Exercise (3.3) 4):

$$I(a) = \int_{-\infty}^{+\infty} e^{-x^2} \cdot \cos(2ax)\, dx \quad (a > 0)$$

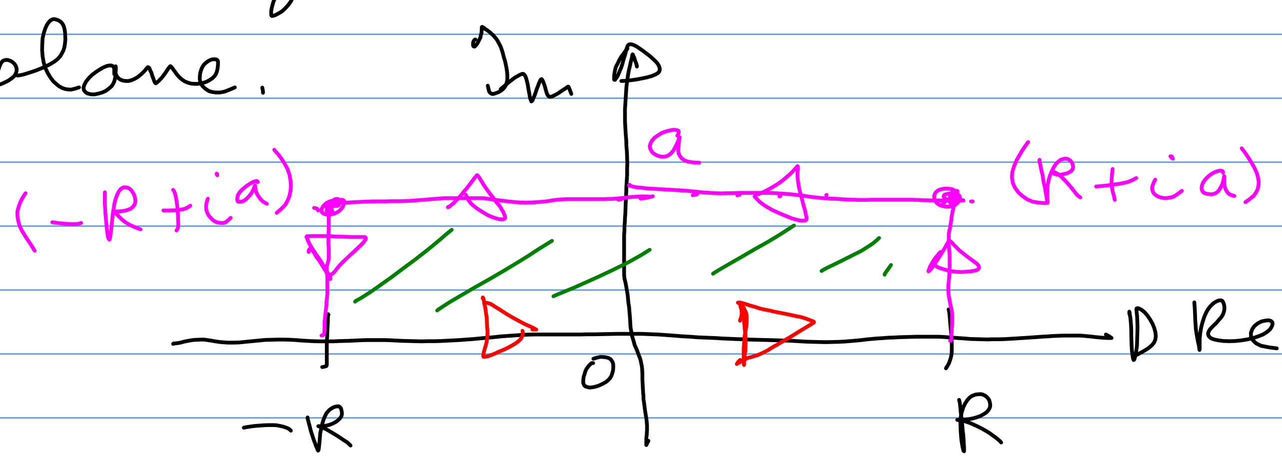
given: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

so $\lim_{R \to \infty} \int_{-R}^{R} e^{-x^2} dx = \sqrt{\pi}$

What can we do with

$$\int_{-R}^{R} e^{-x^2} dx \;??$$

- $\exp(-z^2)$ is holomorphic
- Let's go into the complex plane.



closed curve = contour C_1'

no singularities inside so

$$\int_{C_1'} e^{-z^2}\,dz = 0$$

(if $R \to \infty$: $\sqrt{\pi}$)

$$0 = \boxed{\int_{-R}^{R} e^{-x^2}\,dx} + \boxed{\int_{R}^{R+ia} e^{-z^2}\,dz} + \int_{R+ia}^{-R+ia} e^{-z^2}\,dz + \boxed{\int_{-R+ia}^{-R} e^{-z^2}\,dz}$$

If $z = R + i\cdot t$, $0 \leq t \leq a$,

then $\left|\exp(-z^2)\right| = \left|\exp(-(R+it)^2)\right|$

$$= e^{-R^2} \cdot \left|e^{-2Rit}\right| \cdot e^{t^2} \leq e^{-R^2} \cdot e^{t^2} \leq e^{-R^2} \cdot e^{a^2}$$

so; $\left| \int_{R}^{R+ia} e^{-z^2}\, dz \right| \le a \cdot e^{-R^2} \cdot e^{a^2}$

$\to 0$ if $R \to \infty$

Evenso for

$\left| \int_{-R}^{-R+ia} e^{-z^2}\, dz \right| \le a \cdot e^{-R^2} \cdot e^{a^2}$

$\to 0$ if $R \to \infty$

$$\int_{R+ia}^{-R+ia} e^{-z^2}\, dz = I(R,a)$$

Let $z = +t + ia \qquad -R \le t \le R$

$$I(R,a) = \int_{R}^{-R} e^{-(+t+ia)^2} \cdot (-dt)$$

$$= \int_{-R}^{R} e^{-(t^2 + 2iat - a^2)}\, dt =$$

$$e^{a^2} \cdot \int_{-R}^{R} e^{-t^2 - 2iat}\, dt =$$

$$e^{a^2}\cdot\int_{-R}^{R} e^{-t^2}\cdot\big(\underbrace{\cos(-2at)}_{\text{(even: } f(-x)=f(x)\text{)}}+i\,\underbrace{\sin(-2at)}_{\text{(odd function)}}\big)\,dt$$

Symm. interval

$\int \ldots = 0$

$$= e^{a^2}\cdot\int_{-R}^{R} e^{-t^2}\cdot\cos(2at)\,dt$$

Let $R\to\infty$:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dt = e^{a^2}\cdot\int_{-\infty}^{+\infty} e^{-t^2}\cdot\cos(2at)\,dt$$

So: $$\int_{-\infty}^{+\infty} e^{-x^2}\cdot\cos(2ax)\,dx = e^{-a^2}\cdot\sqrt{\pi}$$

(see figure)

$(\leq\sqrt{\pi})$