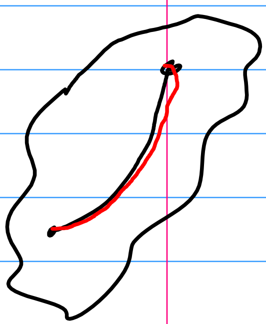
CAI: 200928

- Questions exer., theory? ask them!

This is from home. I hope everything will go well; also new laptop, so you never know.

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(\varphi(t))\cdot\varphi'(t)\,dt\right|$$



$$z = \varphi(t) \qquad dz = \varphi'(t)\,dt$$

$$\leq \int_a^b \underbrace{|f(\varphi(t))|}\cdot|\varphi'(t)|\,dt = M\cdot\int_a^b|\varphi'(t)|\,dt = M\cdot L$$

$$|f(\varphi(t))| \leq M$$

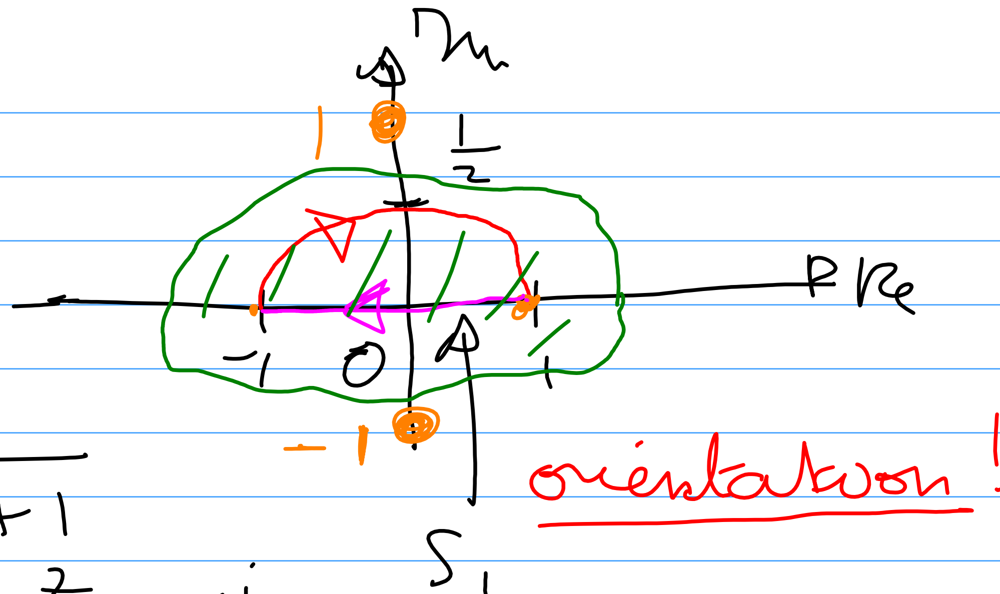
$$z = a + ib \qquad |z| = \sqrt{a^2+b^2}$$

$$|z| = e^{-i\arg(z)}\cdot z$$

$$|z|\cdot\underline{e^{i\arg(z)}} = z$$

(2, 3) 1)

a)



$$f(z) = \frac{1}{z^2+1}$$

$z = i$, $z = -i$

$$\int_K \frac{1}{z^2+1}\, dz = 0$$

$$\int_{K_1 + S_1} \frac{1}{z^2+1}\, dz = 0$$

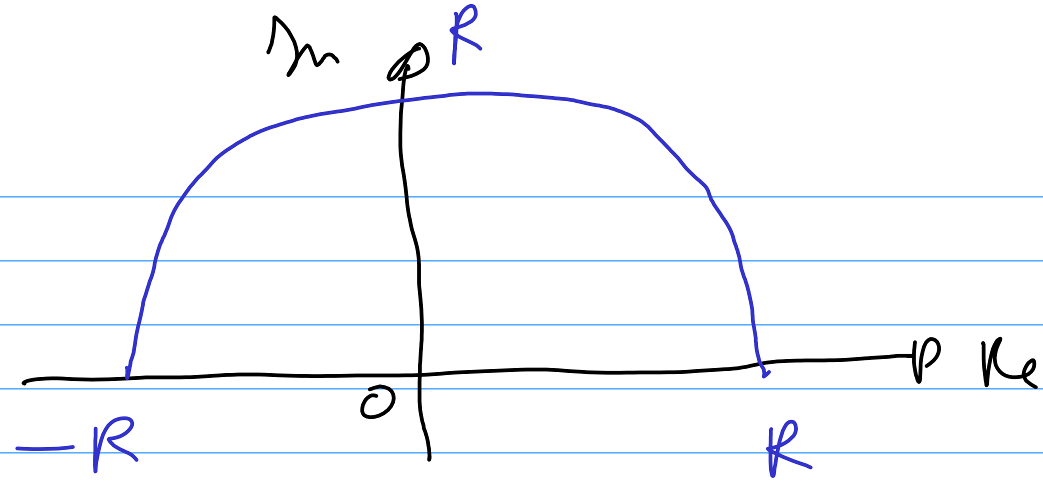
$$\int_{K_1} \frac{1}{z^2+1}\, dz = -\int_{S_1} \frac{1}{z^2+1}\, dz$$

