Exercise 5 exam jan. 2020.

$$f_n(z) = \frac{1}{z^n + 1}$$

(a) $z^n = -1 \Rightarrow \quad |z^n| = |-1| \Rightarrow$

$z = Re^{i\varphi} \qquad |z| = 1 \rightarrow \underline{\underline{R = 1}}$

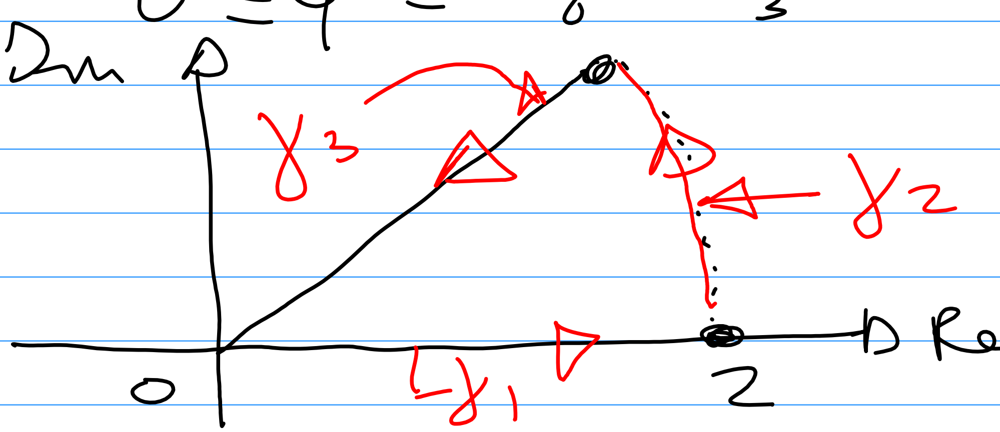
$n\varphi = \arg(z^n) = \arg(-1) = +\pi + k \cdot 2\pi$

$$\underline{\underline{\varphi_n = \left(+\frac{\pi}{n} + k \cdot \frac{2\pi}{n}\right)}}$$

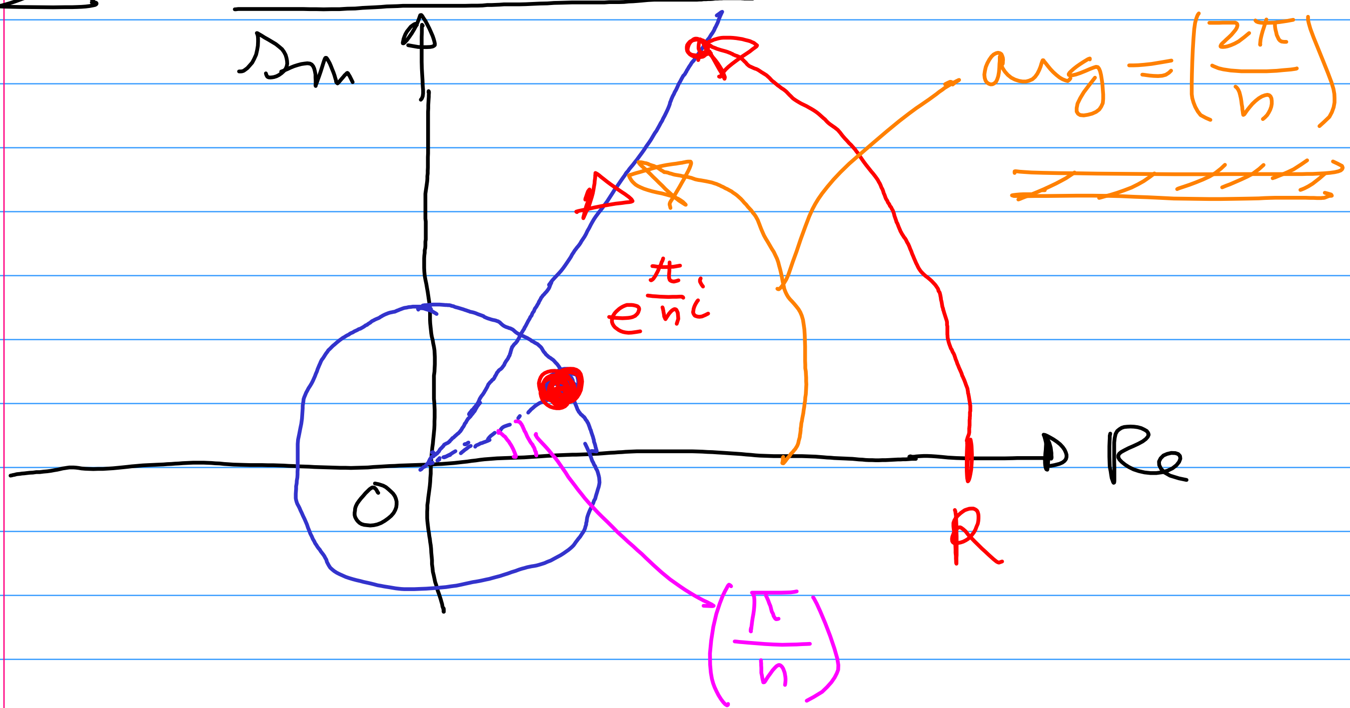
$$\boxed{z_k = \exp\left(\left(\frac{\pi}{n} + k \cdot \frac{2\pi}{n}\right) i\right) \quad k = 0, \dots, (n-1)}$$

