_1) \,+\, \arg(z_2)\right)\right)$$

If $\arg(z_1) \,+\, \arg(z_2) \;=\; \arg(z_1\, z_2)$, then also holds $(z_1)^{\alpha}\,(z_2)^{\alpha} \;=\; (z_1\, z_2)^{\alpha}$ (all in the sense of principal values). But in general is $\arg(z_1) \,+\, \arg(z_2) \;=\; \arg(z_1\, z_2) \,+\, 2\,k\,\pi$, where k can take the values -1, 0 and 1.
For example:

$$(-1)^{\frac{1}{2}}\,(-1)^{\frac{1}{2}} \;=\; i\, i \;=\; -1; ((-1)\,(-1))^{\frac{1}{2}} \;=\; 1 \quad \text{(principal values)}.$$

Exercise: $f(z) = (1+z)^{\frac{1}{2}}(1-z)^{\frac{1}{2}}$, for $-1 < z < 1$ as the ordinary algebraic value defined. Discuss the analytic continuation of $f(z)$ in the complex plane cut open along the rest of the real axis.
Calculate the limits of $f(z)$ on both sides of the cut.
Verify that $(1-z^2)^{\frac{1}{2}}$ has the same analytic continuation.

Exercise: To the following functions, which match with each other for $z > 1$, belong the following cuts:

$g(z) = (z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}$, cut $-1 \le z \le 1$,

$h(z) = (z^2-1)^{\frac{1}{2}}$, cut $-1 \le z \le 1$ ánd the imaginary axes.

Check that $g(z)$ is an odd function; $h(z)$ is naturally even. ($g(z)$ and $-g(z)$ can be united by a two-leaf representation; $h(z)$ and $-h(z)$ also.)

We give another example of analytic continuation. From the real analysis we know the function $\arctan(x)$, is defined for all real x. That we are not being able to continue this function to the whole z-plane is immediately clear! After all $(\arctan(x))' = (1+x^2)^{(-1)}$ and this derivative has in the complex plane singular points. However, this derivative is a simple function: $(1+t^2)^{(-1)}$ has two first order poles (in $t = i$ and $t = -i$). If we now make cuts into the complex plane from i to $i\,\infty$ (positive imaginary axis) and from $-i$ to $-i\,\infty$ (negative imaginary axis) then we keep a region G. In this area depends $\int_0^z (1+t^2)^{(-1)}\,dt$ not of the path of integration.
If z is real we know that the value of the integral is equal to $\arctan(z)$. Moreover, we know (statement page 29) that the integral in G is a holomorphic function. This should be the wanted analytic continuation:

$$\arctan(z) := \int_0^z (1+t^2)^{(-1)}\,dt \quad \text{for} \quad z \in G.$$

Exercise: Let $f(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} (-1)^{(n-1)} \dfrac{z^n}{n^2}$, $|z| \le 1$.

Show that: 1) $f(1) + f(-1) = \frac{1}{2}\,f(-1)$; 2) $\frac{d}{dz}f(z) = \dfrac{\log(1+z)}{z}$; with this continue $f(z)$ analytically (take a suitable cut); 3) $f(z) + f(\frac{1}{z}) = 2f(1) + \frac{1}{2}\log^2(z)$; 4) $f(1) = \dfrac{\pi^2}{12}$ (use 1)); 5) $\sum_{n=1}^{\infty}(-1)^{(n-1)}\dfrac{\cos(n\,\phi)}{n^2} = \dfrac{\pi^2}{12} - \dfrac{\phi^2}{4}$ (compare Fourierseries, Wiskunde 30; use 3)).

7 Various

7.1 Differential equations of Cauchy-Riemann

B. Riemann (1826-1865), A.L. Cauchy (1789-1857)

Let $f(z)$ be holomorphic in a region G; $f(z) = u(x,y) + i\,v(x,y)$ with $z = x + i\,y$.
How is $f'(z)$ expressed in partial derivatives of u and v to x and y?

Consider two special cases of limit transition in the differential quotient:
(1) $z' \to z$ in horizontal direction, (2) $z' \to z$ in vertical direction.
The differential quotient

$$\frac{f(z') - f(z)}{z' - z}$$

is in the first case

$$\frac{u(x',y) - u(x,y)}{x' - x} + i\,\frac{v(x',y) - v(x,y)}{x' - x} \quad (x' \neq x).$$

This expression has to have a limit for $x' \to x$. Therefore it is necessary that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ exist in the point (x,y). The value of this limit has to be equal tot $f'(z)$:

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.$$

In the second case is the differential quotient

$$\frac{u(x,y') - u(x,y)}{i\,(y' - y)} + i\,\frac{v(x,y') - v(x,y)}{i\,(y' - y)} \quad (y' \neq y).$$

Necessarily, therefore also exist

$$f'(z) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}.$$

From the two equalities then follow the identities

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (CR)$$

With this is proved:

Theorem 41

Is $f(z) = u + i\,v$ holomorphic in G, then exist in G the first order partial derivatives of u and v to x and y, and hold the partial differential equations of Cauchy-Riemann (CR).