$$= \int_{-1}^{+1} \frac{1}{1+x^2}\, dx = \left[\arctan(x)\right]_{-1}^{1}$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

b) $|z| = R$ $\quad R \to \infty$

$\operatorname{Im}(z) \geq 0$



$$\left|\int_{\substack{|z|=R \\ \mathrm{Im}\, z \geq 0}} \frac{1}{1+z^2}\,dz\right| \leq \frac{\pi \cdot R}{R^2-1} \to 0 \text{ if } R\to\infty$$

$$\left|\int_C f(z)\,dz\right| \leq M\cdot L$$

$$1+|z|^2$$

$$\underbrace{|1+z^2|}\ \cancel{\leq}\ \geq |1-|z|^2|$$

$$= \underbrace{|1-R^2|} = R^2-1$$

$$|-2| = 2$$

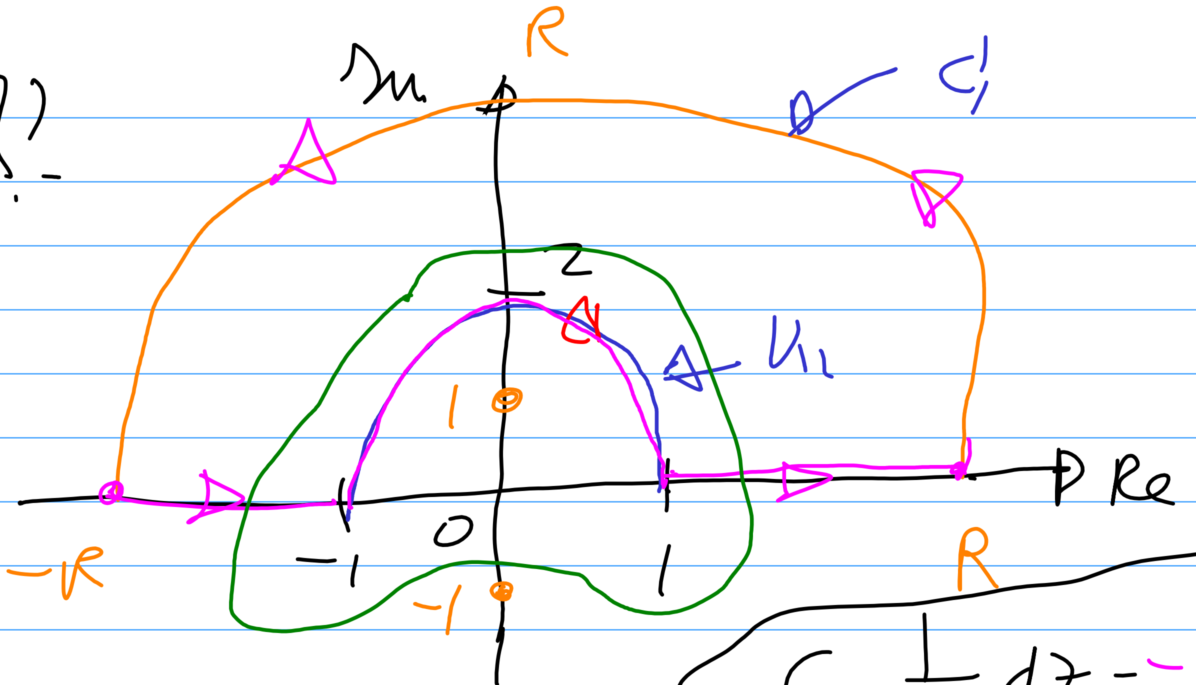
$$|1+z^2| \geq R^2-1$$

$$\frac{1}{|1+z^2|} \leq \frac{1}{(R^2-1)}$$

$$3>2$$

$$\frac{1}{3} < \frac{1}{2}$$

c) ??



