Lecture Notes about Complex analysis

The nicest thing would be if this could be read like a novel.

Every time I say to people "hi, hi", I hear "-1" in mind, so "$i * i = -1$".

The same with "hoj", "hoj" then "$j * j = -1$".

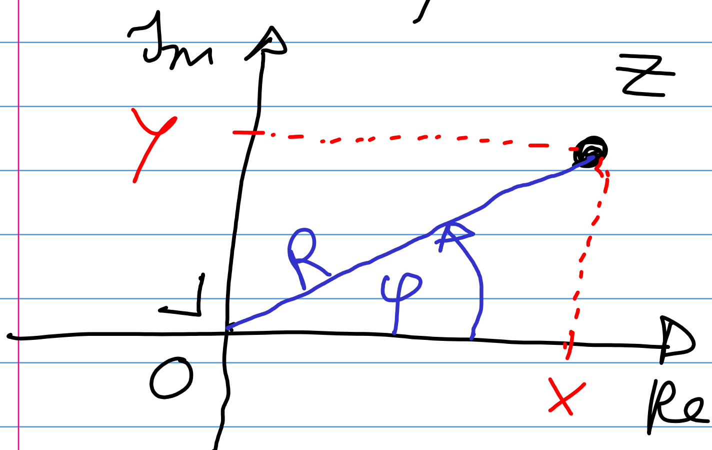
Internet and search:
www.win.tue.nl/~sjoerdr/

Read this for fun.

(I) Complex numbers:

1) Start with: $i^2 = -1$ (or $j^2 = -1$)

$z = x + iy = R \cdot \exp(i\varphi)$



$\begin{cases} R: \text{radius} \\ \varphi: \text{argument} \end{cases}$ $(R, \varphi \in \mathbb{R})$

$(z \in \mathbb{C})$

$x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$ both $\in \mathbb{R}$

real part, imaginary part

Adding and multiplying almost the same as in $\mathbb{R}$, only be careful with the $i^2 = -1$.

2) (R, φ) polar coordinates

$x = R \cdot \cos(\varphi)$, $y = R \cdot \sin(\varphi)$

- $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$

$|\exp(i\varphi)| = 1$ (unit circle of $\mathbb{C}$)

(can be read as a kind of definition)

3) Complex conjugate of z, written by $\overline{z}$

$$\overline{z} = \overline{(x + iy)} = x - iy = R \cdot \exp(-i\varphi)$$

$$z \cdot \overline{z} = |z|^2 = R^2$$

$$Re(z) = \frac{z + \overline{z}}{2}, \quad Im(z) = \frac{z - \overline{z}}{2 \cdot i}$$

$z = \overline{z} \Rightarrow Im(z) = 0$ Re-axis

$z = -\overline{z} \Rightarrow Re(z) = 0$ Im-axis

4) $Arg(z)$ versus $arg(z)$

There is made an appointment about the values of $Arg(z)$

$0 \leq Arg(z) < 2\pi$ or

$-\pi < Arg(z) \leq \pi$ or

something else, important is the length of interval of $Arg(z)$ is: 2π.

So $arg(z) = Arg(z) + k \cdot 2\pi, k \in \mathbb{Z}$.

5) Searching of zeros of polynomial

$p(z)$

order of $p \Rightarrow$ the maximum number of possible different solutions.

coëfficients of p are real then

if $p(z_0) = 0$ then also $p(\bar{z}_0) = 0$

With the zeros, you can <u>factorise p.</u> So searching zeros is also a method to find factors of p.

If $p(z_0) = 0 \Rightarrow$

$$p(z) = (z - z_0) \cdot q(z)$$

order of q, one lower than of p.

6) $(w)^n = (a + ib) \quad a, b \in \mathbb{R}$

$w \in \mathbb{C}$

Use of polar coordinates useful?

α) $|(w)^n| = |a+ib| \Rightarrow |w|$

β) $\arg(w^n) = \arg(a+ib) =$

$\mathrm{Arg}(a+ib) + k \cdot 2\pi$ with

$k \in \mathbb{Z}$: $0 \leq k \leq (n-1)$, or

$1 \leq k \leq n$ or ...

important: n successive k's.

(That gives the n solutions.)

$$\begin{cases} w_k = |w| \cdot \exp(i\varphi_k) \\ \varphi_k = \dfrac{\mathrm{Arg}(a+ib)}{n} + \left(\dfrac{k}{n}\right) \cdot 2\pi. \\ \quad 0 \leq k \leq (n-1) \end{cases}$$

If $w = (z - z_0) \Rightarrow$

$z_k = z_0 + w_k, k \in \mathbb{Z}, 0 \leq k \leq (n-1)$

7) To prove something?

α) Busy and you get some stupid equations, maybe useful to do something else.

β) Try to do something with the given information.

Play with it.

γ) Maybe $z = x + iy$ or

$z = R \cdot \exp(i \cdot \varphi)$ useful to use?

δ) Real and Imaginary part are often important of certain expressions?

ε) What part of the complex plane is used?

But once and for all:

Try to do something.

Too long busy, try another escercise, and then before the traffic light of the Bijenkorf you think: "Oh yes, I see....!"

(s.n.p.)

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8) Triangle inequalities maybe also good to mention:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$$

9) In $\mathbb{C}$ you can not compare numbers with each other so there is no $>$ or $<$!

$\boxtimes$

(II) Series

Sunday morning I always run and always cross the "Snelle Loop", some little river between Aarle-Rixtel and Gemert. There is a bench. During the Corona period, you heard only birds and the leaves of the trees, no noice of traffic and the sky was clear of white lines.

So sitting at that bench, I asked myself: "What do I do with series and why?". So somebody comes with $\sum_{n=1}^{\infty} c_n$, "what is asked?", "what we have to do?".

First of all: $\sum_{n=1}^{\infty} c_n = \lim_{N \to \infty} \sum_{n=1}^{N} c_n$,

so series: a limit of finite sums.
Maybe we can first look if the series converges absolutely? Take series of $|c_n|$, $\left|\sum_{n=1}^{\infty} c_n\right| \leq \sum_{n=1}^{\infty} |c_n|$

Let $S_N = \sum_{n=1}^{N} |c_n|$ then $S_{N+1} \geqslant S_N$
so the sequence $(S_N)_{N\in\mathbb{N}}$ increases.

If we can prove that the sequence $(S_N)_{N\in\mathbb{N}}$ is bounded, we have a bounded increasing sequence, so there excists a limit. We can tell to the people that the series: $\sum_{n=1}^{\infty} c_n$ escists. What comes out of it, that is a completely other question. Most of the time: prove first the escistence and then calcute its value.

$\sum_{n=1}^{\infty}|c_n|$: Criteria to prove existence:

$\lim_{n\to\infty}|c_n|=0$, otherwise you get $*_1$

problems.

Ex: $\underbrace{1-1}_{0}+\underbrace{1-1}_{0}+\underbrace{1-1}_{0}\cdots\cdots = ?$ $= 0$?

or $1+(-1+1)+(-1+1)\cdots\cdots = 1$?

or $1-1+1-1+1-1+1 = 2$?

So we assume that $*_1$ holds.

After some time you get experience by seeing some series of what criterium can be used, or tried.

But in my mind I have a certain order of using those criteria.

See also at internet:

wikipedia convergence tests

Sitting at that bench near the little

river and during the run, I asked myself, shall I let the people see, why those criteria work? Not nice proofs, but just simple examples, that the people get some feeling why? I can not resist to do it. But then there is some series, which you will see very often: $\sum\limits_{n=0}^{\infty} \alpha^n$

$$(1-\alpha)\left(\sum_{n=0}^{N} \alpha^n\right) = 1 + \cancel{\alpha} + \cancel{\alpha^2} + \dots + \cancel{\alpha^N} - \cancel{\alpha} - \cancel{\alpha^2} - \dots - \cancel{\alpha^N} - \alpha^{N+1}$$

$$\sum_{n=0}^{N} \alpha^n = \frac{(1-\alpha^{N+1})}{1-\alpha}$$

Look if $|\alpha| < 1$ then series converges

$$\sum_{n=0}^{\infty} \alpha^n = \left(\frac{1}{1-\alpha}\right)$$

geometric series

$|\alpha| > 1$ then the series diverges (not convergent)

$$|\alpha| = 1 \to \begin{cases} \alpha = 1, & \lim_{n\to\infty} \alpha^n = 1 \neq 0 \\ \alpha = -1, & \text{inconclusive} \end{cases}$$

$$1 - 1 + 1 - 1 + 1 \cdots = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$$

$$= \frac{1}{2}$$

Read and see what agreement is made about $\alpha = -1$. I should say not convergent, because you can get out of it, what you want.

With $\alpha = \frac{1}{3}$, you get the

Corona: $\frac{3}{2} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{1-\left(\frac{1}{3}\right)}\right)$

the "Grabbenhaus constant" α.

Now those criteria, in the order as I should use them:

$*_2$ Ratio test: if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$

then series converges.

Why? For great values of n then

$$\frac{|c_{n+1}|}{|c_n|} \approx \alpha\,(<1) \Rightarrow$$

let $n \geq N$

N fixed

$$|c_{n+1}| \approx \alpha |c_n|$$

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{(N-1)} |c_n| + \sum_{n=N}^{\infty} |c_n| =$$

finite sum so exists

?

$$\sum_{n=N}^{\infty} |c_n| \leq |c_N| + \alpha|c_N| + \alpha^2 |c_N| + \cdots$$

$$= |c_N|(1+\alpha+\alpha^2+\cdots) = |c_N| \cdot \left(\frac{1}{1-\alpha}\right)$$

as small as you want because $\lim_{n\to\infty} |c_n| = 0$, see $*_1$

finite

So if N great enough, the second part can be get as small as you want, so series exists.

When the Ratio Test doesn't work nice I often try to use the

$*_3$ Integral test:

So $|c_n| = f(n)$ $(f : \mathbb{R} \to \mathbb{R})$

If $\int_1^\infty f(x)\,dx < \infty$

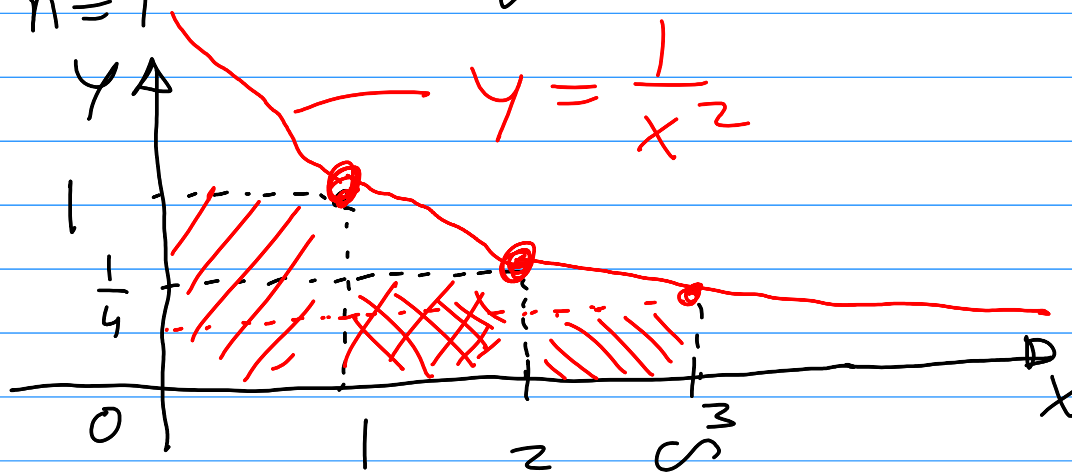
the series converges

(and in certain sense if $\int_1^\infty f(x)\,dx$ not bounded then the series diverges)

Be careful with the lower boundary of the integral, of importance is most of the time the last part of the series.

To keep in mind:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sim f: x \to \frac{1}{x^2}, \quad \frac{1}{n^2} = f(n)$$



So don't take $\int f(x)\,dx$, then it $\frac{1}{4} \to 0$

goes wrong, but

$$1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

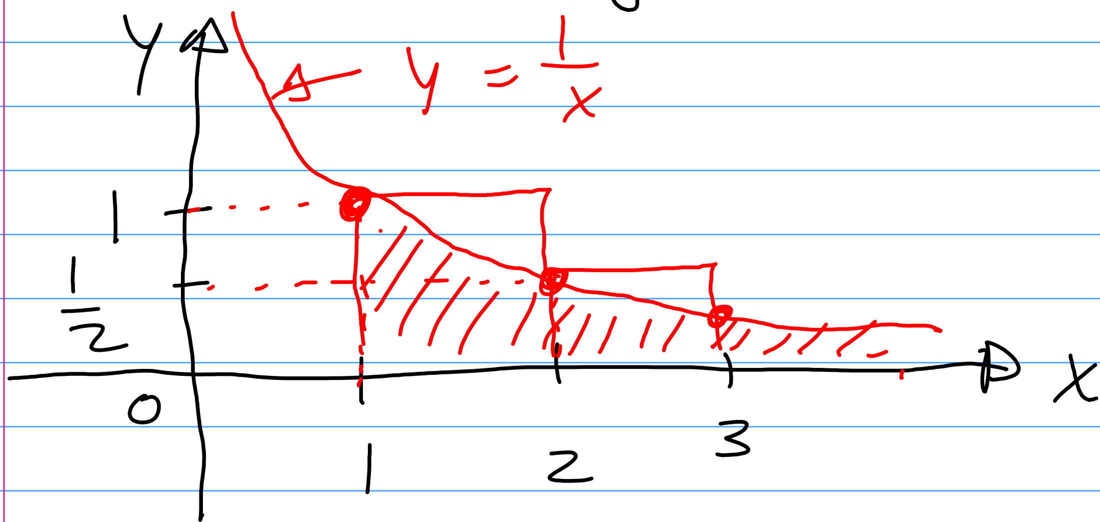
$$= 1 + \left[-\frac{1}{x}\right]_1^{\infty} = 2 < \infty$$

So the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim f: x \to \frac{1}{x} \sim \frac{1}{n} = f(n)$$

$\frac{1}{x}$ something to do with $\ln|x|$

that function is not bounded,
series diverges?



So: $\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx = [\ln|x|]_1^{\infty} \to \infty$

integral is not bounded, so
the series diverges!
When these two criteria
doesn't work, I try the root
test, but it is not my favourite.
This because you have
sometimes terrible limits,

$$\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right) = \lim_{n\to\infty} \exp^{\left(\frac{1}{n}\ln n\right)} = e^{0} = 1,$$

so $\exp(\ln(x)) = x$ is often convenient to use.

$\times_4$ Root test:

If $\lim_{n\to\infty}\left(|c_n|\right)^{\frac{1}{n}} < 1$

then series converges.

Why? If for great values of n:

$$|c_n|^{1/n} \sim \alpha \;(<1) \text{ then}$$

$|c_n| \simeq \alpha^n$, and we know that

$\sum_{n=1}^{\infty} \alpha^n$ converges if $|\alpha| < 1$.

The reason is quite simple, but calculating that limit not always.

*5 Look at wikipedia, comparison test also nice in use.

But now, I'm not running anymore, we have to be aware that we spoke about absolute convergence the whole time. What to do with alternating series, that are series you can write as

$$\sum_{n=1}^{\infty} (-1)^n \cdot c_n, \text{ with } c_n > 0 \ \forall n \in \mathbb{N}.$$

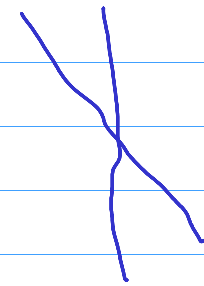
*6 Alternating series test

- 1 $\lim_{n \to \infty} c_n = 0$
- 2 $c_{n+1} < c_n \quad \forall n \in \mathbb{N}.$

then the series $\sum_{n=1}^{\infty} (-1)^n c_n$ converges.

(2 simple conditions $\bullet_1$ and $\bullet_2$)

Why? I look to the sky and I see just two white lines which cross each other and ask myself, shall I write down the idea I had running near the nunnery?



Let's do, but keep in mind it is not a proof of this test *6

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots$$

$$\frac{1}{n} - \left(\frac{1}{n+1}\right) + \cdots\cdots$$

You know this series is not absolute convergent

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$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges!}$$

Look: $c_n = \frac{1}{n}$

$\lim_{n\to\infty} c_n = 0$, $\quad c_{n+1} = \frac{1}{n+1} < \frac{1}{n} = c_n$

Look to: $\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{(n+1)-n}{n(n+1)}$

$= \frac{1}{n(n+1)} \leq \frac{1}{n^2}$ and the

series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

and $(1-\frac{1}{2}) > 0$, $(\frac{1}{3}-\frac{1}{4}) > 0 \dots$

So $\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ is increasing. (increase and bounded)

Let well, in mathematical sense, is not completely well, but I hope, you see that to this series the test works.

Conclusion:

These tests I have in mind and they are also useful working with complex numbers. **But see (I) g)!**

If $c_n \in \mathbb{C}$, you have to work with $|c_n|$, use the absolute value.

Later on those tests will be applied to "power series"

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \text{, then take}$$

$$c_n = |a_n (z - z_0)^n|$$

$\boxtimes$

(III) Example of those "power series"

$$\frac{1}{2-z} \overset{*}{=} \frac{1}{2} \cdot \frac{1}{\left(1-\left(\frac{z}{2}\right)\right)} =$$

$\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$, when do you have convergence?

