CAI-201019:

(2019) - 6)

$$\int_{-\infty}^{+\infty} \frac{1-\cos(x)}{x^2(1+x^2)}\,dx$$

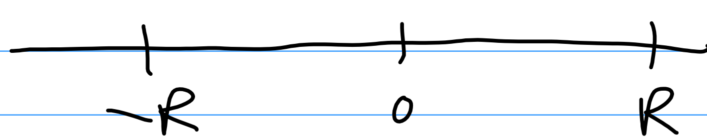
$$\frac{(1-\cos(x))}{x^2} = \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6)\right)}{x^2}$$

$$= \frac{+\frac{x^2}{2}-\frac{x^4}{4!}+O(x^6)}{x^2} =$$

$$= \frac{1}{2} - \frac{x^2}{4!} + O(x^4)$$

$x=0$: removable

$$\int_{-\infty}^{+\infty} \frac{1-\cos(x)}{x^2(1+x^2)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz$$

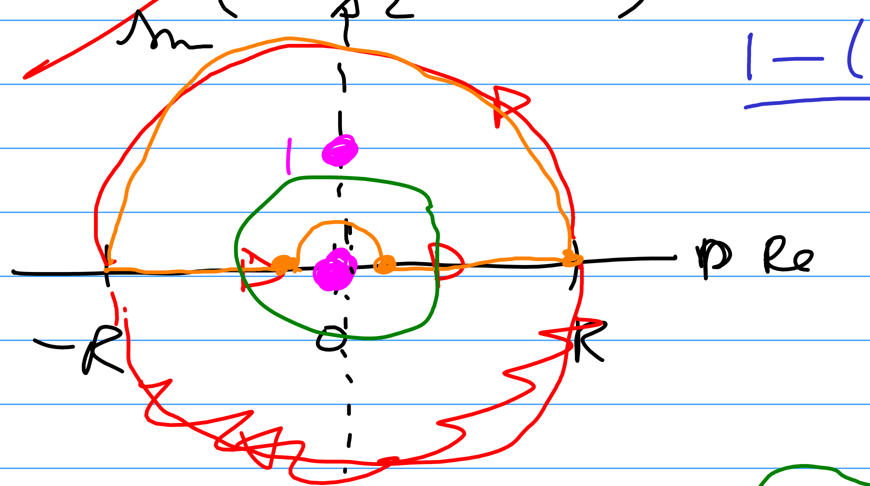


$$\cos(x) = \mathrm{Re}\,(e^{ix})$$

$$\int_{-R}^{R} \frac{1-\mathrm{Re}(e^{ix})}{x^2(1+x^2)}\,dx = \mathrm{Re}\left(\int_{-R}^{R} \frac{1-e^{ix}}{x^2(1+x^2)}\,dx\right)$$

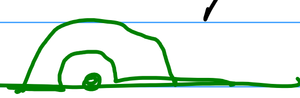
$$\frac{ix}{x^2} = \frac{1}{x}$$

$$\cos(x) = \left(\frac{e^{ix}+e^{-ix}}{2}\right)$$

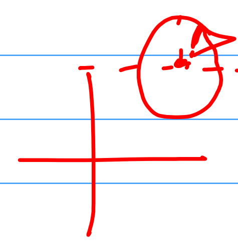


$$\frac{1-(1-ix-\dots)}{x^2}$$

$y \geq 0$



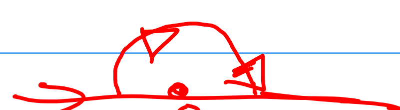
$$\frac{1}{2}\,\mathrm{Res}_{z=0}\left(\frac{1-e^{ix}}{x^2(1+x^2)}\right) =$$



$$\frac{1-(1-ix+O(x^2))}{x^2(1+x^2)} =$$

$$\rightsquigarrow \frac{+ix+O(x^2)}{x^2(1+x^2)} \simeq \frac{i}{x} + \cdots$$

? Res ?



$\boxed{z=0} \Rightarrow \boxed{-\frac{i}{2}}$

$$\frac{1-e^{ix}}{x^2(x-i)(x+i)} =$$

Res $z = i$?