$$\int_{C_1'} \frac{1}{1+z^2}\,dz = 0$$

$$\int_{U_{1,2}} \frac{1}{1+z^2}\,dz = -\frac{\pi}{2}$$

$$\int_{-R}^{-1} \frac{1}{1+x^2}\,dx + \int_{K_2} \frac{1}{1+z^2}\,dz + \int_{1}^{R} \frac{1}{1+x^2}\,dx + \int_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} \frac{1}{1+z^2}\,dz = 0$$

$\frac{\pi}{4}$

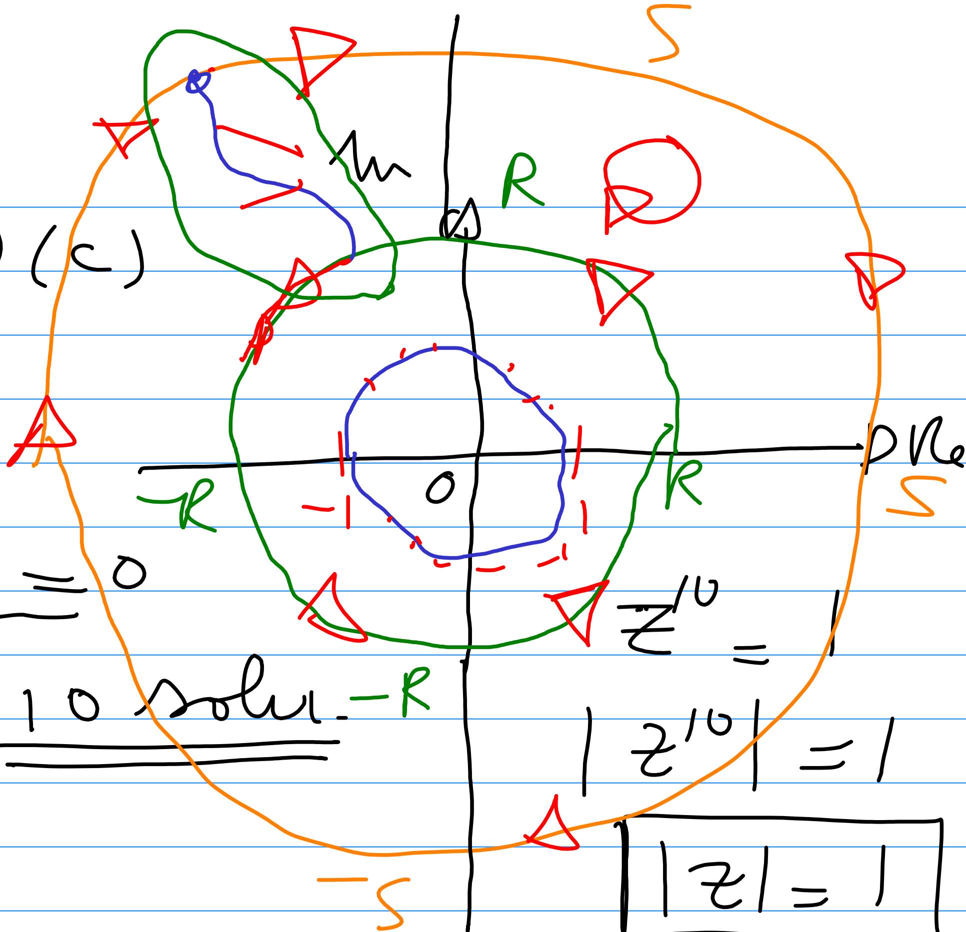
$\frac{\pi}{4}$

$(R \to \infty)$ $\to 0$

$$\int_{1}^{\infty} \frac{1}{1+x^2}\,dx = \left[\arctan(x)\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(2.3) 3) (c)

(a)



