Ex.(2.3) (3) :

First of all $R > 1$, see first line of exercise,
and $n \in \mathbb{N}$.

$$A_n = \int_{|z|=R} \frac{z^n}{z^{10}-1}\, dz$$

In section (2.3), you can read about Cauchy integral theorem

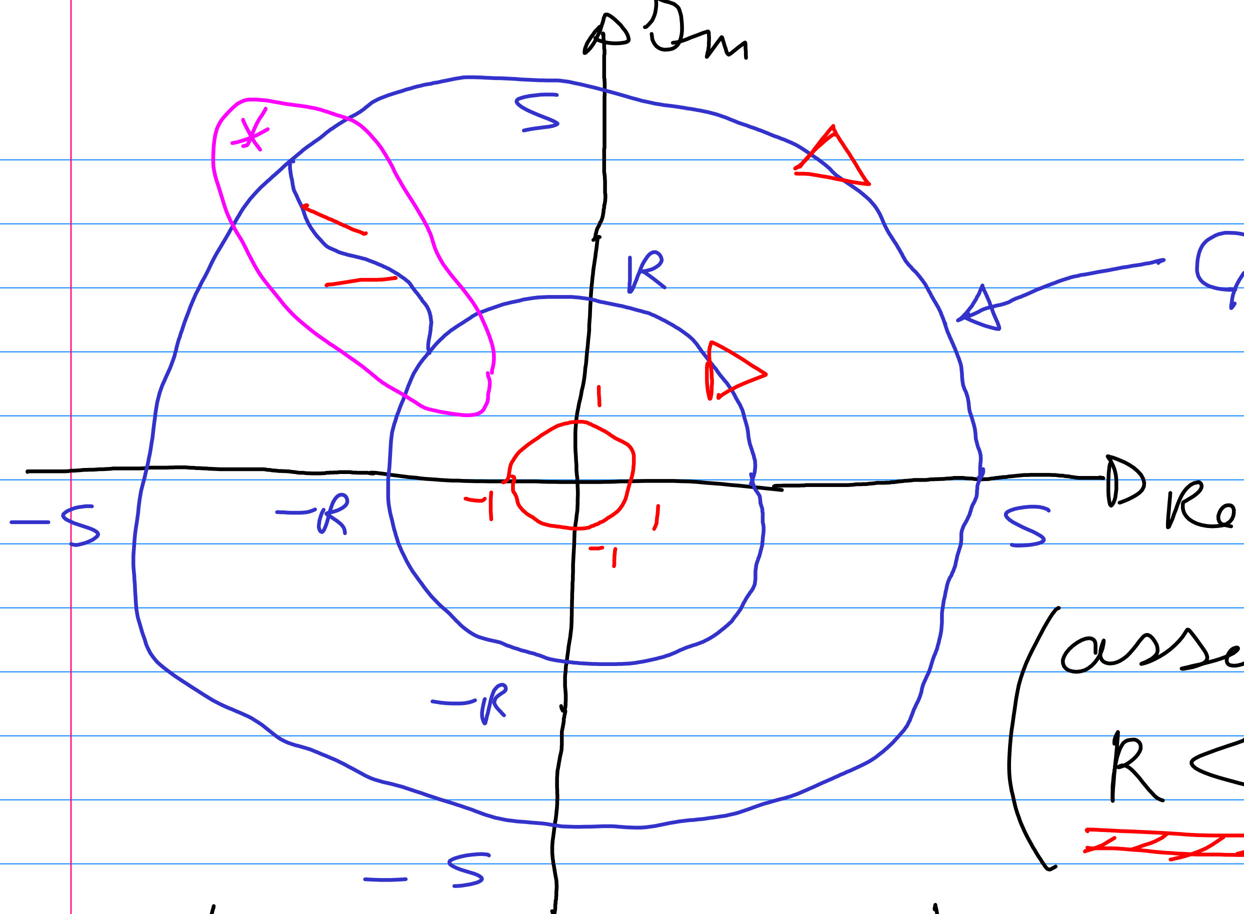
Important is the following:

Let C be a closed contour (Jordan curve) and let f be holomorphic at the inner side of C and on C!

then $\oint_{C} f(z)\,dz = 0$

$R > 1$ and is fixed!

a) $0 \le n \le 8$: Let's construct a closed contour C, without the zeros of $z^{10} - 1 = 0$ in it!