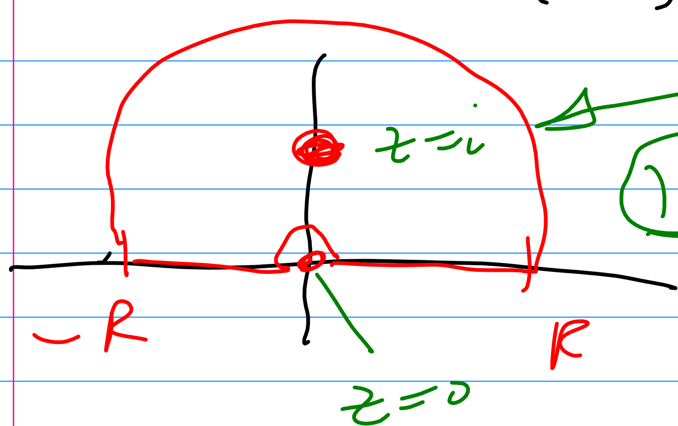
$(z-i)$

$(z-i)\cdot f(z) =$

$$\lim_{z\to i} \cancel{(z-i)}\cdot \frac{1-e^{iz}}{z^2\cancel{(z-i)}(z+i)} = \left(\frac{1-e^{-1}}{-2\cdot i}\right)$$

$i^2(2i) = -2i$

Res in $z=i$



$|z| = R$

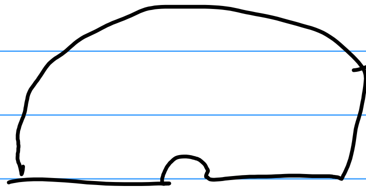
??

$$\left|\frac{1-e^{iz}}{z^2(1+z^2)}\right| \le \frac{2}{R^2(R^2-1)} \approx \frac{1}{R^4}$$

$|z| = R$ $\quad z = x+iy \quad y>0$

$|e^{iz}| = |e^{ix-y}|$

$$\int_{C_R^+} f(z)\,dz = \pi \cdot R \cdot \frac{2}{R^2(R^2-1)} \to 0 \quad \underline{\text{if } R \to \infty}$$

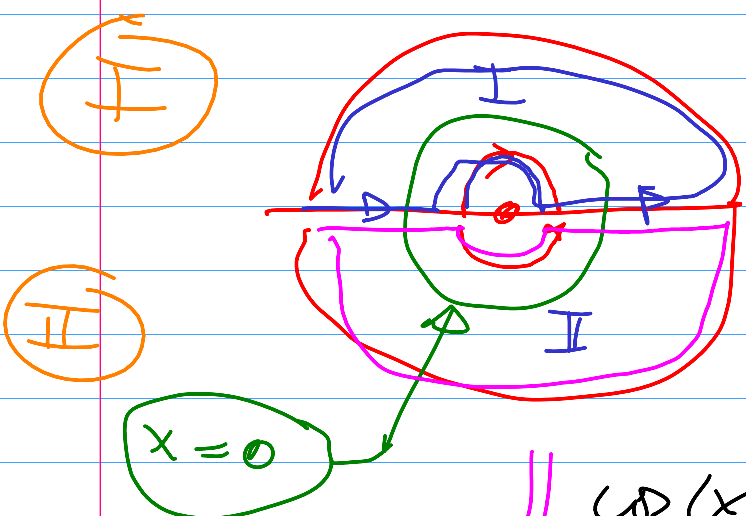


$\to 0$ <u>if $R \to \infty$</u>

<u>sum residues</u>: $2\pi i\left(\frac{-i}{2} + \frac{1-e^{-1}}{(-2i)}\right)$

$$= \pi + -\pi\cdot(1-e^{-1}) = \pi \cdot e^{-1} = \frac{\pi}{e}$$

$$\frac{1-\cos(x) \geq 0}{x^2(1+x^2) \geq 0} \qquad \int_{-\infty}^{\infty} \ldots\, dx \geq 0$$



$f(x) =$

$x \to 0$ remov. sing.

$\cos(x) = \mathrm{Re}(e^{ix})$

<u>control with wolfram alpha</u>! integrate(f(x), x, -inf, +inf)

- Week 8: Fourier → last lecture!!
  ask Luc!!