"geometrical series" so:

$$\left|\frac{z}{2}\right| < 1 \rightarrow \underline{|z| < 2}$$

But what to do, if $z = 4$?

$\frac{1}{2-4} = -\frac{1}{2}$ exists, but

$|4| = 4 > 2$, so given series diverges!! See $*$, why divided by: 2? We can also divide by z, if $z \neq 0$.

Let's do:

$$\frac{1}{2-z} = \frac{1}{\left(\frac{2}{z}-1\right)z} =$$

$$-\frac{1}{z}\cdot\left(\frac{1}{1-\left(\frac{2}{z}\right)}\right) = -\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

we have convergence if

$$\left|\frac{2}{z}\right| < 1 \rightarrow |z| > 2$$

$$\frac{1}{2-z} = \begin{cases} \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \text{ if } |z|<2 \text{ (i)} \\ -\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \text{ if } |z|>2 \text{ (ii)} \end{cases}$$

So, if $z = 4 \rightarrow |z| = 4 > 2$

$$-\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{4}\right)^n = -\frac{1}{4}\cdot\frac{1}{\left(1-\frac{1}{2}\right)} =$$

$$-\frac{1}{2} \quad \left(= \frac{1}{2-4}\ !\right)$$

But see the differences, (i) has <u>positive powers of z</u> and

(ii) has negative powers of z

So you see that you can play with those series.

Let's limit to positive powers of z.

What can we do more?

Somebody wants to have powers of $(z+1)$?

Let's do the following,

let's $w = z+1 \Rightarrow z = (w-1)$

We know: $\frac{1}{2-z} = \frac{1}{2-(w-1)} =$

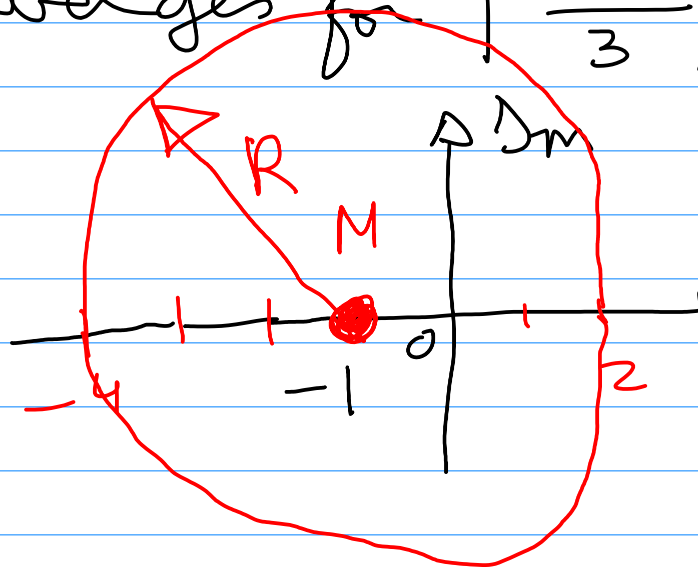
$$\frac{1}{3-w} = \frac{1}{3} \cdot \frac{1}{\left(1-\frac{w}{3}\right)} =$$

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{w}{3}\right)^n = \frac{1}{3} \cdot \sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

which converges for $\left|\frac{z+1}{3}\right|<1$

$|z+1|<3$

circle in $\mathbb{C}$-plane.



$M=-1$
$R=3$

The series of $\frac{1}{(2-z)^2}$?

We know: $\frac{1}{2-z} = \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

and

$$\frac{1}{(2-z)^2} = +\frac{d}{dz}\left(\frac{1}{(2-z)}\right) =$$

$$\frac{d}{dz}\left(\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) = \frac{1}{2}\cdot\sum_{n=0}^{\infty} n\cdot\left(\frac{z}{2}\right)^{n-1}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{n}{2^{n-1}}\right)\cdot z^n$$

It looks nice, but can it be

done without any problems?

- differentiate to z $(z \in \mathbb{C})$

what does it mean?

- $\frac{d}{dz}\left(\sum\right) = \sum\left(\frac{d}{dz}\right)$

can that be done?

!! Let now: $z \in \mathbb{R}$: (also $w \in \mathbb{R}$)

$$\int^{z} \frac{1}{2-w}\, dw = -\ln|2-z| + C$$

?

$$\int_{0}^{z} \frac{1}{(2-w)}\, dw = -\ln|2-z| + \ln 2$$

!

$$\| $$

$$\int_{0}^{z} \left(\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{w}{2}\right)^{n}\right) dw =$$

$$\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{z}{2}\right)^{n+1} \cdot 2 = \sum_{n=0}^{\infty} \frac{1}{(n+1)}\left(\frac{z}{2}\right)^{n+1}$$

So we have:

$$\ln|2-z| \overset{*}{=} \ln(2) - \sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(\frac{z}{2}\right)^{n+1}$$

if $z \in \mathbb{R}$ and $|z| < 2$ then

$\in \mathbb{R}$ $2 - z > 0$ and $|2-z| = (2-z)$

?? So $\ln|2-z| = \ln(2-z)$. $\in \mathbb{C}$

I'm asking myself, if somebody gives me that series $*$, I can fill in $z \in \mathbb{C}$, with $|z| < 2$, what I'm calculating?

Logarithm of complex numbers??

Be aware that we have then

differentiated to z,

integrated to z.

Can we do that in $\mathbb{C}$?

It looks so easy, but......?

That we have to study!

(IV) Differentiation

Given a function $f: \mathbb{C} \to \mathbb{C}$, what can we do with it?

A graph is not possible to construct, $\mathbb{C} \to 2$ dimensions, for a graph you need 4 dimensions. To imagine the behaviour of $f: \mathbb{C} \to \mathbb{C}$ is difficult.

If $f: \mathbb{R} \to \mathbb{R}$ then the derivative is defined by

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{(x+h) - x} \right)$$

$$= \lim_{w \to x} \frac{f(w) - f(x)}{(w - x)},$$

provided that the derivative exists!

At the same way we can define a derivative for $f: \mathbb{C} \to \mathbb{C}$,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{(w - z)}$$

$$= \lim_{h \to 0} \frac{f(z+h) - f(z)}{(z+h) - z},$$

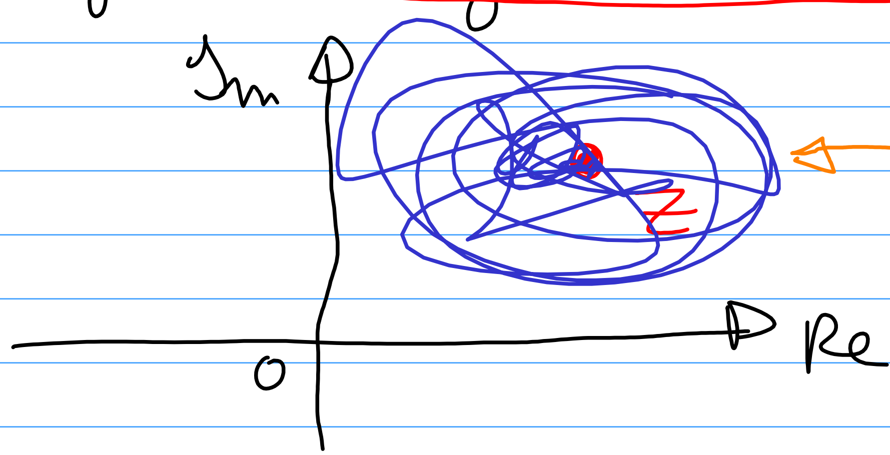
be aware $z, w, h \in \mathbb{C}$, and we assume that the derivative exists.

Questions:

"When everything goes well?"

"What does $\lim_{w \to z}$ or $\lim_{h \to 0}$ mean?"

Be aware of the fact that when there is written $w \to z$, there is not given some way of how to go with w to z.



Here is given some way.

To repeat it? I cann't, because I walked in $\mathbb{C}$ and at a certain moment I came into a very neighbourhood of z, to do a little step and I was at z.

This remembers me to the sentence:
"You can not step twice into the same river".
Keep in mind, that calculating some limit, the next time you are doing it, you maybe have taken quite another way.
It is better to take another way and see if you get the same value out of it, if not then the limit not excists.
Maybe it is better to do it path-independent.

Example: $f: z \to \dfrac{z^2}{z \cdot \bar{z}}$

$\lim_{z \to 0} f(z) = ?$ (in $\mathbb{C}$)

Let $z \in \mathbb{R}$, then $z = x \in \mathbb{R}$

and $x \to 0$ so

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{x^2} = 1$$

Have we calculated the limit?

Let $z = iy$, with $y \in \mathbb{R}$ and

$y \to 0$, then

$$\lim_{y \to 0} f(iy) = \lim_{y \to 0} \frac{(iy).(iy)}{(y)^2} = -1.$$

Compare the resuets!

Here some other way, $(R \in \mathbb{R})$

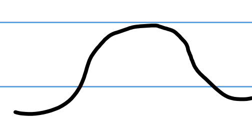
$$\lim_{R \to 0} f(R.\exp(i\varphi)) =$$

$$\lim_{R \to 0} \frac{R^2 \cdot e^{2i\varphi}}{R^2} = e^{2i\varphi}$$

so the limit depends on the angle, the $\arg(z)$.

Keep in mind:

"Walking from the station to the university."

There is not told to you, how you have to walk! Some people cross the traffic way near Chemical Engineering; other people use the traffic lights; other people like to walk their morning walk; I don't like to cross that bridge  over the Aa, incredible smooth if it has frozen!

But all we meet each other at the same point. And now during Corona time, think about all those different connections with the server, there is just one place of which the information is taken from. Maybe the signals have just gone around the earth?

Taking limits, be careful, path-independent is of importance.

If you know that a limit exist, and you have to calculate it, take an easy path. But read well, first you have to know that the limit excists.

Try yourself:

$$\lim_{z\to 0}\left(\frac{1}{2i}\left(\frac{z}{\bar z} - \frac{\bar z}{z}\right)\right)$$ exists or not?

Let's write out and look to it,

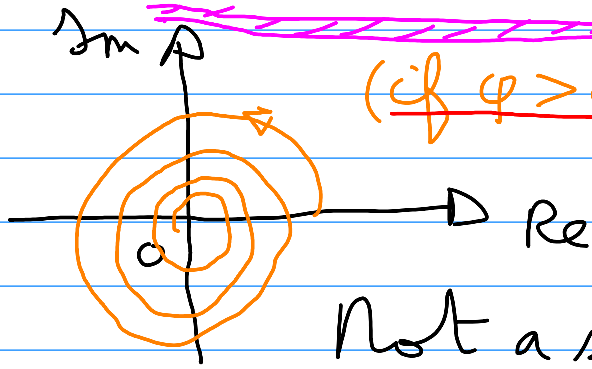
$$\frac{1}{2i}\left(\frac{z}{\bar z}-\frac{\bar z}{z}\right) = \frac{1}{2i}\left(\frac{z}{\bar z} - \overline{\left(\frac{z}{\bar z}\right)}\right) \in \mathbb{R}.$$

$z = x+iy$, $\mathrm{Re}(z)=0$, $\mathrm{Im}(z)=0$,
$\mathrm{Re}(z) = \mathrm{Im}(z)$, $z = R\cdot\exp\left(i\frac{\varphi}{R}\right)$

something to do with: $$\frac{2\cdot x\cdot y}{x^2+y^2} = \frac{2\cdot R\cos\left(\frac{\varphi}{R}\right)\cdot R\sin\left(\frac{\varphi}{R}\right)}{R^2} = 2\cdot\cos\left(\frac{\varphi}{R}\right)\sin\left(\frac{\varphi}{R}\right)$$

$$= \sin\left(2\cdot\frac{\varphi}{R}\right) \to ?? \text{ if } R\to 0$$

(if $\varphi>0$ and constant)



Not a straight line but a spiral and $|z| = R \to 0$, if $R\to 0$, so $z\to 0$.

Why soo much attention to limits?

We want to differentiate complex functions $f: \mathbb{C} \to \mathbb{C}$.

If $g: \mathbb{R} \to \mathbb{R}$ then derivative in $x = x_0$ defined by $\lim_{x \to 0} \frac{g(x) - g(x_0)}{(x - x_0)} =$

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = g'(x_0)$$

if the limit excists.

Exc. $\mathbb{R}$: path dependent!

if $h > 0$: $\frac{h}{h} = 1$

if $h < 0$, $\frac{-h}{h} = -1$

i) $g: x \to |x|$, $x_0 = 0$, $\lim_{h \to 0} \frac{|h| - 0}{h}$ does not excist, g not differentiable in $x_0 = 0$.

ii) $g: x \to x^2$ $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$

$$\lim_{h \to 0} \frac{(x^2 + 2h \cdot x + h^2) - x^2}{h} = 2 \cdot x$$

limit exsists, for every $x \in \mathbb{R}$.

Let's do the same for $f: \mathbb{C} \to \mathbb{C}$

The derivative is defined by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} =$$

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

if those limits exist and $z_0, z, h \in \mathbb{C}$.

Let's look if those limits always exist? If they are path-independent?

Ex. $\mathbb{C}$:

i) $f: z \to \bar{z}$ differentiable?

Look if: $z \in \mathbb{R} \leadsto f(z) = z \leadsto f'(z) = 1$

if $z \in i\mathbb{R} \leadsto z = iy,\ y \in \mathbb{R}$ then

$f(z) = \overline{(iy)} = -iy = -z \leadsto f'(z) = -1$

Maybe not good calculated but I see some difference, so I want to calculate those limits with different paths. For instance (α) parallel to the real-axis, so if $z = x + iy$, then x-direction; (β) parallel to the imaginary-axis, or y-direction.

Let's try and see!

Arbitrary $z \in \mathbb{C}$ is given, to calculate

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{(z+h) - z}$$

(α) ~ (x-direction), let $h \in \mathbb{R}$:

$$\frac{f(z+h) - f(z)}{(x+h+iy) - (x+iy)} = \frac{\overline{(x+h+iy)} - \overline{(x+iy)}}{h}$$

$$= \frac{((x+h) - iy) - (x - iy)}{h} = \frac{h}{h} = 1$$

(β) ~ (y-direction), let $h \in \mathbb{R}$:

$$\frac{f(z+ih) - f(z)}{(z+ih) - (z)} = \frac{\overline{(x+i(y+h))} - \overline{(x+iy)}}{ih} =$$

!

$$\frac{(x-i(y+h))-(x-iy)}{ih} = \frac{-ih}{ih} = -1$$

Compare results: so $f: z \to \bar{z}$ is not differentiable to z.

ii) $f: z \to z^2$, let $z \in \mathbb{C}$ be fixed,

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} =$$

$$\lim_{h\to 0} \frac{(z^2+2z\cdot h+h^2)-z^2}{h} = 2\cdot z = f'(z)$$

This limit is not path-dependent, if $z \in \mathbb{C}$ is fixed, you will always get the same value out of that limit, so

$f': z \to 2z$, or $f'(z) = 2\cdot z$, or $\frac{\partial f}{\partial z}(z) = 2\cdot z$.

May be we can treat all the real function, we know, with some $z \in \mathbb{C}$ in it, and may be the derivatives at the same way?

But what we have to think by

$\ln(z)$, $\sin(z)$, $\cos(z)$, $\arctan(z)$?

At this moment an exciting business!

For instance:

? $\ln(z)$? if $z = R\cdot e^{i\varphi} \leadsto f(z) = \ln(R\cdot e^{i\varphi})$

$= \ln(R) + i\varphi = \ln|z| + i\arg(z)$ and

$\exp(\ln|z| + i\arg(z)) =$

$\exp(\ln(|z|)) \circ e^{i\arg(z)} = |z| \cdot e^{i\arg(z)} = z$

So: $\exp(\ln(z)) = z$, just as in the real case! But are the exp- and ln-function really its inverse functions?

Look to: $\arg(z)$, a lot values can be given to that! If you take $\arg(z) = \gamma$ then I take: $(\gamma + 2\pi)$, also good!

Maybe Arg(z) can help us? We shall see later on.

Sin(z) no idea how to calculate?

We know: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$;

why not $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots$

and then $\frac{d}{dz}(\sin(z)) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots$

$= \cos(z)$

At these pages we did a lot, just playing with mathematics. In the past they did the same and then comes the problem of proving those things.

Can you do it always? Or only in certain cases? Let's ask ourself that question in the case of taking the derivative.

To get some feeling for possible problems, let's look to $f: z \to |z|^2$.

Maybe of importance for later on, in this specific case: $f: \mathbb{C} \to \mathbb{R}$ $(\neq \mathbb{C})$.