(assume: $R < S$)

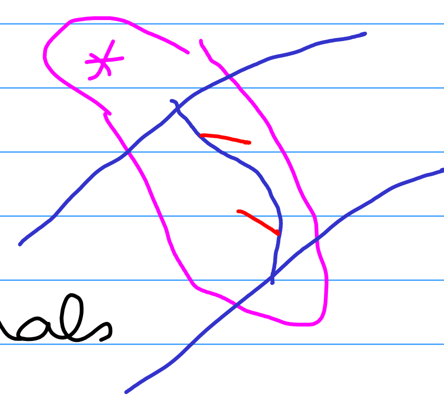
C' is a closed curve!

In the region $R \leq |z| \leq S$ there are no singularities.

C_R with a positive orientation!

so C_S has a negative orientation

See the curve(s) in:



they have opposite orientation, so the sum over these integrals becomes zero.

Using Cauchy Int. th. means that

$$A_n = \oint_{C_R} \frac{z^n}{(z^{10}-1)}\,dz = \oint_{C_S} \frac{z^n}{(z^{10}-1)}\,dz$$

pos. oriented pos. oriented

$$|A_n| = \left| \oint_{C_S} \frac{z^n}{z^{10}-1}\,dz \right|$$

$$\leq \frac{S^n \cdot 2\pi S}{(S^{10}-1)} = \frac{S^{n+1} \cdot 2\pi}{(S^{10}-1)}$$

$1 \leq n+1 \leq 9$ and

$$\lim_{S \to \infty} \frac{S^{(n+1)} \cdot 2\pi}{(S^{10}-1)} = 0$$

So: $A_n = 0$ for $0 \leq n \leq 8$.

b) $\underline{n = 9?}$ If we would do the same as in a) then we get

$|A_g| \leq 2\pi$. Maybe good, but asked is A_g and not some upper boundary.

But still we have that

$$A_g = \oint_{C_S} \frac{z^9}{(z^{10}-1)} dz \quad (\text{pg. } 3)$$

Maybe with some parametrisation

$$z = S \cdot e^{it}, \ dz = i \cdot S e^{it} dt, \ 0 \leq t \leq 2\pi$$

positive orientation!

$$A_g = \int_0^{2\pi} \frac{(Se^{it})^9 \cdot i \cdot Se^{it}}{(Se^{it})^{10} - 1} dt$$

$$= i \cdot \int_0^{2\pi} \frac{S^{10} e^{i \cdot 10 \cdot t} - 1 + 1}{(S^{10} \cdot e^{i \cdot 10 t} - 1)} dt$$

$$= i \int_0^{2\pi} \left(1 + \frac{1}{(S^{10} e^{i 10 t} - 1)} \right) dt$$

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$$= 2\pi i + \frac{i}{S^{10}} \cdot \int_0^{2\pi} \frac{1}{\left(e^{i10t} - \frac{1}{S^{10}}\right)} dt$$

we know: $1 < R < S$ so

$$1 = |e^{i10t}| \neq \frac{1}{S^{10}}$$

The equality for A_9 holds for every $S > R$, so take limit for $S \to \infty$ and the result becomes: $\underline{\underline{A_9 = 2\pi i}}$

$$\left(\begin{array}{l} \left|e^{i10t} - \frac{1}{(S^{10})}\right| \geq \left(1 - \frac{1}{(S^{10})}\right) \quad (S > 1) \\ \frac{1}{\left|e^{i10t} - \frac{1}{S^{10}}\right|} \leq \frac{1}{1 - \frac{1}{(S^{10})}} \end{array} \right)$$

c) $A_{10m+k} = A_{10(m-1)+k}$?

$m \geq 1, \quad 0 \leq k \leq 9$

$$n = (10m + k)$$

$$\int_{|z|=R} \frac{z^{10m+k}}{z^{10}-1}\, dz =$$

$$\int_{|z|=R} \frac{z^{10} \cdot z^{10(m-1)+k}}{(z^{10}-1)}\, dz =$$

$$\int_{|z|=R} \frac{(z^{10}-1+1)\, z^{10(m-1+k}}{(z^{10}-1)}\, dz =$$

$$\int_{|z|=R} \left(z^{10(m-1)+k}\right) dz + A_{10(m-1)+k}$$

$$\left.\begin{matrix} m \geq 1 \\ 0 \leq k \leq 9 \end{matrix}\right\} \Longrightarrow \underline{10(m-1)+k \geq 0}$$

$$\Longrightarrow \int_{|z|=R} z^{10(m-1)+k}\, dz = 0 \quad (m \geq 1,\ 0 \leq k \leq 9)$$

so $A_{10m+k} = A_{10(m-1)+k}$.