$\underbrace{z^{10} - 1 = 0}$ $\to$ $\underline{\underline{10 \text{ solu.}}}$

$z^{10} = 1$

$|z^{10}| = 1$

$\boxed{|z| = 1}$

$R > 1$

$$\int_{|z|=R} \frac{z^n}{(z^{10}-1)}\, dz = 0$$

(R is fixed) $R = 10^{10}$

$$\left| \int_{|z|=S} \frac{z^n}{z^{10}-1}\, dz \right| \leq \frac{S^n \cdot 2\pi S}{S^{10}-1} =$$

$$|z^{10} - 1| \geq (S^{10} - 1) \qquad \frac{S^{n+1} \cdot 2\pi}{S^{10}-1}$$

$$\frac{1}{|z^{10}-1|} \leq \frac{1}{S^{10}-1}$$

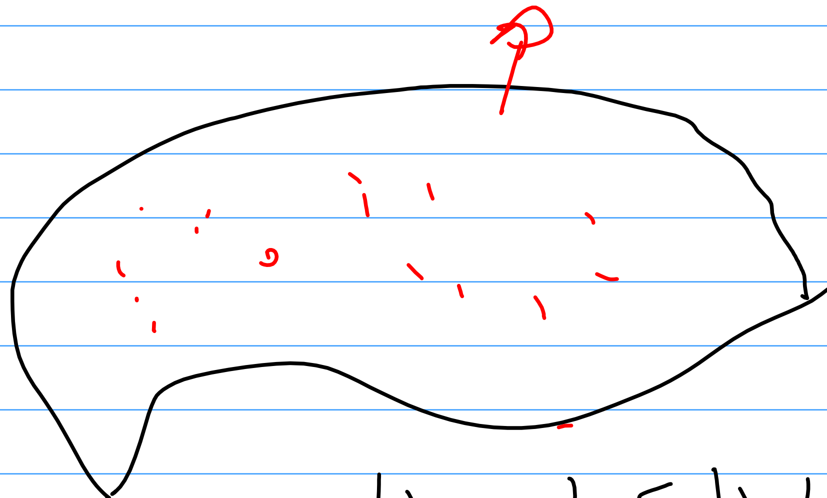
$0 \le n \le 8$

$1 \le n+1 \le 9$

$$\frac{s^{n+1}}{s^{10}-1} \qquad \frac{s^{9}}{s^{10}-1} \longrightarrow 0$$

$(s \to \infty)$

$n = 9$



$$|x+y| \le |x|+|y|$$

$$|z^{10}-1| \le |z^{10}|+1$$

$$\frac{1}{|z^{10}-1|} \ge \frac{1}{1+|z|^{10}} \quad ?$$

$$\left| \int_{|z|=R} \frac{z^n}{z^{10}-1}\,dz \right| \le M \cdot L$$

$$|x+y| \geq ||x|-|y||$$

$$|z^{10}-1| \ \cancel{\leq}\ \geq$$

$$|z^{10}-1| \geq ||z^{10}|-1|$$

$$\frac{1}{|z^{10}-1|} \leq \frac{1}{||z^{10}|-1|} = \frac{1}{(R^{10}-1)}$$

$R^{10}-1>0$

$$|z^{10}| = |z|^{10} \qquad |z| = R \quad (R>1)$$

(2.5) (3)

$$|z| < \frac{1}{2} \qquad \sum_{n=1}^{\infty} \frac{1}{(z^2-n^2)}$$

$$|z^2-n^2| \geq |\underbrace{|z^2|-n^2}| = n^2-|z^2|.$$

$$|z| < \frac{1}{2}$$

$n=1, 2, 3, \ldots$

$$-1\cdot\left(|z|^2 < \frac{1}{4}\right) \qquad -|z|^2 > -\frac{1}{4}$$

$$|z^2-n^2| \geq \left(n^2-\frac{1}{4}\right)$$

$$\frac{1}{|z^2 - n^2|} \le \frac{1}{\left(n^2 - \frac{1}{4}\right)}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}}}_{\sum \frac{1}{n^2} \text{ conv.}} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)} \sim \frac{1}{n^2}$$

$$\left| \sum_{n=1}^{\infty} \frac{1}{(z^2 - n^2)} \right| \le$$

$$\sum_{n=1}^{\infty} \frac{1}{|z^2 - n^2|} \le \sum_{n=1}^{\infty} \frac{1}{\left(n^2 - \frac{1}{4}\right)}$$

the case if $|z| < \frac{1}{2}$ $\sim \frac{1}{n^2}$

(2.5) (8)

(d) $f(z) = \cosh(z) = \dfrac{e^{z} + e^{-z}}{2}$

$\underline{a = \left(\frac{\pi}{2} i\right)}$, $\underline{z = a + h}$

$f(a+h) = \dots \quad \underline{h}$

$h = \underline{(z - a)}$

$\frac{1}{2}\left(e^{(\frac{\pi}{2} i + h)} + e^{-(\frac{\pi}{2} i + h)}\right) =$