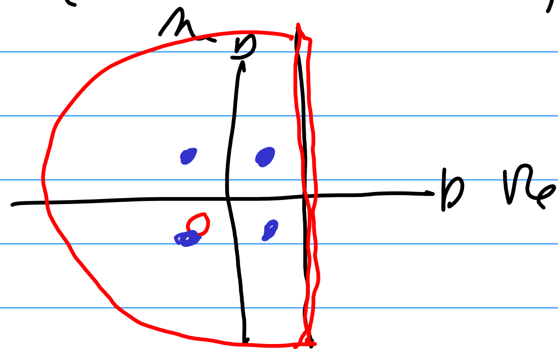
For exam exercises,
see my website!!
Other-Exec-and-Solutions

(3.5)(1 b)

$$F(s) = \left(\frac{1}{1+s^4}\right)$$

$$f(t) = \frac{1}{2\pi i} \cdot \int_L F(s) \cdot e^{s \cdot t} \, ds$$

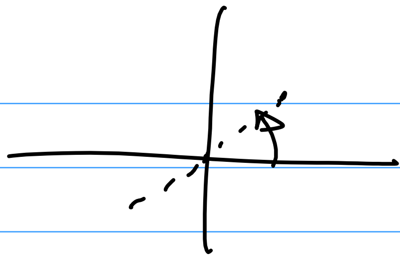
$$L = \{ s = \sigma + i\omega \mid \sigma > \alpha \quad \omega \in \mathbb{R} \}$$



$$1 + s^4 = 0$$

$$s^2 = \pm i$$

$$s^2 = i$$



$$s^2 = i$$
$$s^2 = -i$$

$$e^{\frac{\pi}{4}i} \qquad -e^{\frac{\pi}{4}i}$$

$$1 + s^4 = 0$$

$$e^{-\frac{\pi}{4}i} \qquad -e^{-\frac{\pi}{4}i}$$

$$e^{i\frac{7\pi}{\cancel{8}}}$$

$$\cancel{4 \cdot \frac{7}{8}\pi} = \cancel{\frac{2\pi}{8}\pi}$$

$$\cancel{\left(\frac{7}{8}\pi\right)4 = \frac{7}{2}\pi i)} \quad e \quad \cancel{\pi - 1}$$

$$\boxed{e^{\frac{\pi}{5}i},\; e^{\frac{3\pi}{4}i},\; e^{\frac{5}{4}\pi i},\; e^{\frac{7}{4}\pi i}}$$

$$\left(e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i} = -1\right)$$

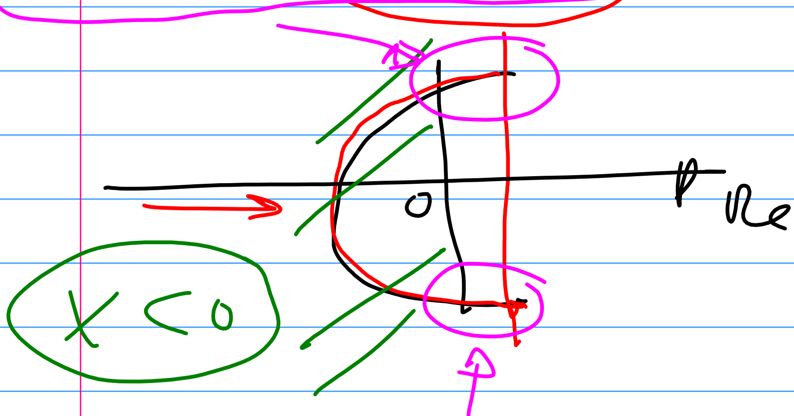
$$\frac{1}{2\pi i} \cdot \int \frac{1}{(s^4+1)} \cdot e^{+st} \, \underline{ds}$$

?