A kind of inversion of this statement is:

Theorem 42

If in a region G the real functions $u(x, y)$ and $v(x, y)$ are together continuous differentiable to x and y, and satisfy they in G to the equations of Cauchy-Riemann, then is $f(z) = u + i\,v$ a holomorphic function of z in G.

Remark: Continuous-differentialble want to say: the foiur partial derivatives of the first order of u and v to x and y exist and are continuous functions of x and y.

Exercise: Prove the last theorem.

Out of earlier results we know that a holomorphic function is arbitrary often differentiable. Then are the components u and v of a holomorphic functions also arbitrary often differentiable, and the derivatives of all orders are also continuous functions of x and y.
In particular hold the formulas:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{\partial^2 v}{\partial x \, \partial y} = \frac{\partial^2 v}{\partial y \, \partial x} = \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = -\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial^2 v}{\partial x^2} = \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad = -\frac{\partial^2 v}{\partial y^2}.$$

Theorem 43

Is $f(z)$ holomorphic in G, then satisfy $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ as functions of x and y in G to the potential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

They say that: u and v are harmonic functions (i.e. solutions of the two-dimensional potential equation) in G.

Conclusion: Theory of complex functions of one complex variable is equivalent with the potential theory in two dimensions. Hence the importance of complex function theory for physics and engineering.

Theorem 44

A holomorphic function is unambiguously determined except for a constant by its real (or its imaginary) part.

Proof

Let $f(z) = u(x,y) + i\,v(x,y)$, and $u(x,y)$ be known in a certain surrounding of (x_0, y_0). Then holds in that surrounding the following. Because $v_y = u_x$, is also $v_y(x,y)$ known. Then we can integrate to y:

$$\int_{y_0}^{y} u_x(x,\eta)\,d\eta = v(x,y) - v(x,y_0).$$

So

$$v(x,y) = \int_{y_0}^{y} u_x(x,\eta)\,d\eta + F(x), \text{ with } F(x) = v(x,y_0).$$

We are ready if we can indicate $F(x)$. $F(x)$ is certainly differentiable to x and it does apply

$$F'(x) = \frac{\partial v}{\partial x} - \frac{\partial}{\partial x} \int\limits_{y_0}^{y} u_x(x,\eta)\, d\eta = v_x - \int\limits_{y_0}^{y} u_{xx}(x,\eta)\, d\eta =$$

$$- u_y - \int\limits_{y_0}^{y} u_{xx}(x,\eta)\, d\eta = \text{known.}$$

Does this known function only depends on x? Yes, because

$$-\frac{\partial}{\partial y} F'(x) = \frac{\partial}{\partial y} \left[u_y + \int\limits_{y_0}^{y} u_{xx}(x,\eta)\, d\eta \right] = u_{xx} + u_{yy} = 0.$$

So $F'(x)$ is a known function of x. By integration we find $F(x)$, unambiguously determined except for a constant. So $v(x,y)$ is known except for an additive constant.

Exercise: Determine $f(z)$ if $u(x,y) = x / (x^2 + y^2)$.

Exercise: Prove that the curves u =constant and v =constant through a point s with $f'(s) \neq 0$ cut each other perpendicular.

7.2 Conformal mapping

Let $w = f(z)$ be holomorphic in region G of the z-plane. To every point of G belongs a point of the w-plane. So the functional relationship $w = f(z)$ gives an image of part of the z-plane on part of the w-plane. Let G' be the image of G. One can prove that G' is a region, if $f \neq$constant.
Let $f(z) = u + i\,v$; $u = u(x,\, y)$, $v = v(x,\, y)$. As is well known, the funktional determinant of the transformation plays a role. This one is

$$D = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \left\{ \begin{matrix} \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} \\ \\ \begin{vmatrix} v_y & u_y \\ -u_y & v_y \end{vmatrix} \end{matrix} \right\} \begin{matrix} = \\ = \end{matrix} \begin{matrix} (u_x)^2 + (v_x)^2 \\ (u_y)^2 + (v_y)^2 \end{matrix} = |f'(z)|^2.$$

Assume $f'(z) \neq 0$ for $z = z_0$. Call $w_0 = f(z_0)$ the image of z_0. Out of $D \neq 0$ follows (no proof here) that the transformation $(x,y) \to (u,v)$ is unambiguous and in both directions continuous differentiable, in the respective environments of the points z_0 and w_0. In these environments we set