$\underbrace{e^{\frac{\pi}{2} i}}_{i} \cdot e^{h} + \underbrace{e^{-\frac{\pi}{2} i}}_{-i} \cdot e^{-h} =$

$i e^{h} - i e^{-h} =$

$i \sum_{n=0}^{\infty} \frac{h^n}{n!} - i \sum_{n=0}^{\infty} \frac{(-h)^n}{n!}$

$$h^n - (-h)^n = 0 \qquad n = \text{even}$$

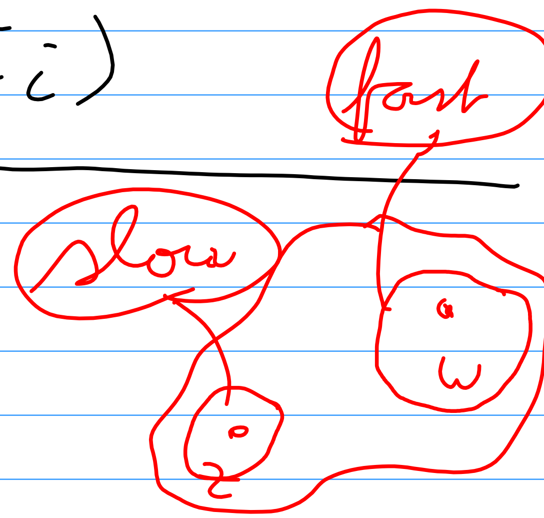
$$h^n - (-h)^n = 2 \cdot h^n \qquad n = \text{odd}$$

$$i \sum_{n=0}^{\infty} \frac{h^{2n+1}}{(2n+1)!}$$

$$h = \left(z - \frac{\pi}{2} i\right)$$

(2.5)(4)

$$\sum_{n=1}^{\infty} \frac{(1-e^{-z})^n}{n}$$



u. c. on U?? What to do??

uniform convergence??

→ "equal"

$$U = \{ z \in \mathbb{C} \mid |1 - e^{-z}| < \tfrac{1}{2} \}$$

$$\left| \sum_{n=1}^{\infty} \frac{(1-e^{-z})^n}{n} \right| \le$$

$$\sum_{n=1}^{\infty} \frac{|1-e^{-z}|^n}{n} \le$$

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n}$$ conv.

$$\alpha = \sum_{n=1}^{\infty} a_n(z) \quad \text{conv}$$

$\varepsilon > 0 \rightsquigarrow \exists N(\varepsilon, z)$ $N(\varepsilon)$

$$\left| \sum_{n=1}^{N(\varepsilon)} a_n - \alpha \right| < \varepsilon$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad 0 < x < \frac{1}{9}$$

(1.3.15)

$\underline{\sum f_n(z)}$ $\qquad |f_n(z)| \le a_n$

$$\sum_{n=1}^{\infty} a_n \quad \underline{\underline{\text{conv}}}$$

then $\displaystyle\sum_{n=1}^{\infty} f_n(z)$ $\underline{\underline{\text{u.c}}}$

$$f_n(z) = \frac{(1-e^{-z})^n}{n}$$

$$a_n = \left(\frac{\left(\frac{1}{2}\right)^n}{n}\right)$$

use $\underline{\underline{1.3.15}}$ $\rightsquigarrow$ $\underline{\underline{\text{u.c. on } U}}$

b) $F(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(1-e^{-z})^n}{n}$

What to ~~put~~ let away?

__Derivative:__

$$F'(z) = \frac{d}{dz}\left(\sum_{n=1}^{\infty} \frac{(1-e^{-z})^n}{n}\right)$$

$$= \sum_{n=1}^{\infty} (1-e^{-z})^{(n-1)} \cdot (e^{-z}) =$$

$$e^{-z} \cdot \sum_{n=1}^{\infty} (1-e^{-z})^{(n-1)} =$$

$$e^{-z} \cdot \underbrace{\sum_{k=0}^{\infty} (1-e^{-z})^{k}} =$$

$$(|1-e^{-z}|<\tfrac{1}{2}) \qquad \frac{e^{-z}}{1-(1-e^{-z})} = 1$$

$$\underline{F'(z) = 1} \qquad F(z) = z + C$$

$$\underline{F(0) = 0} \qquad \Rightarrow C = 0$$

$$F(z) = z$$

$$e^{-nz} = (e^{-z})^n$$

$$\sum \frac{w^n}{n}$$