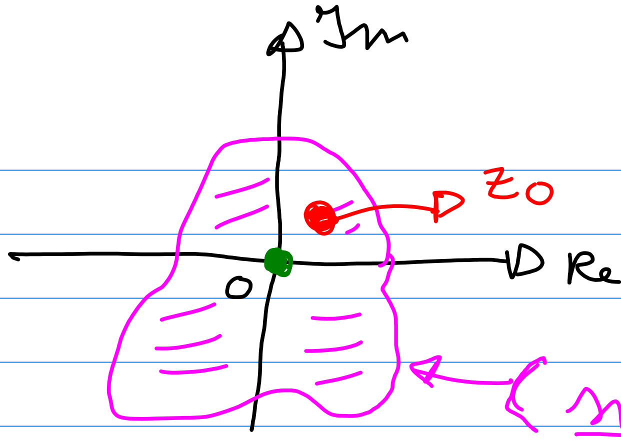
First I want to see if f is differentiable in $z=0$? We need to calculate $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$, $h \in \mathbb{C}$

$$\frac{f(0+h)-f(0)}{h} = \frac{|0+h|^2 - |0|}{h} = \frac{h \cdot \bar{h}}{h} = \bar{h}$$

So $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \bar{h} = 0$,

the derivative in $z=0$ exists!

Can we do the same in a small neighbourhood of $z=0$? See figure on next page,



$f: z \to \frac{|z|^2}{2}$

$(f'(0) = 0)$

(small neighbourhood: V)

Maybe I have taken some magnifying glass (1000x or 10^{18}x??)

$0 \neq z_0 \in V$, and now:

$$(*)\quad \frac{f(z_0+h) - f(z_0)}{h} =$$

$$\frac{1}{2}\cdot\frac{(z_0+h)\overline{(z_0+h)} - z_0\cdot\overline{z_0}}{h} =$$

$$\frac{1}{2}\cdot\frac{(z_0\cdot\overline{z_0} + z_0\cdot\overline{h} + h\cdot\overline{z_0} + h\cdot\overline{h}) - z_0\cdot\overline{z_0}}{h}$$

$$= \frac{1}{2}\left(z_0\cdot\left(\frac{\overline{h}}{h}\right) + \overline{z_0} + \overline{h}\right).$$

$$\frac{-ik}{ik} = -1$$

What happens if $h \to 0$? See pg. 37-38?

(α) if $h \in \mathbb{R} \rightsquigarrow \lim\limits_{h\to 0}(*) = \frac{z_0 + \overline{z_0}}{2} = \mathrm{Re}(z_0)$

(β) if $h = ik \in i\mathbb{R}$, with $k \in \mathbb{R}$, $\rightsquigarrow$

$$\lim_{h\to 0}(*) = \lim_{k\to 0}\frac{1}{2}\left(z_0\cdot(-1) + \overline{z_0}\right) = -i\,\mathrm{Im}(z_0)$$

(!) $z_0 \in V$ is fixed, so we see that $f: z \to |z|^2$ is not differentiable in z_0, different paths give different values!

Conclusion:

$f: z \to |z|^2$ is differentiable in $z = 0$, but not differentiable in a (small) neighbourhood of $z = 0$.

We have already seen (pg. 38), that $f: z \to z^2$ is differentiable for every $z \in \mathbb{C}$. So also in $z = 0$ and so in a neighbourhood of $z = 0$.

Let's give them some name.

If $f: \mathbb{C} \to \mathbb{C}$ differentiable in $z = z_0$ and also in a neighbourhood of $z = z_0$ then f is called holomorphic.

For the fun: google on internet with: differentiable holomorphic.

Also a nice site:

Paul Math Online Notes (not soo much about $\mathbb{C}$), or just google with: Complex Analysis

Don't forget: z-lib, just google and search inside with: complex analysis, be careful, just 5 books at a day!

But how to look if a function is holomorphic, easier then just done?

Let's first write $f: \mathbb{C} \to \mathbb{C}$ at some other way:

$f(z) = u(z) + i\, v(z)$

with $u: \mathbb{C} \to \mathbb{R}$ and $v: \mathbb{C} \to \mathbb{R}$,

maybe, if we take $z = x + iy$,

$f(z) = u(x,y) + i\, v(x,y)$

with $u: \mathbb{R}^2 \to \mathbb{R}$ and $v: \mathbb{R}^2 \to \mathbb{R}$

also a nice way, $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

In the cases:

$$f: z \to |z|^2 \rightsquigarrow \begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$$

$$f: z \to z^2 \rightsquigarrow \begin{cases} u = x^2 - y^2 \\ v = 2 \cdot x \cdot y \end{cases}$$

If $f(z)$ given, fill in $z = (x + iy)$ and search real- and imaginary part as functions of x and y, so $\begin{cases} u(x,y) \\ v(x,y) \end{cases}$

The reverse $g(x,y)$ is given, fill in $x = \left(\frac{z+\bar{z}}{2}\right)$ and $y = \left(\frac{z-\bar{z}}{2i}\right)$, try to get expressions with z and $\bar{z}$, so: $g(z)$.

But now? If f is differentiable in z, what kind of relations excist for the functions u and v? Cauchy and Riemann asked themselves the same and they came with the Cauchy-Riemann equations. Let's do, what they did. The derivative f' excists if the limit $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ is path-independent. So assume that

$f'(z)$ excists, then calculating that limit, in the real direction, has to give the same value as calculating it into the imaginary direction.

Let's just do, assume that $h \in \mathbb{R}$,

$$\frac{f(z+h)-f(z)}{h} = \left[\left(u(x+h,y)+i\,v(x+h,y)\right) - \left(u(x,y)+i\,v(x,y)\right)\right]/h =$$

$$\frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{\left(v(x+h,y)-v(x,y)\right)}{h}$$

$$\longrightarrow \left(\frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y)\right), \text{ if } h \to 0$$

$$\left(\frac{1}{i}=-i\right) \quad \frac{f(z+ih)-f(z)}{(z+ih)-(z)} = \left[\left(u(x,y+h)+i\,v(x,y+h)\right) - \left(u(x,y)+i\,v(x,y)\right)\right]/\underline{\underline{ih}} \; !$$

$$= -i\cdot\frac{u(x,y+h)-u(x,y)}{h} + \frac{v(x,y+h)-v(x,y)}{h}$$

$$\longrightarrow -i\,\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y), \text{ if } h \to 0.$$

Because $f'(z)$ excists, both limits have to be equal so:

$(z = x + iy)$

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) =$$

$$-i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$$

There follow the C.R. equations:

$$\begin{cases} \dfrac{\partial u}{\partial x}(x,y) = \dfrac{\partial v}{\partial y}(x,y) \\[2ex] \dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y) \end{cases}$$

applied to: $f: z \to z^2$

$$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases} \Rightarrow \begin{cases} u_x = 2x, \ u_y = -2y \\ v_x = 2y, \ v_y = 2x \end{cases}$$

C.R. equations are satisfied for every $z \in \mathbb{C}$, in this case.

Applied to $f: z \to |z|^2$

$$\begin{cases} u(x,y) = x^2 + y^2 \\ v(x,y) = 0 \end{cases} \Rightarrow \begin{cases} u_x = 2x, \ u_y = 2y \\ v_x = 0, \ v_y = 0 \end{cases}$$

See that C.R. equations is only satisfied if $x=0$ and $y=0$, thus only in $z=0$.

So C.R.-equations are very helpful to determine where a function $f: \mathbb{C} \to \mathbb{C}$ is differentiable.

See pg. 41, there I made some remark about $f: \mathbb{C} \to \mathbb{R}$.

Assume that f is differentiable for every $z \in \mathbb{C}$, then C.R. eqn's are satisfied! What do we know: $f(z) = u(x,y) + i \cdot 0$,

so $v(x,y) = 0 \Rightarrow u_x = 0, u_y = 0$

that means that $u(x,y) = C \in \mathbb{R}$, is constant, so $f(z) = C$ is constant!

<u>Conclusion:</u>

If $f: \mathbb{C} \to \mathbb{R}$, it can be differentiable at certain points but it can not be holomorphic, unless $f(z) = C \ (\in \mathbb{R})$

In mind I have always:

$$f = (u, v) = u + iv$$

$$(x, y) = x + iy$$

$$u_x = v_y$$ and there is an equation with a (-1) in it:

already smooth. for u_x so next one: $u_y = (-1) \cdot v_x$

If I don't know anymore then I write down those calculations on pg. 46.

Now we have for $f: \mathbb{C} \to \mathbb{C}$:
If f holomorphic then C.R. eqn's for every $z \in \mathbb{C}$.
Holds the same the other way around?
Search on internet!
See Goursat's theorem and especially the one with weakened hypothesis:
If f continuous in open set Ω and partial derivatives of u and

v exist in Q and C.R-eqn's are satisfied in Q $\Rightarrow$ f is holomorphic in Q.
(Looman-Menchoff theorem)

Search on internet (wikipedia)

- Cauchy-Rieman equations
- Goursat theorem
- Looman-Menchoff theorem

What will we use often??

Given some $f: \mathbb{C} \to \mathbb{C}$,

- determine u and v?
- are u and v continuous diff.? Y
- C.R. eqn's are satisfied? Y

$\Rightarrow$ f is holomorphic.

Running at a Friday morning I asked myself how the C.R.-eqn's would be in polar coordinates?

So: $f(z) = u(R,\varphi) + i\,v(R,\varphi)$

Derivative into the R-direction?
(φ is fixed and $h \in \mathbb{R}$, $R+h>0$)

$$\frac{f((R+h)e^{i\varphi}) - f(R\cdot e^{i\varphi})}{(R+h)e^{i\varphi} - R\cdot e^{i\varphi}} =$$

$$\frac{(u(R+h,\varphi) - u(R,\varphi)) + i\,(v(R+h,\varphi) - v(R,\varphi))}{h\cdot e^{i\varphi}}$$

$$\longrightarrow \left(\frac{\partial u}{\partial R}(R,\varphi) + i\,\frac{\partial v}{\partial R}(R,\varphi)\right)\cdot e^{-i\varphi} \text{ if } h \to 0.$$

Derivative into the φ-direction?
(R is fixed)

$$\frac{f(R\cdot e^{i(\varphi+h)}) - f(R\cdot e^{i\varphi})}{(R\,e^{i(\varphi+h)} - R\,e^{i\varphi})} =$$

!

$$\frac{(u(R,\varphi+h) - u(R,\varphi)) + i\,(v(R,\varphi+h) - v(R,\varphi))}{R\cdot e^{i\varphi}(e^{ih}-1)}\cdot\frac{h}{h} =$$

$$\left(\frac{u(R,\varphi+h)-u(R,\varphi)}{h}+i\left(\frac{v(R,\varphi+h)-v(R,\varphi)}{h}\right)\right)\cdot$$

$$\frac{1}{R}\cdot e^{-i\varphi}\cdot\frac{h}{(e^{ih}-1)}\longrightarrow\quad \frac{e^{ih}-1}{h}\to i\,e^{0}\ \text{if } h\to 0$$

$$\left(\frac{\partial u}{\partial\varphi}+i\frac{\partial v}{\partial\varphi}\right)\cdot\frac{1}{R}\cdot e^{-i\varphi}\cdot\frac{1}{i}\quad \text{if } h\to 0.$$

Derivative exists, so path-indep.,

$$\left(\frac{\partial u}{\partial R}+i\frac{\partial v}{\partial R}\right)\cdot\cancel{e^{-i\varphi}}=\left(\frac{\partial u}{\partial\varphi}+i\frac{\partial v}{\partial\varphi}\right)\cdot\frac{1}{R}\cancel{e^{-i\varphi}}\cdot(-i)\Longrightarrow$$

$$\begin{cases}\dfrac{\partial u}{\partial R}=\dfrac{1}{R}\cdot\dfrac{\partial v}{\partial\varphi}\\[2mm] \dfrac{\partial v}{\partial R}=-\dfrac{1}{R}\cdot\dfrac{\partial u}{\partial\varphi}\end{cases}$$

Cauchy Riemann eqn's in polar coordinates.

In mind I thought, that I had to do much more.

Let's now look those functions f, u and v, with $z=x+iy$, and let we assume that we can differentiate them at any moment we want to that. If we want to make some theory of it, then we try to prove every step we have done. Let's first see what is maybe possible,

$$f = u + iv \quad \text{with} \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$$u_{xx} = (v_y)_x = (v_x)_y = -u_{yy} \Rightarrow$$

$$u_{xx} + u_{yy} = 0.$$

The function u is called harmonic.

$$v_{xx} = -(u_y)_x = -(u_x)_y = -v_{yy} \Rightarrow$$

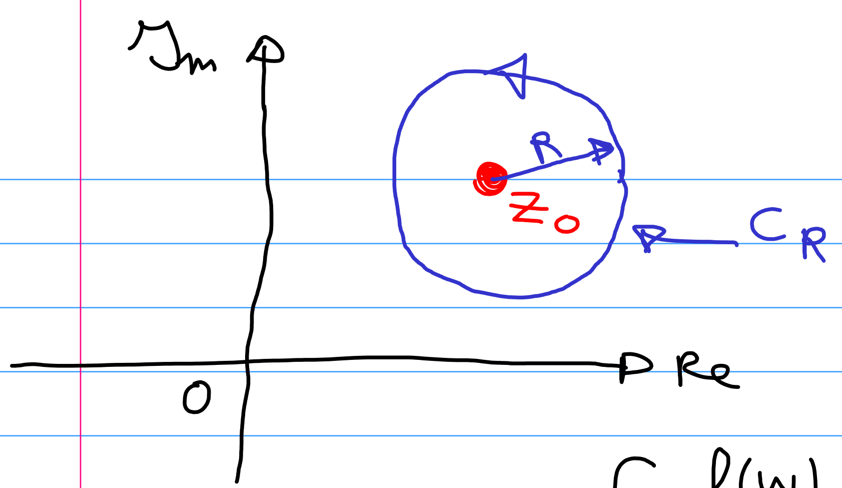
$$v_{xx} + v_{yy} = 0 \quad \text{v is also harmonic}$$

We have soo much differentiated, let's now integrate! Let's do the same as in the real numbers. Why, don't ask me, but I want to calculate:

$$\int \frac{f(w)}{(w - z_0)} dw,$$ see next page!

(V) Integration:

$$\boxed{?\ \oint_C f(z)\,dz = 2\pi i \sum \text{residues}\ ?}$$



circle around z_0, with radius R

What gives $\displaystyle\int_{C_R} \frac{f(w)}{w - z_0}\, dw$?

Let's: $w = z_0 + R \cdot \exp(i\varphi)$,

R fixed and $0 \le \varphi < 2\pi$, φ is variable, then:

$$dw = R \cdot i \cdot \exp(i\varphi)\, d\varphi .$$

After just scaring some pingeons out of my garden, I become crazy of those beasts. And I have taken some coffee, yes, I'm interested in that integral. If I may do it? At this moment not a problem, I just do it!

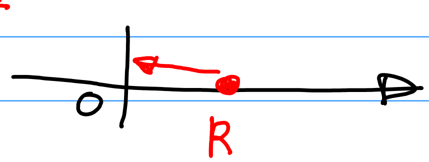
So R fixed, $0 \le \varphi < 2\pi$, dw is known, let's do it anti-clockwise, so with a positive orientation. Just like the low pressure areas we have:

$$\oint_{C_R} \frac{f(w)}{w - z_0}\, dw = \int_0^{2\pi} \frac{f(z_0 + R e^{i\varphi})}{R \cdot e^{i\varphi}}\, R \cdot i \cdot e^{i\varphi}\, d\varphi$$

$$= \int_0^{2\pi} \underline{i} \cdot f(z_0 + R e^{i\varphi})\, d\varphi$$

($\oint$ integral over a closed contour; $\circlearrowleft$ gives orientation most of time positive)

What can we do with that result?

With R can be done something, the only variable which is free, let's $R \to 0$. Remember, R is a radius, so $\underline{R \in \mathbb{R} \text{ and } R > 0}$, so $R \to 0$ means that $R \downarrow 0$,  $\mathbb{R}$.

If R is very small then

$f(z_0 + R e^{i\varphi}) \approx f(z_0)$ almost a constant, so $\int_0^{2\pi} f(z_0 + Re^{i\varphi})\, d\varphi \approx f(z_0) \cdot (2\pi - 0)$

So we get

$$\lim_{R \downarrow 0} \oint_{C_R} \frac{f(w)}{(w - z_0)}\, dw = 2\pi i \cdot f(z_0)$$

Now I got the question in mind, can I make clear to people that

$$\boxed{f(z) = \oint_{C_R} \frac{f(w)}{(w - z)}\, dw.}$$

See: no limit!!

Can that be done for every f? We will see later on. First play with the stuff and it is weekend, so all those official things. I have other things on my mind. Coffee cup is empty!

From vector Calculus/Analysis
I remember me some name: Green.
There was some theorem of Green?
Internet helped me, searching:
theorem of Green. I see somewhere
on my screen:

$$\int_C (M\,dx + N\,dy) = \iint_V \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

V some region in $\mathbb{R}^2$, C is the
boundary of V and there is written
a lot more of all kind of conditions.
About orientation on C (the same as
we have) and V may not have holes in it.
My thoughts go Poirot, who told
somebody that fingerprints were found

somewhere. The guy became angry and admitted to be the criminal. But afterwards Poirot told to people, that he knew nothing of those fingerprints. So I hope that everything, I'm telling you, is true!

Now, I'm interested what out of the following integral in the complex plane:

$$\oint_{C} f(z)\,dz,$$ C a closed contour in $\mathbb{C}$, positive oriented.