b) $R = 2, \; n = 6 \rightarrow$

$$0 \leq \varphi \leq \frac{2\pi}{6} = \frac{1}{3}\pi$$



c) $\underline{\underline{n \geq 2, \quad R > 1}}$;



$$z^n + 1 = (z - z_0)(\cdots\cdots)\cdots\cdots(z - z_{n-1})$$

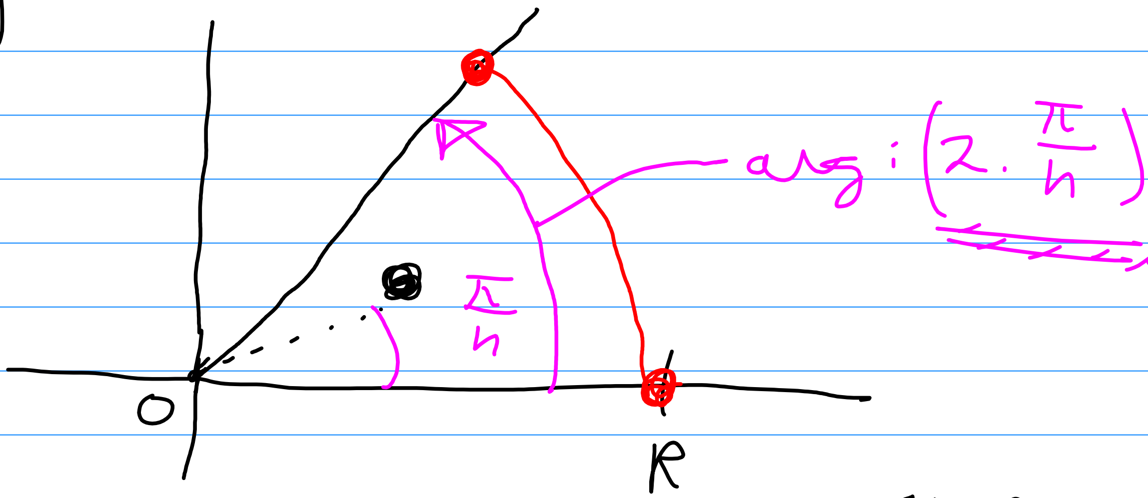
$$\lim_{z \to z_0}\left(\frac{(z - z_0)}{(z^n+1)}\right) = \left(\frac{1}{(z_0 - z_1)(\cdots\cdots)(z_0 - z_{n-1})}\right)$$

$$\int_{\Gamma_R} \frac{1}{(z^n+1)}\,dz = 2\pi i \operatorname*{Res}_{z=z_0}\left(\frac{1}{z^n+1}\right) =$$

$$\frac{2\pi i}{(z_0 - z_1)\cdots\cdots(z_0 - z_{n-1})}$$

no idea how to calculate that product. Maybe with the answer of (d) we can find it!

d)



$$\int_{\gamma_{3,R}} \frac{1}{z^n+1}\, dz = *$$ // fill in param.