t fixed

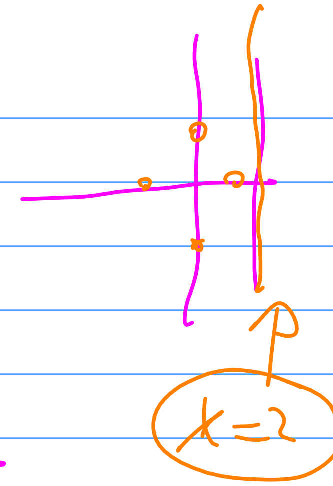
$$\pi R = L$$

$$\left|\frac{1}{1+s^4}\right| \le \frac{1}{(R^4-1)}$$



$s = (x + iy) \qquad (x < 0)$

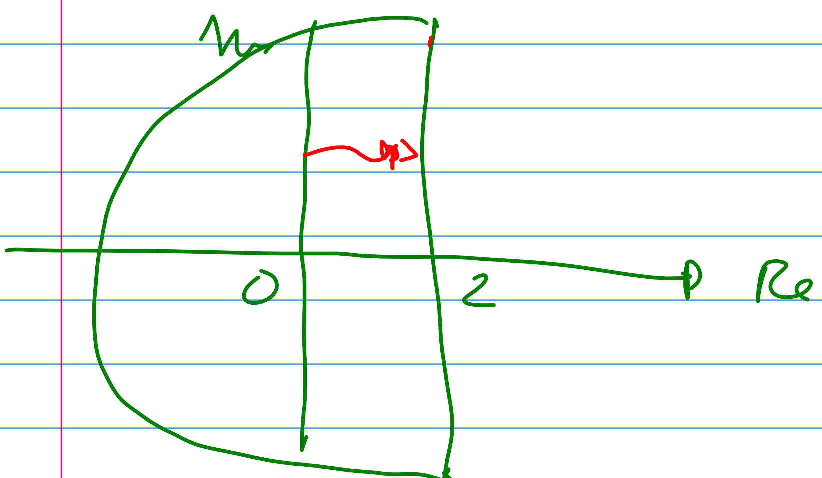
$(x > 0) \qquad t \geq 0$



$$|e^{st}| \leq e^{\mathrm{Re}(s)\cdot t} \leq e^{2t}$$

$$|s| \geq (R-2)$$

$$|F(s)| \leq \frac{1}{((R-2)^4 - 1)}$$



t fixed

arbitrary $t \geq 0$

$$\int \cdots f(s) \cdot e^{s \cdot t} \, ds = f(t)$$

Res

$$\frac{1}{(s - e^{\frac{\pi}{4} i})(s - e \cdots\cdots)} = g(s)$$

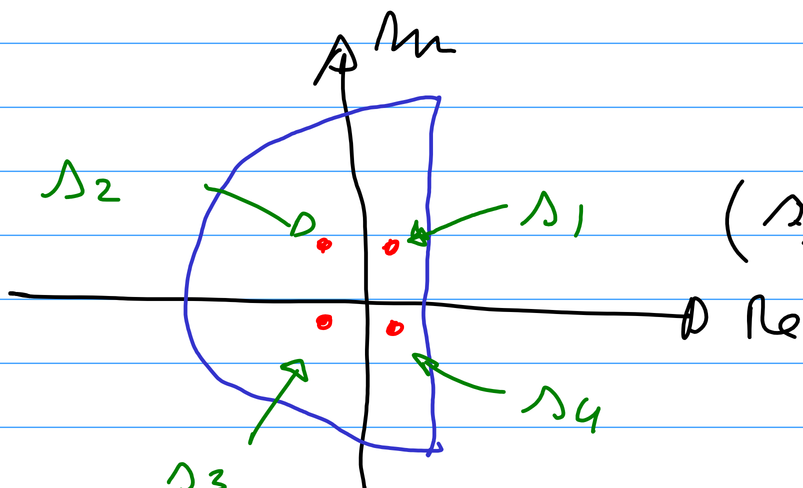
$s^4 + 1 = 0 \qquad (s - s_1)(s - s_2)(s - s_3)(s - s_4) = 0$

$$\lim_{s \to s_i} (s - s_i) \cdot g(s) = \ldots$$

|| **Just follow example (3.5.1)**

Ex. (3.5.1) $\quad F(s) = \dfrac{1}{(s^4 - 1)}$

Exercise we have to do: $\quad F(s) = \dfrac{1}{(s^4 + 1)}$



$$\left(s_k = e^{i\frac{\pi}{4}(1+2k)} \right) \quad k = 0, 1, 2, 3$$

$$\left(\frac{e^{st}}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)} \right)$$

$$\lim_{s \to s_i} \left(\frac{(s - s_i) \cdot e^{st}}{(s - s_1) \cdots (s - s_4)} \right)$$

(from residues)

(it looks me a terrible thing)

$\left(s_i = \left(\pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}} \right) \right)$

(maybe wolfram alpha; some help?)

$$\left\{ e^{-\frac{t}{\sqrt{2}}} \left(\left(e^{\sqrt{2}t} + 1 \right) \cdot \sin\left(\frac{t}{\sqrt{2}} \right) - \left(e^{\sqrt{2}t} - 1 \right) \cdot \cos\left(\frac{t}{\sqrt{2}} \right) \right) / (2\sqrt{2}) \right\}$$

Let it never be an exam exercise!
To correct such kind of things??