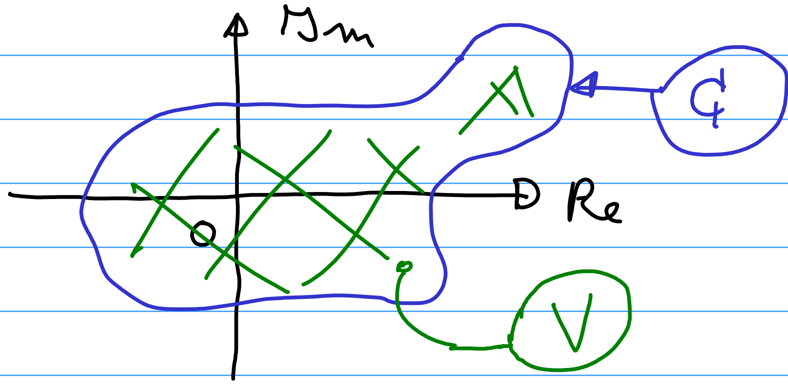
What can we do with Green??

$f(z) = u(x,y) + i \cdot v(x,y)$

$z = x + iy, \quad dz = dx + i\,dy$

$f(z)\,dz = (u + iv)(dx + i\,dy) =$

$$\underbrace{(u\cdot dx - v\cdot dy)}_{\in \mathbb{R}} + i\cdot\underbrace{(u\cdot dy + v\cdot dx)}_{\in \mathbb{R}}$$



$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i\cdot(u\,dy + v\,dx) \overset{\text{"Green"}}{=}$$

$$\iint_V \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dV + i\cdot \iint_V \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dV \overset{\text{"Cauchy-Riemann"}}{=}$$

"Cauchy – Riemann" $\begin{pmatrix} u_x = v_y \\ u_y = -v_x \end{pmatrix}$

$$\iint_V 0\,dV + i\iint_V 0\,dV = \underline{0} \quad !$$

Let $f \neq 0$, and let f be a nice function, and C, a closed contour in $\mathbb{C}$ then

$$\int_C f(z)\,dz = 0.$$

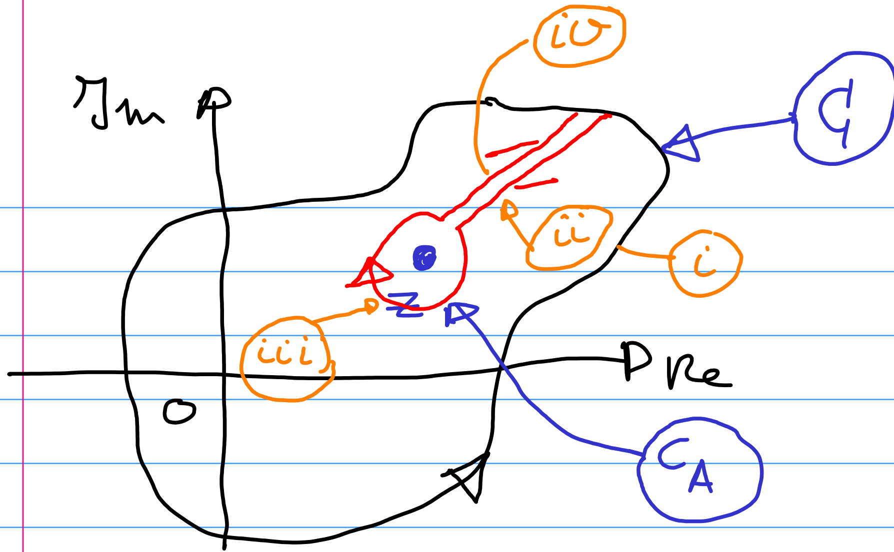
See page 59! May be we can the people tell, that there is no nonsense written. But when we will prove all those steps we have done? That I'm now thinking myself, it is weekend.

So let's calculate: (z inside C)

$$\int_{C} \frac{f(w)}{(w-z)}\, dz,$$ then a problem in $w=z$ dividing by 0, so a singularity of the first order in the language of a mathematican.

What can we do, about that?

Let's try to use what we have done on page 58 and 59. See following figure:



I take some alternative closed contour

z lies not on the inner side of the closed contour, so $\left(\frac{f(w)}{w-z}\right)$ is a nice function on the inner site of C_A.

And we know:

$$\oint_{C_A} \frac{f(w)}{(w-z)}\, dz = 0$$

C_A have all kind of different pieces.

Let's take for piece (iii)

$w = z + R\cdot e^{i\varphi}$, $\quad \varphi$ well chosen.

Let's fall the pieces (ii) and (iii) together, the direction of integration is opposite to each other so adding those results will give us : 0.

The orientation of (iii) is negative, so we have to multiply the result of pg. 58–59, by −1

This means that

$$0 = \lim_{R \to 0} \oint_{C_A} \frac{f(w)}{(w-z)}\, dw =$$

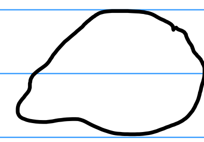
$$\lim_{R \to 0} \oint_{C'} \frac{f(w)}{(w-z)}\, dw + \oint_{(iii)} \frac{f(w)}{w-z}\, dz =$$

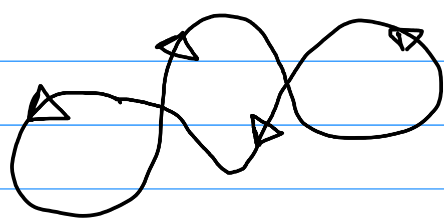
$$\oint_{C'} \frac{f(w)}{(w-z)}\, dz - 2\pi i \cdot f(z)$$

Conclusion:

$$f(z) = \frac{1}{2\pi i} \int_{C'} \frac{f(w)}{(w-z)}\, dw$$

We have more than on page 59!

G on page 59, was a circle and now have just some positive oriented closed contour with z inside G.
But be careful with those contours.
In mind and on paper we saw closed contours as:  ,
but what to do with:



What is inside ?
and outside !

If we prove things, we have to avoid a lot of these problems by using good definitions and go so on.
Or shall we first look about what we can do with it? There are a lot of books with profs and see the lecture

notes that are used.
The lecture notes of Joop Boersma were good. Not soo much text. Those lecture notes were typed by a typewriter. Computers were just emerging.
The lecture notes became bigger and bigger, but the source of Joop Boersma is still found in it.
Try to make a small summary.
In certain sense you can do a lot with the results we have at this moment:

$*_1$: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $\underline{\underline{\text{C.R.}}}$; $*_3$: path-indepent

$*_4$: series

$*_2$: $\left\{ \oint_C \frac{f(w)}{(w-z)} dw = 2\pi i \cdot f(z) \right.$ (Cauchy)

(<u>z inside C</u>)

(VI) holomorphic / analytic functions

$f: \mathbb{C} \to \mathbb{C}$

- f is holomophic in open set $A \subset \mathbb{C}$, if it is differentiable at each point of A.
- f is analytic, if it has a power series reprensentation

Most of the time these terms are interchangeable used, but be careful.

In the lecture notes there is made no difference between.

Here an example to see some difference between these type of functions. But in the example $x \in \mathbb{R}$!

$$f : \mathbb{R} \to \mathbb{R} ; \quad f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

$$f'(x) = +\frac{1}{x^2} e^{-\frac{1}{x}}$$

$$\lim_{x \downarrow 0} f'(x) = 0$$

$$f''(x) = -\frac{2}{x^3} e^{-\frac{1}{x}} + \frac{1}{x^4} \cdot e^{-\frac{1}{x}}$$

$$\lim_{x \downarrow 0} f''(x) = 0$$

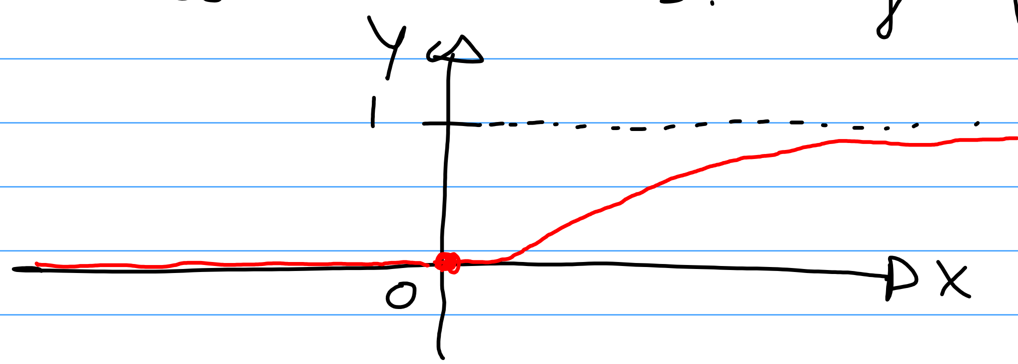
So each derivative a finite sum of $\left(e^{-\frac{1}{x}}/x^m\right)$ so

$$\lim_{x \downarrow 0} f^{(n)}(x) = 0. \quad \forall n \in \mathbb{N}.$$

From the left-hand site the derivative is always zero, that means that

$$f^{(n)}(0) = 0 \quad \forall n \in \mathbb{N}.$$

So $f(x) = e^{-1/x}$ is differentiable in $x = 0$, all the derivatives are continuous. a graph of f



But f is not analytic at origin:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$$

The Taylor series does not equal to f for $x > 0$.

(VII) Back to the integrals

Let's calculate:

$$I(R,p) = \int_{C_R} (z - z_0)^p \, dz$$

with $p \in \mathbb{Z}$,

$0 < R \in \mathbb{R}$

C_R: $\begin{cases} z = z_0 + R \cdot e^{i\varphi} \\ 0 \leq \varphi < 2\pi \\ \text{pos. orientation} \end{cases}$

$dz = R \cdot i \cdot e^{i\varphi}$

$$I(R,p) = \int_0^{2\pi} (R e^{i\varphi})^p \cdot i \cdot R \cdot e^{i\varphi} d\varphi =$$

$$i R^{p+1} \cdot \int_0^{2\pi} e^{i\varphi(p+1)} d\varphi \; ?$$

$I(R,-1) = i 2\pi$; now $p \neq -1$:

$$I(R,p) = R^{p+1} \cdot \left[\frac{i}{(p+1)} \cdot e^{i\varphi(p+1)} \right]_0^{2\pi}$$

$$= \frac{i \cdot R^{p+1}}{(p+1)} \cdot \Big(\underbrace{e^{i(p+1) \cdot 2\pi}}_{(=1)} - e^{i \cdot 0} \Big) = 0$$

for every $0 < R$.

That is an easy integral:

$$I(R,-1) = i \cdot 2 \cdot \pi$$

$$I(R,p) = 0 \quad \forall p \in \mathbb{Z} \setminus \{-1\}$$

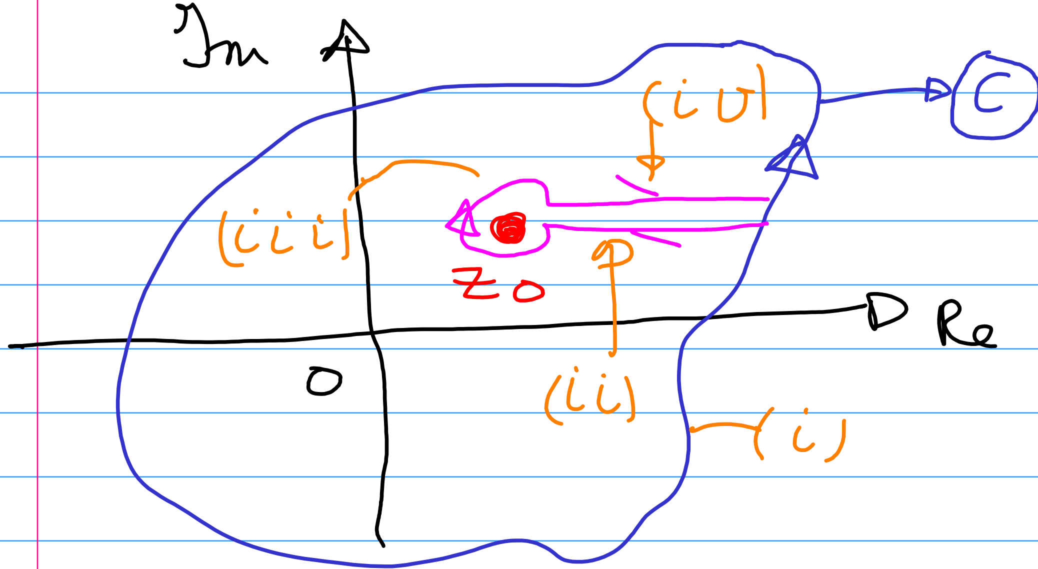
With the help of these integrals there has been done a lot of things and we will do.

Keep in mind, the $\left(\frac{1}{z-z_0}\right)$- term is of importance.

I can not wait anymore, I'm near some important fact. Let's do almost the same as done at page 64:

$$I(z_0) = \oint_C \frac{f(z)}{(z-z_0)}\, dz \qquad z_0 \text{ inside } C,$$

see figure



(ii) and (iv): Let's fall together, direction is opposite so sum integrals becomes: 0.

(iii): Let that be some circle around z0, with a small radius, only clockwise oriented. But with a (−1) before the integral it becomes anti-clockwise. No problems.

Not much space, so see next page:

$$\int_{(i)+(ii)+(iii)+(iv)} \frac{f(z)}{(z-z_0)}\,dz = \int_{(i)+(iii)} \frac{f(z)}{(z-z_0)}\,dz = 0$$

and we see:

$$I(z_0) = -\int_{(iii)} \frac{f(z)}{(z-z_0)}\,dz.$$

One thing, in the right hand side I see nothing of the curve C. I see only a circle with a small radius!

I want to use those $I(k,p)$ written on page 72!

How can that be done?

Till now I have seen no series and I told you that they are of importance pg. 67.

We have f, assume it is holomorphic/analytic then we can express f into a power series. Let's do around the point $z = z_0$,

so $$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n .$$

If there are negative powers, they are called Laurent series.

$$\text{So } I(z_0) = \oint_{C_R} \frac{\sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n}{(z - z_0)} dz$$

(CR positive oriented around $z = z_0$)

$$= (2\pi i) \cdot c_0$$!

There is just one term with power (−1) and that comes to the n = 0 of that series!!

That is impressive only one term of that series is important for that integral, all the other term can be thrown away, in certain sense.

Let's summarise this play. Somebody gives us the following function $f: \mathbb{C} \to \mathbb{C}$

$$f(z) = \frac{1}{(z-2)^2} \cdot \frac{1}{3-z}$$

and wants to know the following integral:

$$\int_C f(z)\, dz, \quad C = \text{circle with midpoint } M = 2 \text{ and radius } R = 1.$$

How to do?

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Maybe f can be written by some Laurent series around $z=2$? So negative powers of $(z-2)$ is no problem.

Try! $f(z) = \frac{1}{(z-2)^2} \cdot \frac{1}{2-(z-2)} =$

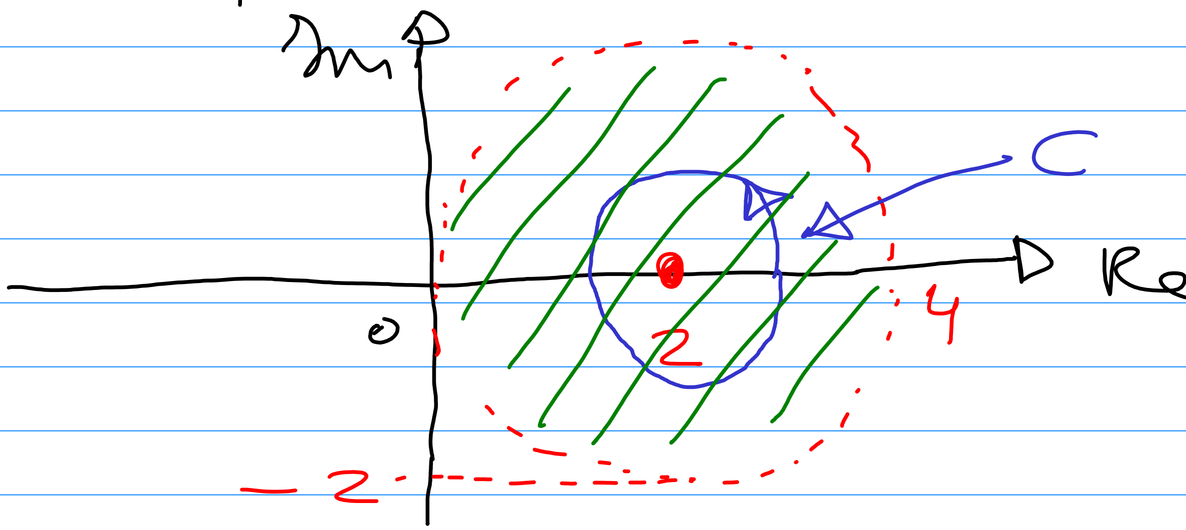
$$\frac{1}{(z-2)^2} \cdot \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z-2}{2}\right)^n =$$

$$\left(\frac{1}{2} \cdot \frac{1}{(z-2)^2} + \boxed{\frac{1}{4} \cdot \frac{1}{(z-2)}} + \frac{1}{8} + \dots\dots\right)$$

!!