$$z = (R-s)\, e^{2\left(\frac{\pi}{n}\right)i} \qquad 0 \le s \le R$$

$$* = \int_0^R \frac{1}{\left((R-s)^n \cdot \underbrace{e^{2\pi i}}_{(=1)} + 1\right)} \left(-e^{\frac{2\pi}{n} i}\right) ds =$$

$$t = (R-s) \rightsquigarrow dt = -ds \quad \begin{pmatrix} s=0 \to t=R \\ s=R \to t=0 \end{pmatrix}$$

$$= \int_R^0 \frac{1}{(t^n+1)} \cdot \left(-e^{\frac{2\pi}{n} i}\right)(-dt) =$$

$$-\int_0^R \frac{e^{\frac{2\pi}{n}i}}{t^n+1}\,dt$$

$$\left|\int_{\gamma_{2R}} \frac{1}{z^n+1}\,dz\right| \leq \frac{\pi R}{(R^n-1)}$$

$$\to 0 \quad \text{if } R \to \infty$$

Take integral of part (c) and let $R \to \infty$:

$$\left(1-e^{\frac{2\pi}{n}i}\right)\cdot\int_0^\infty \frac{1}{x^n+1}\,dx =$$

$$\frac{2\pi i}{(z_0-z_1)\cdot\;\cdots(z_0-z_{(n-1)})}$$

$$\int_0^\infty \frac{1}{x^n+1}\,dx = \frac{1}{\left(1-e^{\frac{2\pi}{n}i}\right)}\cdot\frac{2\pi i}{(z_0-z_1)\cdots(z_0-z_{n+1})}$$

$$= \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)} \quad ?$$

$$\frac{e^{-\frac{\pi}{n}i}}{e^{-\frac{\pi}{n}i}} \cdot \frac{1}{\left(1-e^{\frac{2\pi}{n}i}\right)} = \frac{e^{-\pi/n\, i}}{\left(e^{-\frac{\pi}{n}i} - e^{\frac{\pi}{n}i}\right)} =$$

$$\frac{\left(e^{-\frac{\pi}{n}i}/2i\right)}{-\left(\frac{e^{\frac{\pi}{n}i} - e^{-\frac{\pi}{n}i}}{2i}\right)} = \frac{e^{-\frac{\pi}{n}i}/2i}{-\sin\left(\frac{\pi}{n}\right)}$$

our answer

given answer

$$\frac{e^{-\frac{\pi}{n}i}/\cancel{2i}}{-\cancel{\sin(\pi/n)}} \cdot \frac{\cancel{2\pi i}}{(z_0-z_1)\cdots(z_0-z_{n-1})} \overset{?}{=} \frac{\left(\frac{\pi}{n}\right)}{\cancel{\sin(\pi/n)}}$$

$$-\frac{\cancel{\pi}\cdot e^{-\frac{\pi}{n}i}}{(z_0-z_1)\cdots\cdots(z_0-z_{n-1})} \overset{?}{=} \frac{\cancel{\pi}}{n} \qquad z_0 = e^{\frac{\pi}{n}i}$$

$$-n\cdot\left(e^{-\frac{\pi}{n}i}\right) \overset{?}{=} (z_0-z_1)\cdots\cdots(z_0-z_{n-1})$$

??

$$-\frac{n}{(z_0)} = (z_0-z_1)\cdots\cdots(z_0-z_{n-1})$$

$$|z_k| = 1 \qquad z_1 = z_0^3,\ z_2 = z_0^5 \cdots$$

$$\left(z_0 = e^{\frac{\pi}{n}i},\ z_1 = e^{\left(\frac{\pi}{n}+\frac{2\pi}{n}\right)i},\ \cdots\cdots\right)$$

$$\left(z_0^n + 1 = 0\right)$$

I just wrote something down

and see;

$$(z - z_0)(z - z_1)\cdots(z - z_{n-1}) = z^n + 1$$

$$(z - z_1)\cdots\cdots(z - z_{n-1}) = \left(\frac{z^n + 1}{z - z_0}\right)$$

$$f_n(z) = z^n + 1$$
$$f_n(z_0) = 0$$

$$\lim_{n \to \infty} \left(\frac{f_n(z) - f_n(z_0)}{(z - z_0)}\right) = f_n'(z_0)$$

$$= n \cdot z_0^{(n-1)}$$

$$\left(\begin{array}{l} z_0^n + 1 = 0 \\ \Rightarrow z_0^n = -1 \end{array} \right)$$

$$= n(z_0^n) \cdot \left(\frac{1}{z_0}\right)$$

$$(z_0 - z_1)\cdots\cdots(z_0 - z_{n-1}) = -\frac{n}{z_0}$$

!! We have it !!

it is now 15:00 hour