$z = z_0 + \triangle z;\ w = w_0 + \triangle w.$

Then is $\triangle w = f(z_0 + \triangle z) - f(z_0) = f'(z_0)\triangle z + \epsilon \triangle z$, with $\epsilon \to 0$ for $\triangle z \to 0$.
For $|\triangle z|$ small enough holds approximately $\triangle w = f'(z_0) \triangle z$, because $f'(z_0)$ differs from zero. Then let's say $\triangle z = \xi + i\,\eta$, $\triangle w = \sigma + i\,\tau$, then are (ξ, η) and (σ, τ) local cartesian coordinates, and we have

$$\sigma + i\,\tau = f'(z_0)\,(\xi + i\,\eta).$$

Now we write $f'(z_0)$ out in modulus and argument: $f'(z_0) = \rho \exp(i\,\phi)$. Then we find

$$\sigma = \rho\,(\xi \cos\phi - \eta \sin\phi),$$

$$\tau = \rho\,(\xi \sin\phi + \eta \cos\phi),$$

where ρ and ϕ are independent of (ξ, η) and (σ, τ).
Above transformation $(\xi, \eta) \to (\sigma, \tau)$ is a simple linear transformation with functional determinant p^2. It is a combination of a rotation over the angle ϕ and a multiplication out of the origin by the factor ρ.
Suppose we have two oriented curves in the z-plane which cut each other in z_0 over an angle α (tangent vectors s_1 and s_2; if we rotate s_1 over an angle α in positive sense then we obtain s_2). As an image in the w-plane we get again two curves, through w_0, whose tangent vectors t_1 and t_2 with respect to s_1 and s_2 are both rotated over an angle ϕ. The corner under which they cut each other is so again α. The angle ϕ can be positive or negative, but the order (t_1, t_2) will be the same as (s_1, s_2) as for the rotation.

Conclusion: the transformation is angle-preserving and also directly angle-preserving.

Furthermore, lengths are transformed with the factor $\rho = |f'(z_0)|$. A small triangle (with vertex in z_0, sides 1 and 2 which include the corner α) is transformed into a similar triangle (with vertex in w_0, sides $1'$ and $2'$ which are ρ times as long as the sides 1 and 2, while the corner between sides $1'$ and $2'$ is again α). The image is therefore conform. With that is proved

Theorem 45

Is $f(z)$ holomorphic ikn the point z_0, and $f'(z_0) \neq 0$, the is the transormation $w = f(z)$ in the neighbourhood of z_0 conform and directly angle-preserving.

Theorem 46

There are no other (continuous differentiable) directly angle-preserving transformations than that by holomorphic functions.

Theorem 47

There are no conformal maps in two dimensions other than those due to holomorphic functions of z or of $\overline{z}$.

We will not go into the proof of these two propositions.

Some simple transformations

The linear transformation: $w = a\,z + b \quad (a \neq 0)$.
The inversion: $w = \dfrac{1}{z}$.
This are one to one transformations of the the extended complex z-plane to the extended complex w-plane. With the linear map, the point ∞ remains on its place. At the inversion are 0 and ∞ each other's image.

Transformation of Möbius (A.F. Möbius, 1790-1868)

This is the general broken linear transformation: $w = \dfrac{a\,z + b}{c\,z + d}$, wit a, b, c and d complex constants. The derivative is

$$\frac{d\,w}{d\,z} = \frac{a\,d - b\,c}{(c\,z + d)^2}$$

Therefore suppose

$$a\,d - b\,c = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0,$$

then is $\frac{d\,w}{d\,z}$ nowhere zero (with the exception of ∞). The inversion is

$$z = \frac{d\,w - b}{-c\,w + a}, \text{with determinant} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = a\,d - b\,c \neq 0,$$

$$\frac{d\,z}{d\,w} = \frac{a\,d - b\,c}{(c\,w - a)^2}, \text{(nowhere zero, excepted} \quad w = \infty).$$

In this Möbius transformation the extended complex z-plane becomes unambiguously transformed to the extended complex w-plane. The image of the point $z = \infty$ is $w = \frac{a}{c}$; that of $z = -\frac{d}{c}$ is $w = \infty$. Here too we see the advantage of the introduction of one improper complex number ∞.