Series converges if $\left|\frac{(z-2)}{2}\right| < 1$,

so: $|z-2| < 2$



So $C \subset \{z \mid |z-2| < 2\}$

Look at page 72, the term with $\frac{1}{(z-2)}$ will determine the result of that integral.

So:

$$\oint_C \frac{1}{(z-2)^2(4-z)}\, dz = 2\pi i \cdot \left(\frac{1}{4}\right)$$

with C: $|z-2| = 1$, positive oriented.

That coëfficient which belongs to the term with power (-1), so $(z-2)^{-1} = \frac{1}{(z-2)}$ is called the residue of $f(z) = \frac{1}{(z-2)^2(4-z)}$ at the pole $z = 2$.

In mind I have always

*5 $$\boxed{\oint_C f(z)\,dz = 2\pi i \cdot (\text{sum of residues})}$$

*6 $\boxed{\underline{\text{residue}}\text{: coëfficient of } \frac{1}{(z-a)} \text{ term.}}$

also good for the summary.
Keep in mind: integration on closed contours, only the coëff. of the $\left(\frac{1}{z-a}\right)$ terms give a contribution, all the other terms seem not of importance.

Now it is Sunday, just made my run to Brabantse Kluis and back. My watch told me 41750 steps, so I think 38-40 km. Indeed I stopped at the bench and thought about some nice integral, which can be calculated by those residues..

Searching some nice curve - in $\mathbb{C}$, I came to

$$C_1 = \left\{ x + i \sin\left(\frac{x}{1+x^2}\right) \mid x \in \mathbb{R} \right\}$$

maybe the sun helped me?

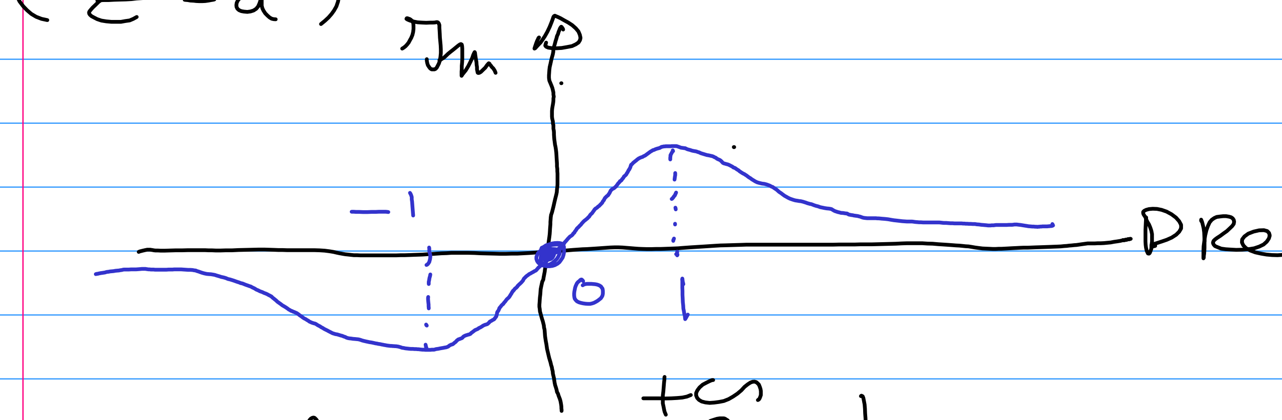
During the run I could not get out of my mind: $\frac{1}{1+x^2}$.

So it has become:

$$I = \int_C \frac{1}{1+z^2}\, dz \,, \quad \text{with}$$

$$C = \left\{ x + i \cdot \sin\left(\frac{x \cdot \pi}{1+x^2}\right) \,\middle|\, x \in \mathbb{R} \right\}$$

In a figure, always try to make some sketch of it. Important if you search singularities, in special those $\left(\frac{1}{z-a}\right)$ terms!



I think: $I \approx \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx =$

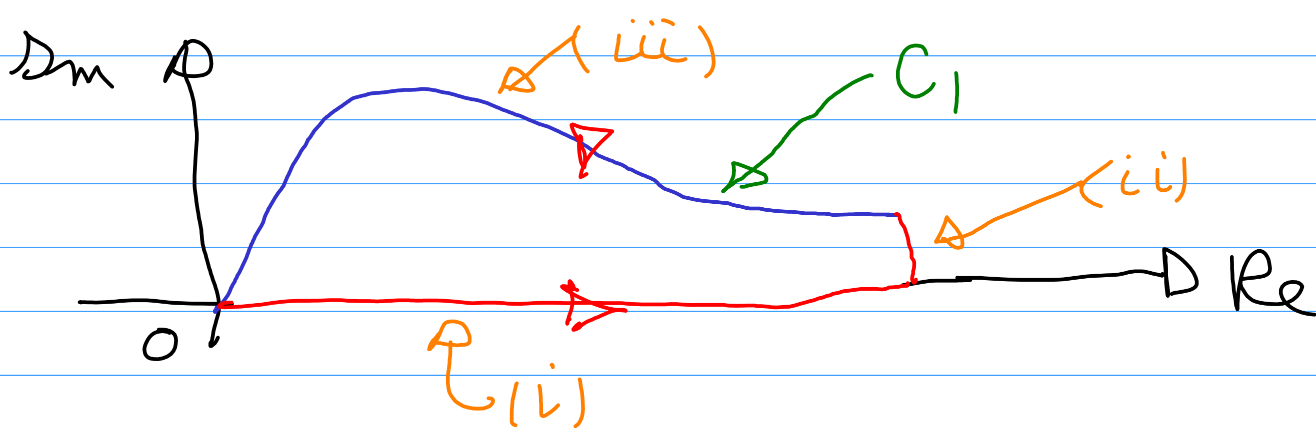
$$\left[\arctan(x)\right]_{-\infty}^{+\infty} = \pi \,,$$

why, take a hammer and

give at the right a thumb from above and at the left from below and we have

$$I \approx \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \pi$$

Or is a hammer not needed?
We are busy with integrals in $\mathbb{C}$ and we know that on closed contours, where you have no singularities inside the integral $= 0$!!
What can we do:



We have some closed contour C_1

$$f(z) = \frac{1}{1+z^2}$$ behaves well

inside, no singularities, so

$$\oint_{C_1} f(z)\,dz = 0$$

$C_1 = i + ii + iii$

maybe instead of + to write $\cup$ (union) but we play!

When I ever started with complex anal., I had often the idea to integrate about areas, but that is not the case

We are integrating about one-dimensional curves in the $\mathbb{C}$-plane. If those curves are closed, the inside (or outside) is of importance, because of those singularities, but of the area itself you see nothing!

So for the ~~curve~~ closed curve we take $i = \{x + i0 \mid 0 \le x \le R\}$, $ii = \{R + iy \mid 0 \le y \le \sin(\frac{R}{1+R^2})\}$ and $iii = \{x + i\sin(\frac{x}{1+x^2}) \mid 0 \le x \le R\}$

be careful with the orientation!

Then we have, see page 83,

$$\int_{(i)} \frac{1}{1+x^2}\,dx + \int_{(ii)} \frac{1}{1+z^2}\,dz + \int_{iii} \frac{1}{1+z^2}\,dz = 0 \Longrightarrow$$

?

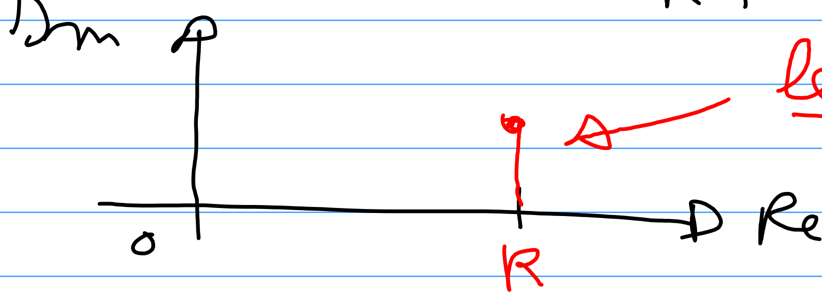
$$\int_{(i)} \frac{1}{1+x^2}\,dx = -\int_{(iii)} \frac{1}{1+z^2}\,dz - \int_{(ii)} \frac{1}{1+z^2}\,dz$$

Will $\int_{ii} \frac{1}{1+z^2}\, dz$ give some contribution if $R \to \infty$?

$\boxed{R>1}$

$z = R + iy$, with

$$0 \le y \le \sin\left(\frac{R}{1+R^2}\right) \le \left(\frac{R}{1+R^2}\right)$$



length?

abs. value!

$$\text{Length} \le \frac{R}{1+R^2}$$

Maximum value of $\left|\frac{1}{1+z^2}\right|$ at that interval?

$$|1+z^2| \ge \big|1-|z|^2\big| =$$

$$\big||z|^2-1\big| \ge (R^2-1)$$

$\boxed{R>1}$

$$\Longrightarrow \frac{1}{|1+z^2|} \le \frac{1}{(R^2-1)}$$

Conclusion:

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$$\left| \int_{(ii)} \frac{1}{1+z^2} dz \right| \leq \frac{R}{1+R^2} \cdot \frac{1}{R^2-1} \longrightarrow 0$$

if $R \to \infty$

Do the same at the left side of the imaginary ascis, but be careful with the orientation.

Conclusion:

$$\int_C \frac{1}{1+z^2} dz = \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx$$

One problem away I thought so, but then some windflow gave me some other question!

What is meant by: $\int_{-\infty}^{+\infty} f(x)\, dx$?

an improper integral

I told you:
series are limits of finite sums
and so we can say that:
So you can tell me:
an improper integral is the
limit of proper integrals.
Most of the time we do:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx .$$

Why we choose an interval
symmetric around the origin?
So during my run I had
constant in mind:

$$\int_{-\infty}^{+\infty} \frac{x}{1+x^2}\,dx$$ what happens
if I choose a

non-symmetric interval?

First of all if $f(x) = \left(\frac{x}{1+x^2}\right)$ then $f(-x) = -f(x)$, so f is odd,

so $\int_{-R}^{R} f(x)\,dx = 0 \quad \underline{\forall R \in \mathbb{R}.}$

But now:

$$\int_{-R}^{(2R)} f(x)\,dx = \int_{-R}^{R} f(x)\,dx + \int_{R}^{2R} f(x)\,dx$$

$$= \left[\frac{1}{2}\ln(1+x^2)\right]_R^{2R} =$$

$$\frac{1}{2}\cdot\ln\left(\frac{1+4R^2}{1+R^2}\right) \to \frac{1}{2}\ln(4) = \ln 2$$

for $R \to \infty$

or

$$\int_{-R}^{R^2} f(x)\,dx = \int_{R}^{R^2} f(x)\,dx = \frac{1}{2}\ln\left(\frac{1+R^4}{1+R^2}\right)$$

$(R>1)$ not exist for $R \to \infty$

Choosing the boundaries well,

you can get out of it, what
you want!

But if we look to $f(x) = \frac{x}{1+x^2}$
for great values of x,
we see $f(x)$ behaves like: $\frac{1}{x}$
That will be the cause of the
problem.

At Perron 15 at Oarle-Riestel
I saw some pencil and paper
and I had almost written the
ideas of above on paper, but
could restrain myself.
But here at home they are
written down, with the music
of "I will survive" at my

earphones.

We were busy with: $\frac{1}{1+z^2}$;

$$|1+z^2| \geq \left| |z|^2 - 1 \right|$$

so $\left|\frac{1}{1+z^2}\right| \leq \frac{1}{(|z|^2-1)}$ if $|z|>1$

that has a behaviour like $\frac{1}{|z|^2}$, nice integrable, if $|z|>1$

So the curve C_1' is already beaten flat, so

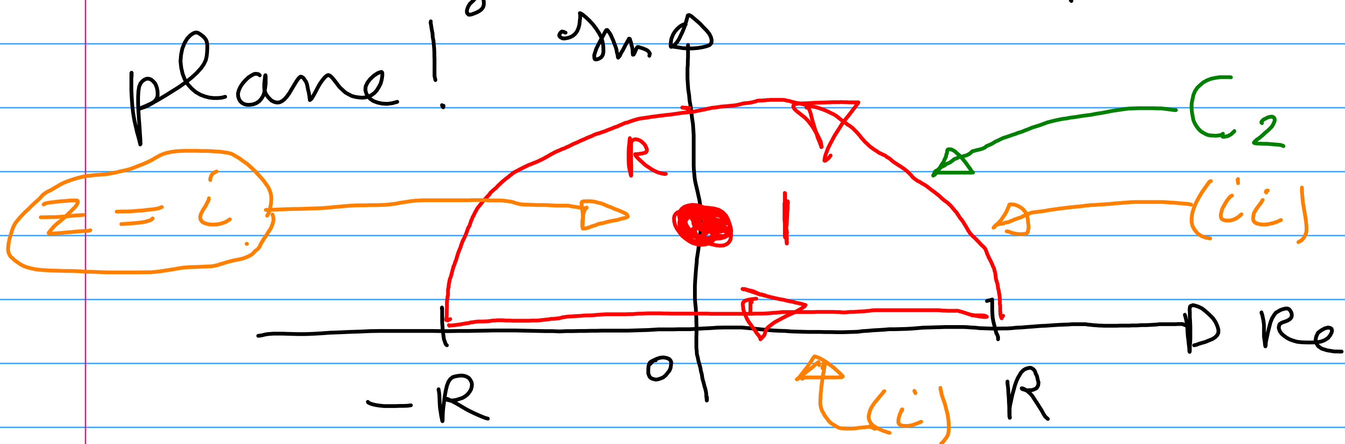
$$I = \int_{C_1'} \frac{1}{1+z^2}\, dz = \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx$$

but I want to do it with residues! We know:

improper = limit of proper

so $I = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1+x^2}\, dx$

Let's go to the complex plane!



Let's make a closed contour C_2

I should do it in the upper half plane, with a circle and let R be great $(R>1!!)$

What can we do with

$$\frac{1}{1+z^2}? \qquad z^2=1 \qquad z=\pm i$$

so $\qquad z^2+1=(z+i)(z-i)$

and $$\frac{1}{1+z^2}=\frac{A}{(z+i)}+\frac{B}{(z-i)}$$

$$A(z-i)+B(z+i)=1$$

$$A+B=0, (-A+B)i=1$$

$A = -B \rightsquigarrow B \cdot 2i = 1 \quad B = \frac{1}{2i} = \frac{-i}{2}$

$A = \frac{i}{2},\ B = \frac{-i}{2}$ &

$$\frac{1}{1+z^2} = \frac{i}{2} \cdot \frac{1}{z+i} - \frac{i}{2} \frac{1}{(z-i)}$$

Inside the closed curve we see a singularity $z = i$.

We have the residue!

Residue at $z = i$ is $-\frac{i}{2}$, just the coëff. of $\left(\frac{1}{z-i}\right)$.

!

Now we have:

$$\int_{C_2} \frac{1}{1+z^2}\, dz = (2\pi i) \cdot \left(\frac{-i}{2}\right) = \pi$$

But $C_2 = [-R, R] \cup \{|z| = R \text{ and } \operatorname{Im} z \geq 0\}$

So: $\int_{C_2} \frac{1}{1+z^2}\,dz = \pi$

$$\int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\substack{|z|=R \\ \operatorname{Im} z>0}} \frac{1}{1+z^2}\,dz$$

what happens if $R \to \infty$

*7 $\boxed{|\text{integral of } f \text{ on } C| \leq (\max |f| \text{ on } C) \cdot (\text{length of } C)}$

Good for in the summary!

Let $|z| = R$ then

$$|1+z^2| \geq \big||z|^2 - 1\big| \geq R^2 - 1$$

if $R > 1$, so

$$\frac{1}{|1+z^2|} \leq \frac{1}{(R^2-1)}$$

$\underline{\max |f|}$
$\underline{\text{on } C}$

length of C: $\frac{2\pi R}{2} = \pi R$

So

$$\left| \int_{(ii)} \frac{1}{1+z^2}\, dz \right| \leq \frac{\pi R}{R^2-1}$$

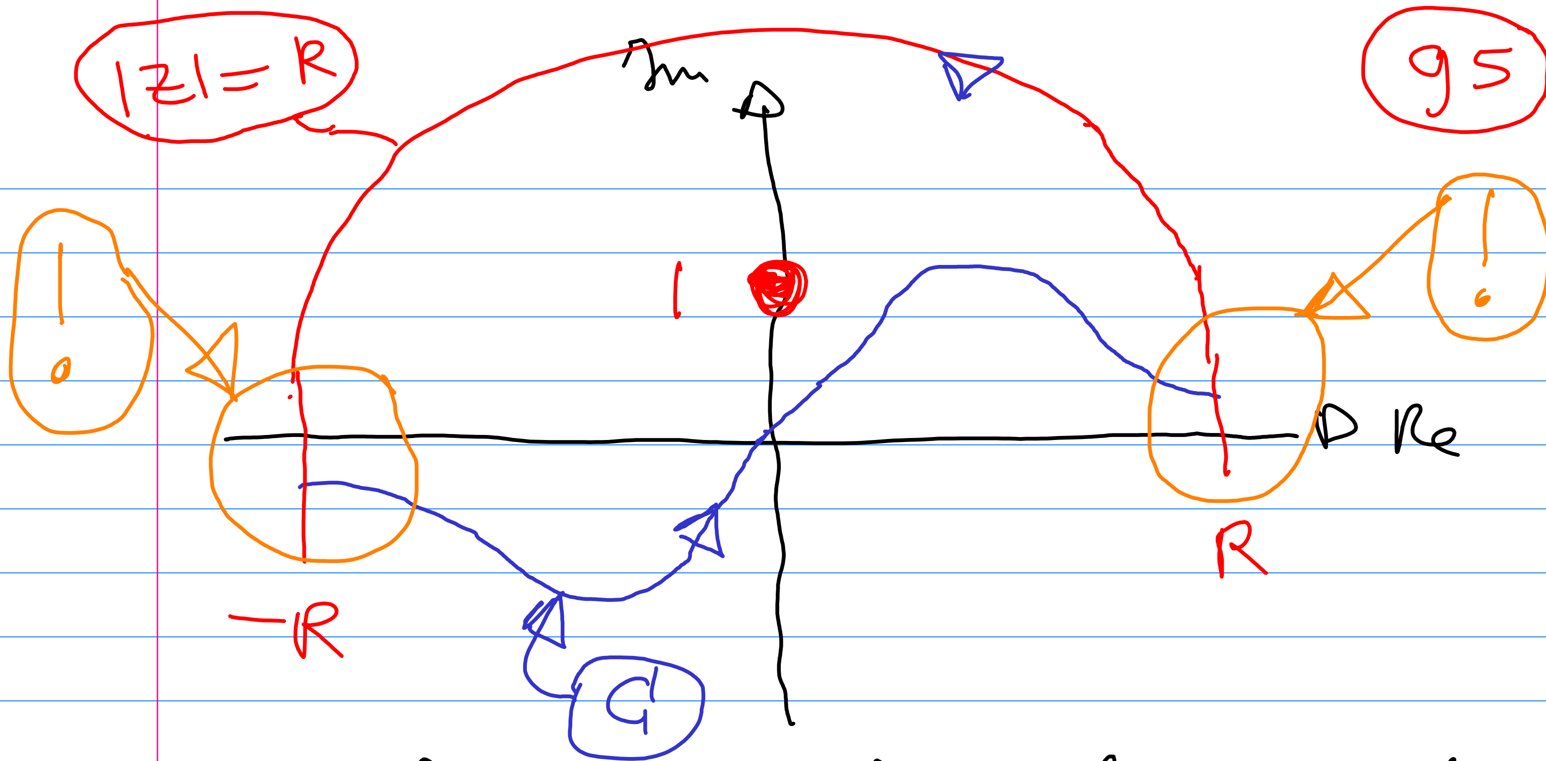
$\longrightarrow 0$ if $R \to \infty$

And if we take the limit of $R \to \infty$:

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \pi \text{ and}$$

we have the answer of the problem at page 81.

Remark: we could also do it at once! See beneath



But be careful at those little pieces near $z=-R$, and $z=R$, most of the time they go well, but sometimes??

So I would have looked if

$$\int_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} f(z)\,dz \to 0 \quad \text{if } R \to \infty$$

and if that goes well:

$$\int_{C_1^+} f(z)\,dz = 2\pi i \cdot \operatorname{res}_{z=i}(f) \quad ?$$

but be careful, those little pieces

And now? Maybe another run to get other ideas? Running in the nature, but next time a little less sun.

It is again almost weekend, what I'm thinking now that is about those Cauchy-Riemann equations. Just the question of about how to use them.

For instance, there is given that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\mathrm{Im}(f) = (-x + x^2 - 2y)$, asked is to determine the whole f as function of $z \in \mathbb{C}$ ($z = x + iy$, $x, y \in \mathbb{R}$).

What to do? We have seen that f can written as:

$f(z) = u(x,y) + i\,v(x,y)$,

so $v(x,y) = (-x + x^2 - y^2)$.

If f is holomorphic then v has to be harmonic, let' see if that is the case:

$v_x = -1 + 2x,\ v_{xx} = 2$
$v_y = -2y,\ v_{yy} = -2$

$v_{xx} + v_{yy} = 2 - 2 = 0$,

that is oké,

May be we can use those C.R-eqn's?

$$\text{C.R.} \begin{cases} u_x = v_y & (i) \\ u_y = -v_x & (ii) \end{cases}$$

We need both equations, but use them in a good order.
What should I do? Look to the first order derivatives, an easy one is:

$$v_y(x,y) = -2y,$$

that means, see C.R.(i), that

$$u_x(x,y) = -2y$$

Integrate to the variable x and we get

$$u(x,y) = -2y \cdot x + g(y).$$

!

Not an integration constant but a function which depends on y, because we integrated to x.

Differentiate u to x and we get back where we started.

The question becomes, how to determine the unknown function g? Till now we only used C.R.(i), so the other one has also to be used.

To get something new, the only thing we can do that is to calculate:

$$u_y(x,y) = -2x + g'(y)$$

and now use C.R.(ii), so:

$$u_y(x,y) = \underline{-2x + g'(y)} =$$

$$-v_x(x,y) = -(-1 + 2x) =$$

$$\underline{1 - 2x}.$$

We are searching for the unknown function g, which only depends on the variable y

We get:

$$-2x + g'(y) = 1 - 2x,$$

everything with an x it has to fall away and see

$$g'(y) = 1 \Longrightarrow \quad \in \mathbb{R}$$

$$g(y) = y + C.$$

We have already integrated to x, then we integrated to y, so in this last step we can only get some integration constant. So we have found:

$$U(x,y) = -2 \cdot y \cdot x + y + C_1.$$

$u_x = -2y$ $\quad u_{xx} = 0$ $\quad v_x = -1 + 2x$, $v_y = -2y$

$u_y = -2x + 1$ $\quad u_{yy} = 0$ $\quad v_{xx} = 2$, $v_{yy} = -2$

We have found the real part of f, that means:

$$f(x,y) = (-2 \cdot y \cdot x + y + C_1)$$
$$+ i \cdot (-x + x^2 - y^2)$$

u has to be harmonic, so some control, if u is possible good,

$u_x = -2y$, $u_{xx} = 0$
$u_y = -2x + 1$, $u_{yy} = 0$ $\Big\}$ $u_{xx} + u_{yy} = 0$ that is oké.

But now we to write f, as a function of z. We have taken $z = x + iy$. What you do is the following, write: $x = (z - iy)$.
Calculate: $f(z - iy, y)$, if everything goes well, then

y has to fall away. So let's try

$$f(z-iy,y) =$$

$$-2y(z-iy)+y+C+$$

$$i\left(-(z-iy)+(z-iy)^2-y^2\right)=$$

$$\cancel{-2yz}+\cancel{2iy^2}+\boxed{\cancel{y}}+C+$$

$$i\left(-z+\boxed{\cancel{iy}}+z^2\cancel{-2iyz}\cancel{-y^2-y^2}\right)=$$

$$C+i(-z+z^2),$$ result:

$$f(z)=-iz+iz^2+C,$$

with $C\in\mathbb{R}$

<u>Some remarks:</u>

- If $f(x,y)$ found, another possibility to find $f(z)$, that is to substitute:

$$x=\left(\frac{z+\bar{z}}{2}\right) \text{ and } y=\frac{z-\bar{z}}{2i}.$$

So if you have filled in x, and y and you written it out, all $\bar{z}$ have to be fallen away.
I think that what I have done above: $x = z - iy$ or $y = -i(z - x)$ are the easiest ways to do.

- Realize yourself, $\bar{z}$ is not holomorphic, and expressions $|z|$ in it are not holomorphic.
- And a holomorphic function $f: \mathbb{C} \to \mathbb{R}$ can only be a constant function.

If $f: \mathbb{C} \to \mathbb{R}$ then

$$f(x,y) = u(x,y) + i \cdot 0,$$

use C.R-eqn's and see that

$u_x = 0$ and $u_y = 0 \Rightarrow$

$u(x, y) = C \quad (\in \mathbb{R})$

so $f(x, y) = C$.

- Use both C.R. eqn.'s, by filling in the result of the first one used, into the other one. Don't solve these equation separately of each other.
- If you think you know a certain function $f(z)$, which satisfies to all what is given, let then see that everything goes well.

The next pages: (till the end?)
"I had in mind holomorphic"
but I have not written, sorry!

It's Sunday, till I reached the bench, I had the melody of "I will survive" in mind.
At the bench listening to the nature, I got the stupid question in mind: "How can I get

$\frac{\partial^n f}{\partial z^n}(z)$ of $f(z)$, without

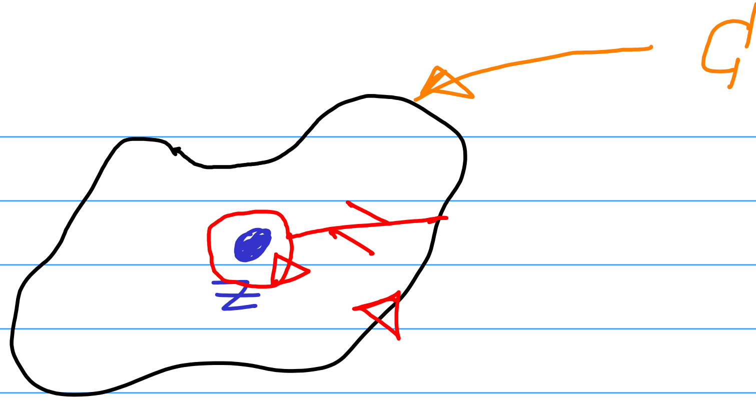
differentiating f to z?".
We have, see pg. 65:

$$f(z) = \oint_{C} \frac{f(w)}{(w-z)}\,dw$$

and see pg. 72:

$$\begin{cases} I(R,-1) = 2\pi i, \\ I(R,p) = 0, \quad p \in \mathbb{Z}\setminus\{-1\} \end{cases}$$

and the idea:



$$\oint_{C} f(w)\,dz = \oint_{C(z,R)} f(w)\,dw$$

C some arbitrary contour and $C(z_0,R)=\{z\in\mathbb{C} \mid |z-z_0|=R\}$ with $0<R$ small enough, so $C(z_0,R)$ a circle inside the contour C.

We play, so let we see what we get out of it. But I think we have to be careful, later on, by the use of the Laurent series. First I want to have some escpression and then we see if that is always correct?

So we have:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)} \, dw$$

let's assume that f can expanded in a Laurent series around z, then (z fixed)

$$f(w) = \sum_{n=-\infty}^{+\infty} c_n \cdot (w - z)^n, \text{ then}$$

$\begin{pmatrix} p \text{ fixed} \\ \text{and } p \in \mathbb{Z} \end{pmatrix}$ $\displaystyle \frac{f(w)}{(w - z)^p} = \sum_{n=-\infty}^{+\infty} c_n \cdot \frac{(w-z)^n}{(w-z)^p}$,

there will be some $n \in \mathbb{Z}$ with $(n - p) = -1$. Keep in mind $I(R, -1) = 2\pi i$ and that gives:

$$\oint_{C'} \frac{f(w)}{(w - z)^p} \, dw = 2\pi i \cdot c_{(p-1)}.$$

So $c_{p-1} = \dfrac{1}{2\pi i} \oint_{C} \dfrac{f(w)}{(w-z)^p}\, dw,$

that is nice, but have we found that good formula ourselves?
I have the idea we have to look to those Laurent series, what is their domain of convergence?

Let $f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$ then

$$f(z) = \underbrace{\sum_{n=-\infty}^{-1} c_n (z-a)^n}_{\text{neg. powers}} + \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{pos. powers}}$$

$$= \underbrace{\sum_{1}^{\infty} c_{-n} (z-a)^{-n}}_{(\mathrm{I})} + \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{(\mathrm{II})}$$

Let's look to (I):

Domain of conv.? Ratio test for instance:

$$\lim_{n\to\infty}\left|\frac{c_{-(n+1)}(z-a)^{-(n+1)}}{c_{-n}(z-a)^{-n}}\right|<1$$

$$\Rightarrow \frac{1}{|z-a|}\cdot\lim_{n\to\infty}\left|\frac{c_{-(n+1)}}{c_{-n}}\right|<1$$

$$\Rightarrow |z-a| > \lim_{n\to\infty}\left|\frac{c_{-(n+1)}}{c_{-n}}\right| = R_1$$

From (II), doing the ratio test, we know that

$$|z-a|\cdot\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|<1, \text{ so:}$$

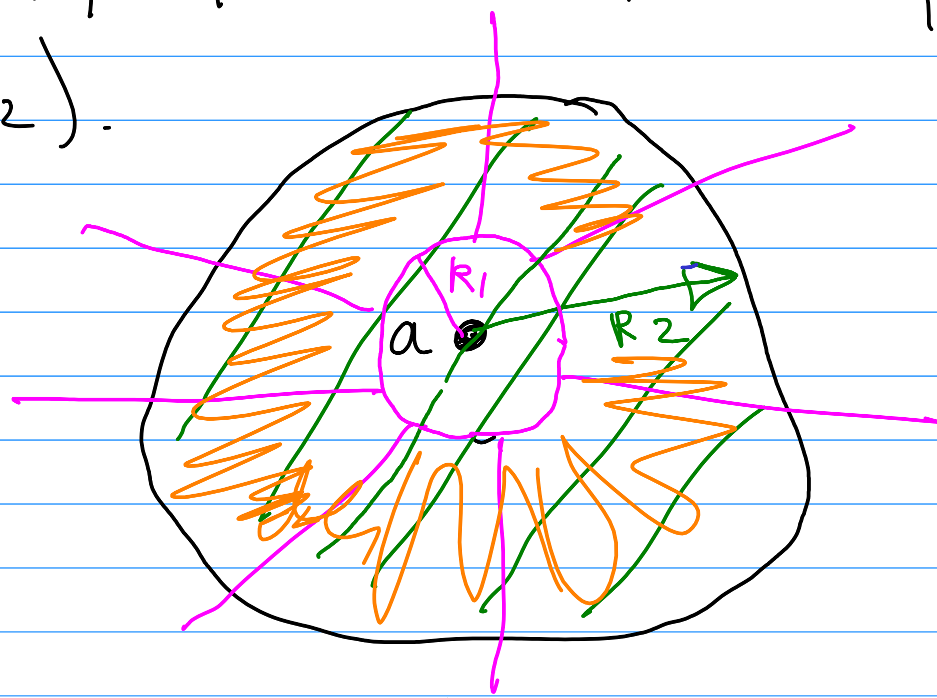
$$|z-a| < \left(1\Big/\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|\right) = R_2$$

The Laurent series only converges if $R_1 < R_2$, and then we get some

annular domain of convergence

$\{ z \in \mathbb{C} \mid R_1 < |z-a| < R_2 \}$

$(R_1 < R_2)$.



the overlap of the inner region and the outer region.

To make the formula on pg. 108 correct, do 2 things:

i) Replace C by a curve in the annular domain encircling a in positive direction.

ii) Replace z by a.

G has to be Jordan curve, so a curve which is closed, without intersection points. Search on google for more information.

So:

$$c_n = \frac{1}{2\pi i} \cdot \oint_G \frac{f(w)}{(w-a)^{(n+1)}}\, dw$$

with $n \in \mathbb{Z}$.

But the first question was about the $\frac{\partial^n f}{\partial z^n}(z)$? How to do that without taking the derivative. Let f be holomorf on some simple connected domain. A domain without holes in it. Every closed curve can be contracted to a point of that domain. Search on internet for more escamples.

Keep in mind that we work in $\mathbb{C}$, so a 2-dim. space.

We have already seen several times: $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)} \, dw$,

We know the Taylor expansion of f around z (so z is fixed)

$$f(w) = f(z) + f'(z)(w-z) + \dots$$
$$\dots + \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(z)(w-z)^n + O\left((w-z)^{n+1}\right)$$

with w in the neighbourhood of z.

Divide $f(w)$ by $(w-z)^{n+1}$, that gives us

$$\frac{1}{(2\pi i)} \oint_{C_1} \frac{f(w)}{(w-z)^{n+1}} \, dw =$$

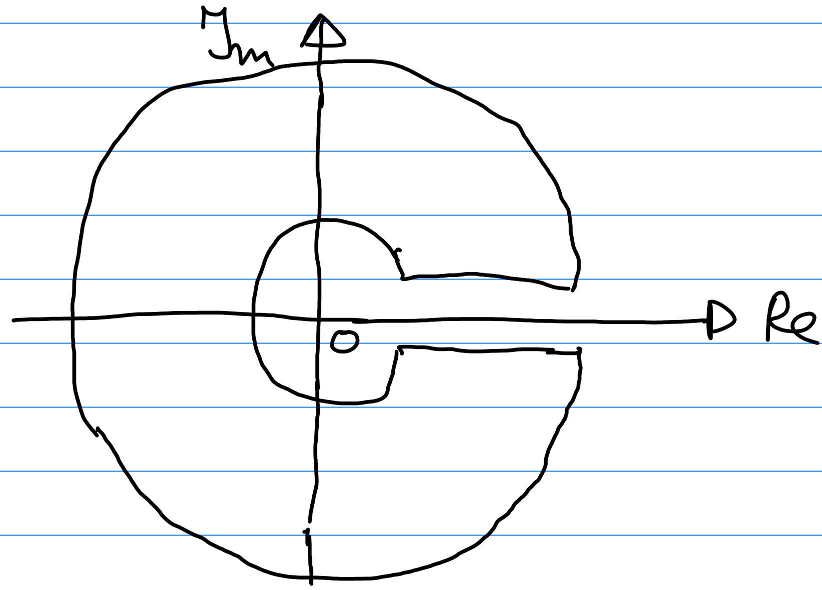
$$\frac{1}{2\pi i} \cdot \oint_{C_1} \frac{1}{n!} \frac{\partial^n f}{\partial z^n}(z) \cdot \frac{1}{(w-z)} \, dw =$$

$$\frac{2\pi i}{2\pi i} \cdot \frac{1}{n!} \cdot \frac{\partial^n f}{\partial z^n}(z^n) \Longrightarrow$$

$$\frac{\partial^n f}{\partial z^n}(z) = \frac{n!}{2\pi i} \cdot \oint_{C_1} \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Here we found what we sought.