The Möbius transformation can be regarded as a "product" of elementary transformations. If $c = 0$ then it is linear; if $c \neq 0$ then

$$(6) \quad w = \frac{a\,z + b}{c\,z + d} = \frac{a}{c} - \frac{a\,d - b\,c}{c^2}\,\frac{1}{z + \dfrac{d}{c}}.$$

The Möbius transformations form a so-called group. If

$$w = \frac{a\,z + b}{c\,z + d}, \quad u = \frac{\alpha\,w + \beta}{\gamma\,w + \delta},$$

then holds

$$u = \frac{A\,z + B}{C\,z + D} \quad \text{with} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(Proof ?)

Theorem 48

In a Möbius transformation, circles are transferred into circles (a straight line is understood as a degenerate circle).

Proof

With the above product split (see formula 6).
The theorem holds for each of the individual factors.
The only non-trivial verification is for the inversion $w = \frac{1}{z}$.

□

Exercise: The general equation of the circle is

$p\,z\,\overline{z} + \alpha\,z + \overline{\alpha}\,\overline{z} + q = 0, \quad p$ and q real.

Prove the foregoing theorem by showing that this equation changes into a corresponding equation in w at every Möbius transformation.

Of the four parameters that occur in the general Möbius transformation, there are three independent (it only depends on the ratio of the parameters). Then is clear that

Theorem 49

There is one and only one Möbius transformation where three given points A, B, C in the z-plane are depicted on three given points P, Q, R in the w-plane. Explicitly one can write this transformation as

$$(7) \quad (\frac{w-P}{w-R}):(\frac{Q-P}{Q-R}) = (\frac{z-A}{x-C}):(\frac{B-A}{B-C}) \quad (= t).$$

For the verification, the t-plane is taken as an intermediate step, where t is equal to the common value of left and right members:

$w = P \leftrightarrow t = 0 \leftrightarrow z = A$

$w = Q \leftrightarrow t = 1 \leftrightarrow z = B$

$w = R \leftrightarrow t = \infty \leftrightarrow z = C.$

Double ratio of four points

Take four points in the complex plane: z, A, B, C. The double ratio (z, A, B, C) of these four ordered points is by definition the number that is the image of z under the Möbius transformation which transposes A, B, C into respectively about 0, 1, ∞.

Exercise:

$$(z,\, A,\, B,\, C) \;=\; (\frac{z\,-\,A}{x\,-\,C}) : (\frac{B\,-\,A}{B\,-\,C})$$

Name the four points in the z-plane: z, A, B, and C. Let their images be, in this order: w, P, Q, and R. Then is

$$(w,\, P,\, Q,\, R) \;=\; (\frac{w\,-\,P}{w\,-\,R}) : (\frac{Q\,-\,P}{Q\,-\,R}) \text{ and } (z,\, A,\, B,\, C) \;=\; (\frac{z\,-\,A}{x\,-\,C}) : (\frac{B\,-\,A}{B\,-\,C}).$$

The right-hand members are equal (see explicit formula for the Möbius transformation (formula 7)). So also the left members.

One can use Möbius transforms, where circles pass within circles, to prove all kinds of geometric properties of systems of circles. As an example one may try one's strengths on the

Steiner's closure problem

Given: two circles, of which the smallest is inside the largest.
Start drawing a circle somewhere that touches the smallest circle on the outside and the largest circle on the inside.
Construct a second circle tangent to all three.
Fill the gap between the two original circles with tangent circles.

There are now two possiblities:

1. the chain closes (the last touches exactly to the first),

2. the chain does not close.

One might think that this depends on the choice of the first circle in the chain. However, that is not the case. It is always (1) or always (2), independent of the starting circle.
The evidence is very transparent. For a set of concentric circles, the is statement trivial. And one can easily transform the two given circles in two concentric circles. To that transformation (a Möbius transformation) circles turn into circles, and tangent circles into tangent circles, because the transformation is conform.

8 Literature

1. Ackermans, S.T.M. en Van Lint, J.H., Algebra en Analyse, Wolters-Noordhoff, 1970.

2. Knopp, Funktionentheorie I, II, Sammlung Göschen Nrs. 668, 703. Aufgabensammlung zur Funktionentheorie I, II. Sammlung Göschen Nrs. 877, 878.

3. Spiegel, M.R., Theory and Problems of Complex Variables, Schaum Publishing Co., 1964.

4. Whittaker-Watson, A course of modern analysis.

5. Polya, G. en Latta, G., Complex Variables, J. Wiley & Sons, 1974.

1 Index

f
functions, entire *54*
function, entire rational *55*
function, transcedent *55*

i
Identy Theorem *54*
Inequality of Cauchy *45*
Integralformula of Cauchy *39*
integrals, definite *67*

m
Main Theorem of Algebra *57*
multiplicity *52*

p
points, branch *60*
points, isolated *60*
pole order k *37*
property* *17*

s
series, Laurent *60*
series, main part *60*
singular, essential *61*

t
Theorem of Morera *44*