While I googled at internet, I saw a nice contour, called the "Pac-Man" integration contour:



I'm searching some integral where this Pac-Man contour can be used. Maybe this one:

$\int_0^\infty \frac{x^{p-1}}{(1+x)}\,dx$? Maybe we can try to solve it, little by little.

Let $0<p<1$, so $p\in\mathbb{R}$, and if we want to use that Pac-Man contour we have to jump into the complex plane.

What is z^{p-1} with $z\in\mathbb{C}$?

In the past they solved quadratic equations as follows:

$x^2+2x+8=0 \Rightarrow (x+1)^2=-7$

$x_{1,2}=-1\pm\sqrt{-7}$. It went well, if you fill in the equation, it went well.

$(-1+\sqrt{-7})^2+2(-1+\sqrt{-7})+8=$

$1-\cancel{2\sqrt{-7}}+(\sqrt{-7})^2-2+\cancel{2\sqrt{-7}}+8=$

$1+(-7)-2+8=0$. Maybe somebody thought $\sqrt{-7}=\sqrt{7}\cdot\sqrt{-1}$

and then why still writing $\sqrt{-1}$?
$\sqrt{-1}$ is something imaginary, so
$\sqrt{-1} = i$, and $i^2 = -1$.

Let we try to do the same with: $z^{(p-1)}$
$z \in \mathbb{C}$. Let's find something that
works well, If $x \in \mathbb{R}$, then we
know: $x^{p-1} = \exp(\ln(x^{p-1}))$, so
let $x > 0$, otherwise problems with
ln-function. So $x^{p-1} = \exp((p-1)\ln(x))$,
if we could replace the x by z, it
would work, but what is meant by
$\ln(z)$? (We play and try! We know
$z = R.\exp(i\varphi) = |z| \cdot \exp(i \arg(z))$.
We can discuss about $\arg(z)$, or shall
we take $\mathrm{Arg}(z)$? $\mathrm{Arg}(z)$ is some
interval of length: 2π. Here we

have some freedom. Inside the Pac-Man contour, I don't want to have jumps of certain functions, so may a good choice will be: $0 \leq \mathrm{Arg}(z) < 2\pi$.

So we have one value for $\mathrm{Arg}(z)$ and not infinetly many:

$$\arg(z) = \mathrm{Arg}(z) + k \cdot 2\pi, \quad k \in \mathbb{Z}.$$

So we write $z = |z| \cdot \exp(i\,\mathrm{Arg}(z))$, and our ln becomes:

$$\ln(z) = \ln(|z| \cdot \exp(i\,\mathrm{Arg}(z))) =$$

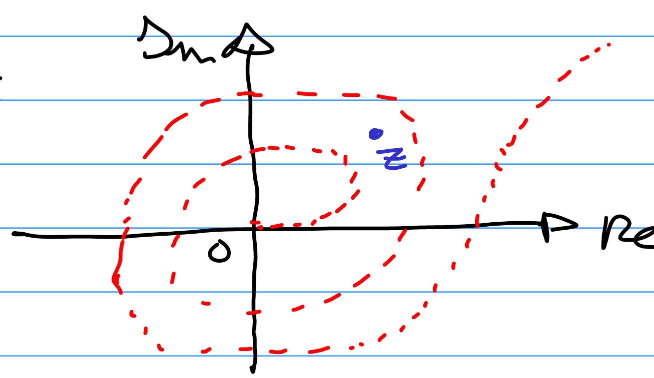
$$\ln|z| + i \cdot \mathrm{Arg}(z),$$ just one value!

It works just like the $\sqrt{7}$ of above:

$$\exp(\ln(z)) = \exp(\ln|z| + i\,\mathrm{Arg}(z)) =$$

$$\exp(\ln|z|) \cdot \exp(i\,\mathrm{Arg}(z)) = z.$$

So with that Arg(z) you have a certain freedom. "certain" maybe better a lot of freedom? You want to have for each $z \in \mathbb{C}$, just one value for Arg(z) and not infinitely many. Now we have chosen the real axis as a kind of barrier. But in another problem, it can be of interest to take another barrier, and maybe it has not to be always a straight line. At that bench I thought to the following barrier:



Each $z \in \mathbb{C}$ has just one Arg(z),

otherwise you have to cross that barrier! But don't ask me to describe that given barrier, some spiral?

So we use: $\ln(z) = \ln|z| + i\,\mathrm{Arg}(z)$

with: $0 \leq \mathrm{Arg}(z) < 2\pi$, and so we

have: $z^{p-1} = \exp(\ln(z^{(p-1)})) =$

$\exp((p-1)\cdot \ln(z)) =$

$\exp[(p-1)\cdot(\ln|z| + i\,\mathrm{Arg}(z))] =$

$\exp((p-1)\cdot\ln|z|)\cdot\exp(i(p-1)\cdot\mathrm{Arg}(z)) =$

$|z|^{(p-1)}\cdot\exp(i(p-1)\cdot\mathrm{Arg}(z))$.

If $z = x \in \mathbb{R}$, and $x > 0$ then everything goes well! $\mathrm{Arg}(x) = 0$, if $0 < x \in \mathbb{R}$.

What happens with $z^{(p-1)}$, near the positive real axis? See those two parallel lines above and beneath the real axis, with $x > 0$. I'm interested.

So let $f(z) = z^{(p-1)}$ and now I want to calculate $\lim\limits_{h \downarrow 0} f(x + hi)$ and

$\lim_{h \downarrow 0} f(x - ih)$ with $x > 0$.

What do we see? If $x > 0$ and $h > 0$, then $\lim_{h \downarrow 0} \mathrm{Arg}(x + ih) = 0$ and $\lim_{h \downarrow 0} \mathrm{Arg}(x - ih) = 2\pi$. What does it mean for those limits of f near the real axis? $(x > 0)$

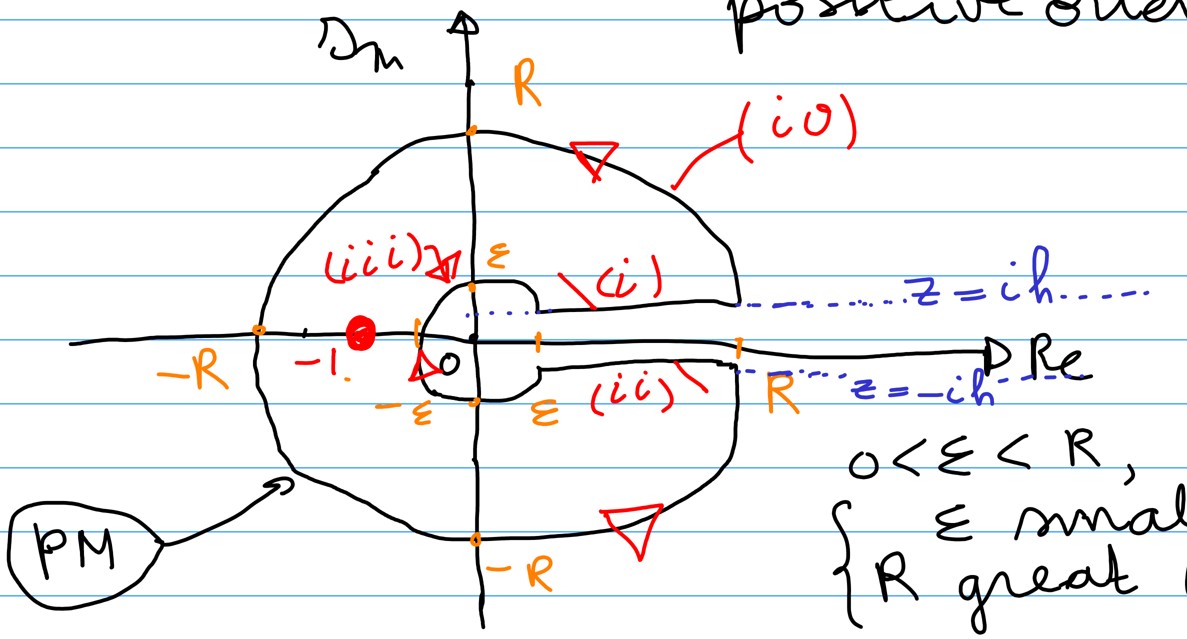
$$\lim_{h \downarrow 0} f(x + ih) = x^p \quad \text{and}$$

$$\lim_{h \downarrow 0} f(x - ih) = x^p \cdot \exp(i(p-1) \cdot 2\pi)$$

This function f makes some jump near the real axis! Take $p = \frac{1}{2}$, or take $p = \frac{1}{4}$ and you get a complex number out of it. The $\ln(z)$ jumps also across the real axis, have you realised that?

That Pac-Man contour how many

pieces do we have? Keep careful: positive oriented.



$0<\varepsilon<R$,
$\begin{cases} \varepsilon \text{ small } (\varepsilon \to 0?) \\ R \text{ great } (R\to\infty?) \end{cases}$

So let' sum up:

i) $z = x + ih$, $\varepsilon < x < R$, $h > 0$.
ii) $z = x - ih$, $\varepsilon < x < R$, $h > 0$.
(iii) $|z| = \varepsilon$, small part out of it.
(iv) $|z| = R$, small part out of it.

and look to the point: $z = -1$.

What to do first? Maybe, in this case the easiest one:

$$\oint_{PM} \frac{z^{p-1}}{1+z}\, dz.$$

In $z = -1$ a singularity of the

first order, and we can directly calculate the residue of $\left(\frac{z^{p-1}}{1+z}\right)$ in $z=-1$. We get $(-1)^{p-1}$, we are in the complex plane so:

$$(-1)^{(p-1)} = |-1|^{p-1} \cdot \exp(i(p-1)\cdot \operatorname{Arg}(-1))$$
$$= 1 \cdot \exp(i(p-1)\cdot \pi).$$

And that means:

$$\oint_{PM} \frac{z^{p-1}}{1+z}\, dz = 2\pi i \cdot \exp(i(p-1)\pi)$$

Let's look to (i):

$$\lim_{h \to 0} \int_{\varepsilon}^{R} \frac{(x+ih)^{p-1}}{1+(x+ih)}\, dx = \int_{\varepsilon}^{R} \frac{x^{p-1}}{1+x}\, dx$$

and now to (ii), be careful with orientation,

$$\lim_{h \to 0} - \int_{\varepsilon}^{R} \frac{(x-ih)^{p-1}}{1+(x-ih)}\, dx =$$
$$- \int_{\varepsilon}^{R} \frac{x^{(p-1)} \cdot \exp(i(p-1)\cdot 2\pi)}{(1+x)}\, dx =$$

$$- \exp(i(p-1)2\pi) \cdot \int_{\varepsilon}^{R} \frac{x^{p-1}}{(1+x)} dx$$

So we have the following result, so far:

$$(i) + (ii) = (1 - \exp(i(p-1)2\pi)) \int_{\varepsilon}^{R} \frac{x^{p-1}}{1+x} dx$$

Now those integrals (iii) and (iv), of interest is, what happens, if $\varepsilon \to 0$ and $R \to \infty$. Will they go to zero, or will they distribute some value to the ingral?

Let's do (iii) first. What else to use then: $\left| \int_{C_1} f(z) dz \right| \leq \max_{C} |f(z)| \cdot \text{length}(C_1)$?

Let's try; with $0 < \varepsilon << 1$ (ε is small)

$$|1 + z| \geq ||1 - |z|| = |1 - \varepsilon|, \text{ so:}$$

$$\frac{1}{|1+z|} \leq \frac{1}{(1-\varepsilon)}, \text{ further } 0 < p < 1$$

so $(1-p) > 0$, and:

terrible! $(p-1) < 0$

$$|z^{p-1}| \leq |z|^{(p-1)} \leq \varepsilon^{(p-1)}$$

So we have:

$$\left|(iii)\right| \leq \left| \int_{C_\varepsilon} \frac{z^{(p-1)}}{1+z} dz \right| \leq \left(\frac{\varepsilon^{p-1}}{1+\varepsilon} \right) \cdot \underbrace{2\pi\varepsilon}_{\text{(perimeter circle)}}$$

$$= \left(\frac{2\pi \cdot \varepsilon^{p}}{1+\varepsilon} \right) \to 0 \text{ if } \varepsilon \downarrow 0,$$

because: $0<p<1$, the perimeter of the circle helped us!

The last one: (iv) (Take: $R \gg 1$ $(R \to \infty)$)

$$|1+z| \geq |1-|z|| = |1-R| = (R-1)$$

$$|z^{1-p}| \leq |z|^{1-p} = R^{1-p}$$

So we have:

$$|(iv)| \leq \frac{R^{(p-1)}}{R-1)} \cdot \underbrace{2\pi R}_{\text{(perimeter circle)}} \to 0 \quad \text{if } R \to \infty,$$

because $0<p<1$, also here the perimeter of the circle was doing us a favor!

So let $\varepsilon \downarrow 0$ and $R \to \infty$ and we obtain that:

$$(1-\exp(i(p-1)2\pi))\cdot\int_0^\infty \frac{x^{p-1}}{1+x}\,dx =$$

$$2\pi i\cdot\exp(i(p-1)\cdot\pi)\ ,\ \text{so:}$$

$$-\sin((p-1)\pi)$$

$$\frac{(\exp(-i(p-1)\pi)-\exp(i(p-1)\cdot\pi))}{2i}\cdot$$

$$\left(\int_0^\infty \frac{x^{p-1}}{1+x}\,dx\right)=\pi \Rightarrow$$

$$\int_0^\infty \frac{x^{p-1}}{(1+x)}\,dx = \frac{\pi}{-\sin((p-1)\pi)}\ ,$$

$$\sin((p-1)\pi)=\sin(p\cdot\pi-\pi)=-\sin(p\pi)$$

$$\int_0^\infty \frac{x^{p-1}}{(1+x)}\,dx = \frac{\pi}{\sin(p\cdot\pi)}\ ,\ 0<p<1.$$

If we take $p=\frac{1}{2}$, then

$$\int_0^R \frac{1}{\sqrt{x}}\cdot\frac{1}{1+x}\,dx = \left[2\cdot\arctan(\sqrt{x})\right]_0^R$$

$$2\cdot\arctan(R)\to 2\cdot\frac{\pi}{2}=\pi \quad \text{with } R\to\infty,$$

it corresponds with the result we found!

But now, here behind the table, arises the question can I describe the arcsin, arccos and arctan with the logarithm?

Let's try: $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} = w$

$\Rightarrow \left(e^{iz} - e^{-iz} = 2i \cdot w\right) \cdot e^{iz}$

$e^{2iz} - 2i \cdot w \cdot e^{iz} - 1 = 0 \Rightarrow$

$(e^{iz} - wi)^2 + w^2 - 1 = 0 \Rightarrow (e^{iz} - wi)^2 = (1 - w^2) \Rightarrow$

$e^{iz} = \left(wi + \sqrt{1 - w^2}\right) \Rightarrow$

$z = -i \ln\left(i \cdot w + \sqrt{1 - w^2}\right) = \arcsin(w)$

Maybe, we have to take another interval for Arg in the ln we use?

And what to do with that $\sqrt{1-w^2}$, if $w \in \mathbb{C}$?

But we have something. Try yourself for the $\arccos(w)$ $\left[= -i \ln\left(w + \sqrt{w^2 - 1}\right)\right]$.

And we have

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{i}\left(\frac{\exp(iz) - \exp(-iz)}{\exp(iz) + \exp(-iz)}\right)$$

$= w \quad \Rightarrow$

$$z = \frac{i}{2}\left(\ln(1-iw) - \ln(1+iw)\right) = \arctan(w)$$

$$\frac{d}{dw}(\arctan(w)) = \frac{i}{2}\cdot\left(\frac{(-i)}{1-iw} - \frac{i}{1+iw}\right) =$$

$$\frac{i}{2}\cdot\frac{-i(1+iw) - i(1-iw)}{(1-iw)(1+iw)} = \frac{-i\cdot i}{1+w^2} = \frac{1}{1+w^2}$$

But be careful these functions are not defined everywhere in the complex plane. We have to do with so-called branches. But that we saw earlier on pg. 118, where we defined our complex logarithm. There we called it a barrier, but branch is nice mathematical term for it. But be always careful, there are general accepted branches, but sometimes they can be chosen at other places.

In real analysis, the ln-function is defined by $\ln(x) = \int_1^x \frac{1}{t}\,dt$, and it doesn't exist for $x < 0$. Going to the complex plane, the negative real axis can be chosen as a branch, but then

$$\ln(z) = \ln|z| + i\,\mathrm{Arg}(z), \text{ with}$$

$$-\pi < \mathrm{Arg}(z) < \pi.$$

See difference with pg. 118. With the ln just defined, we had not obtained the result on pg. 124. Then we would have some jump of the ln inside the domain surrounded by the Pac-Man contour, but also on the contour itself. That we have to avoid and the singularity $z = -1$, would by at the branch! Be careful.

That gives me the question can I do something if a singularity lies at the boundary of a contour? It was a little bit searching but in one of the lecture notes of Boersma I saw the following example

$$\int_0^\infty \frac{1-\cos(x)}{x^2}\,dx,$$ so what to do?

Near $x=0$ no problems:

$$\frac{1-\cos(x)}{x^2} = \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)}{x^2}$$

$$= \frac{1}{2} + O(x^2).$$

For instance $\left(\frac{\sin x}{x^2}\right)$ is not integrable near $x=0$, because

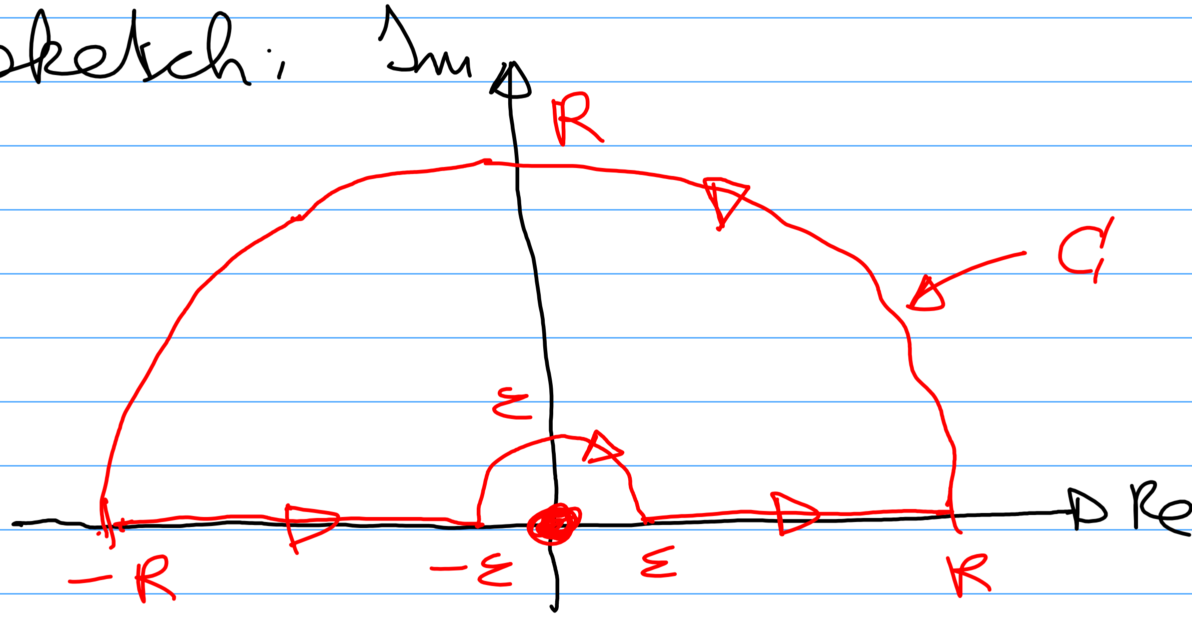
$$\frac{\sin(x)}{x^2} = \frac{1}{x} - \frac{x}{3!} + O(x^3).$$

and $\int_0^\infty \frac{1-\cos(x)}{x^2}\,dx = \mathrm{Re}\left(\int_0^\infty \frac{1-e^{ix}}{x^2}\,dx\right)$, we want to go to the complesc plane, but what part of it? Look if $z = x+iy$ then $e^{iz} = e^{i(x+iy)} = e^{-y} \cdot e^{ix}$,

if $y>0$ then $\lim_{y\to+\infty} e^{-y} = 0$, so let' take the upper half plane.

We need some nice closed contour C, and inside that contour we want to have no singularities.

See sketch:



Take the closed contour C always such that when $\varepsilon \downarrow 0$, the singularity will be thrown out of the inner region of the closed contour.

Be always careful with the orientation. In this we know already that the answer has to be positive because $\frac{1-\cos(x)}{x^2} > 0$ for $x > 0$. If we get something negative, somewhere is made a mistake. So we have

$$\int_{C} \frac{1-e^{iz}}{z^2}\,dz = 0.$$

If $|z| = R$, and $\mathrm{Im}(z) > 0$: C_R^+ then

$$\left| \int_{C_R^+} \frac{1-e^{iz}}{z^2}\, dz \right| \le \frac{2}{R^2}(\pi R) \to 0 \text{ if } R \to \infty.$$

Then we have that little "circle" around $z=0$, $z = \varepsilon \cdot e^{i(\pi-\varphi)}$, $0<\varphi<\pi$, let call C_ε^+, look to orientation and $\Im(z)>0$! Don't fill into the integral, let look to the behavior of the integrand near $z=0$:

$$\frac{1-e^{iz}}{z^2} = \frac{1-\left(1+iz-\frac{z^2}{2!}-\dots\right)}{z^2}$$

$$= \frac{-i}{z} + \underbrace{\frac{1}{2} + O(z)}_{\text{will have contribution to the integral if } \varepsilon \downarrow 0.}$$

$$\int_{C_\varepsilon^+} \frac{-i}{z}\, dz = \int_0^\pi \frac{-i\,\overbrace{\left(-i\cdot\cancel{\varepsilon}\,\cancel{e^{i(\pi-\varphi)}}\right)}^{(dz)}}{\cancel{\varepsilon}\,\cancel{e^{i(\pi-\varphi)}}}\, d\varphi$$

$$= \int_0^\pi -1\, d\varphi = -\pi$$

$$\operatorname*{res}_{z=0}\left(\frac{-i}{z}\right) = -i$$

$$\int_C \frac{-i}{z}\,dz = (2\pi i)\cdot(-i) = 2\pi$$

C - closed curve around $z=0$ pos. oriented

$$\int_{C_\varepsilon^+} \frac{-i}{z}\,dz = (-1)\cdot\left(\frac{2\pi}{2}\right) = -\pi$$

(neg. orientation)

residue

(half circle)

!

$-1\cdot$ (half residue)

(orientation!)

We have:

$$\int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2}\,d + \int_{C_\varepsilon^+} \frac{1-e^{iz}}{z^2}\,dz +$$

$$\int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2}\,dx + \int_{C_R^+} \frac{1-e^{iz}}{z^2}\,dz = 0$$

$$\int_{-R}^{-\varepsilon} \frac{1-\cos(x)}{x^2}\,dx = \int_{R}^{\varepsilon} \frac{1-\cos(-y)}{(-y)^2}\,(-1\cdot dy)$$

!

$(y=-x)$

$$= \int_{\varepsilon}^{R} \frac{1-\cos(x)}{x^2}\,dx$$

So: $2 \cdot \int_{\varepsilon}^{R} \frac{1-\cos(x)}{x^2} dx =$

$$\mathrm{Re}\left(+ \pi + \int_{C_R^+} \frac{1-e^{iz}}{z^2} dz \right),$$

let $\varepsilon \to 0$ and $R \to \infty$ and we conclude

$$\int_0^\infty \frac{1-\cos(x)}{x^2} dx = \left(\frac{\pi}{2}\right)$$

In short, if you have a half circle you get also a half residue. Be careful with the orientation!

What to do now?

In the exam excercises I see often an excercise, where properties of a function f are given. There is also given that f is holomorphic and there is asked to reconstruct that function f.

No idea if the next one is easy or difficult, let's just try. May be find some theorem which is used.

Of some holomorphic function

$f : \mathbb{C} \setminus \{-i, 0, i\}$ is given:

i) in $z = 0$ a pole of order 2

in $z = i$ and $z = -i$ a pole of order 1,

ii) $z^2 \cdot f(z)$ is bounded for $|z| > 1$,

iii) f is even: $f(-z) = f(z)$

iv) $\operatorname{res}_{z=i}(f) = \frac{i}{2}$

(v) $\operatorname*{res}_{z=0}(h) = 1$, with $h(z) = z \cdot f(z)$.

Naturally I have looked to other exams and their solutions. I noticed that the solutions were always something of:

$$f(z) = \frac{(\text{polynomial in } z)}{(\text{polynomial in } z)}$$

Let's try if we can get the same out of this example.

If you read i), it looks that

$$f(z) = \frac{p(z)}{(z-0)^2 \cdot (z-i) \cdot (z+i)} \quad \text{i)}$$

p a polynomial in z,

only $p(0) \neq 0$, $p(i) \neq 0$, $p(-i) \neq 0$

if $p(i) = 0 \rightsquigarrow p(z) = (z-i) \cdot q(z)$

and the pole in i falls away.

Out of (ii)?

$$z^2 f(z) = \frac{\cancel{z^2} \cdot p(z)}{\cancel{z^2}(z-i)(z+i)}$$

is bounded for $|z|>1$. This gives me the idea that $p(z)$ only can be a polynomial of order 2.

So I think: $p(z) = Az^2 + Bz + C$

$$f(z) = \frac{Az^2+Bz+C}{z^2(z-i)(z+i)} \quad \text{ii)}$$

Using (iii)? What will it tell about A, B, C the constants?

$$(z-i)(z+i) = (z^2+1)$$

!

$$f(-z) = \frac{Az^2 - Bz + C}{z^2(z^2+1)} =$$

$$\frac{Az^2+Bz+C}{z^2(z^2+1)} = f(z) \quad \text{iii)}$$

Numerators have to be equal for all values of z, so $B=0$

Result: $f(z) = \dfrac{Az^2 + C}{z^2(z-i)(z+i)}$

Let's try to use i)

To calculate the residue in $z=i$:

$$\lim_{z \to i} (z-i) f(z) = \frac{i}{2}$$

$$(z-i) \cdot f(z) = \left. \frac{Az^2 + C}{z^2(z+i)} \right|_{z=i}$$

$$= \frac{-A+C}{-1 \cdot (2i)} = \frac{i}{2} \quad \text{so}$$

$$-A + C = 1.$$

We have to get another equation for A and C, so let use ii)

$$h(z) = z \cdot f(z) = \frac{Az^2 + C}{z(z^2+1)}$$

$$1 = \operatorname*{res}_{z=0} h(z) = \lim_{z \to 0} z \cdot h(z) = C$$

That means $C = 1$, and

$-A + 1 = 1 \Rightarrow \underline{A = 0}$

So we have:

$$f(z) = \frac{1}{z^2(z^2+1)}$$

We see that the idea of

$$f(z) = \frac{\text{polynomial in } z}{\text{polynomial in } z}$$

works well. The only thing is the order of those polynomials?

See the theorems of Liouville!

Next time, with an exercise like this one, I will first write that I use the theorems of Liouville!

Important is the fact that:

If f is entire (analytic on the whole $\mathbb{C}$) and $\lim_{z \to \infty} f(z) = c$

$\Rightarrow$ $f(z) = c \quad \forall z \in \mathbb{C}$

In our example:

$z^2 \cdot f(z)$ bounded for $|z| > 1$,

see (ii). $\Rightarrow$

$$\frac{p(z)}{(z^2+1)} \text{ bounded} \Rightarrow$$

$$p(z) = Az^2 + Bz + C.$$

used: gen. th. of Liouville

In a lot of exams I see this kind of questions.

Now it is the day of the run and sitting for a moment at the bench, listening to the nature.

There I asked myself what to do?
May be an integral with cos and sin functions is an idea?

But first a nice sentence, I read it in a book of my wife:

"It is not difficult to get to know something; what is difficult is knowing how to use that knowledge."

Let's do some integral with a sin-function:

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+\sin(4\varphi)}\, d\varphi$$

What becomes the question?
We are busy with complex

analysis. Let's try to rewrite the integral to complex variables.

What do we know? If $z \in B_1(0)$: the unit circle in the complex plane then we know: $Re(z) = \cos(\arg(z))$ and $Im(z) = \sin(\arg(z))$ and $z \cdot \bar{z} = |z|^2 = 1$ at the unit circle. Further we know that: $Re(z) = \frac{z + \bar{z}}{2}$, $Im(z) = \frac{z - \bar{z}}{2i}$,

but if $|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z \cdot \bar{z} = 1$ so $\bar{z} = \frac{1}{z}$

So if $|z| = 1$, $z = e^{i\varphi}$ then

$$\cos(\varphi) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin(\varphi) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

What can we use of it?

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{3 + \sin(4\varphi)} \, d\varphi$$

I see: $0 \leq \varphi \leq \frac{\pi}{2}$ then

$$0 \leq (4\varphi) \leq 2\pi$$

Maybe first new variable

$\theta = (4\varphi)$ if $0 \leq \varphi \leq \frac{\pi}{2}$

then $0 \leq \theta \leq 2\pi$

$\sin(\theta) \rightsquigarrow$ may be do something with the knowledge above about that unit circle?

$\theta = 4\varphi \Rightarrow \varphi = \left(\frac{\theta}{4}\right) \Rightarrow$

$d\varphi = \frac{1}{4} d\theta$

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+\sin(4\varphi)} d\varphi = \int_0^{2\pi} \frac{1 \cdot \frac{1}{4}}{3+\sin(\theta)} d\theta$$

Now go to the complex plane and construct a function which gives at the unit circle in the complex plane the same values as the function

$\frac{1}{4} \cdot \frac{1}{3+\sin(\theta)}$ at the interval $[0, 2\pi]$

How that function further behaves, so in the complex plane without that unit

circle doesn't interest me.
"Doesn't interest me" is not completely true, because I want to have a holomorphic function, such that I can do something with residues.

I know if $|z|=1$ then

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\theta = \arg(z),\ z = e^{i\theta} \Rightarrow$$

$$dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{i \cdot z} dz$$

$$\frac{d\theta}{(3+\sin(\theta))} = \frac{1}{iz} \cdot \frac{dz}{3 + \frac{1}{2i}\left(z - \frac{1}{z}\right)}$$

$(0 \leq \theta \leq 2\pi)$

(if $z = e^{i\theta}$!)

So;

$$I = \frac{1}{4}\int_0^{2\pi} \frac{1}{3+\sin(\theta)}\,d\theta =$$

$$\frac{1}{4}\int_{|z|=1} \frac{1}{iz}\cdot\frac{1}{3+\frac{1}{2i}\left(z-\frac{1}{z}\right)}\,dz =$$

$$= \int_{|z|=1} \frac{1}{z}\cdot\frac{1}{(z^2+i\cdot 6\cdot z-1)}\,dz$$

Keep always in thought, that we integrate about a curve in the complex plane. At that curve the values have to the same at other places "in certain sense" it doesn't interest us.

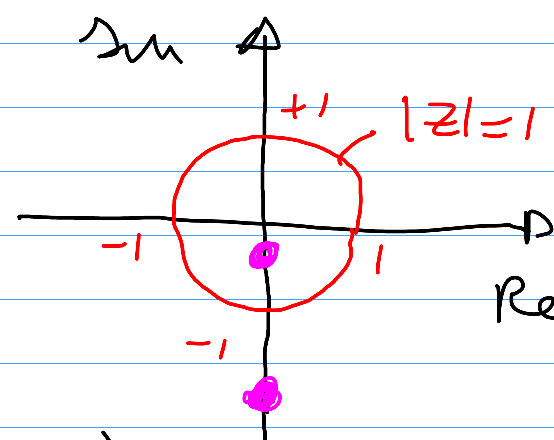
"in certain sense" $\rightsquigarrow$ we want to have a holomorphic function, don't use $\bar{z}$ for instance, not a holomorphic operation, but if $|z|=1$ then $\bar{z} = \frac{1}{z}$ and everything goes well.

Let go to the singularities and hopefully also residues?

$$z^2 + i6z - 1 = 0$$

$$(z+3i)^2 + 8 = 0$$

$$z = -3i \pm \sqrt{8}\, i$$



$2 < \sqrt{8} < 3$, $(-3-\sqrt{8}) < -1$,

$$-1 < -3 + \sqrt{8} < 0$$

- $z = (-3+\sqrt{8})i$ inside $|z|=1$

other one outside $|z|=1$.

$$I = \int_{|z|=1} \frac{1}{z} \cdot \frac{1}{(z-(-3+\sqrt{8})i)} \cdot \frac{1}{(z-(-3-\sqrt{8})i)}\, dz$$

$$= \frac{2\pi i}{2} \cdot \frac{1}{((\cancel{-3}+\sqrt{8})i - (\cancel{-3}-\sqrt{8})i)} =$$

$$= \frac{\pi i}{2 \cdot \sqrt{8}\, i} = \frac{\pi}{4\sqrt{2}}$$

Some checks:

I has to be a real value → good.

I has to be positive → good

$(|3+\sin(4\varphi)| > 0)$

$I \sim \frac{\pi}{2} \cdot \frac{1}{3} = \frac{\pi}{6}$, our result:

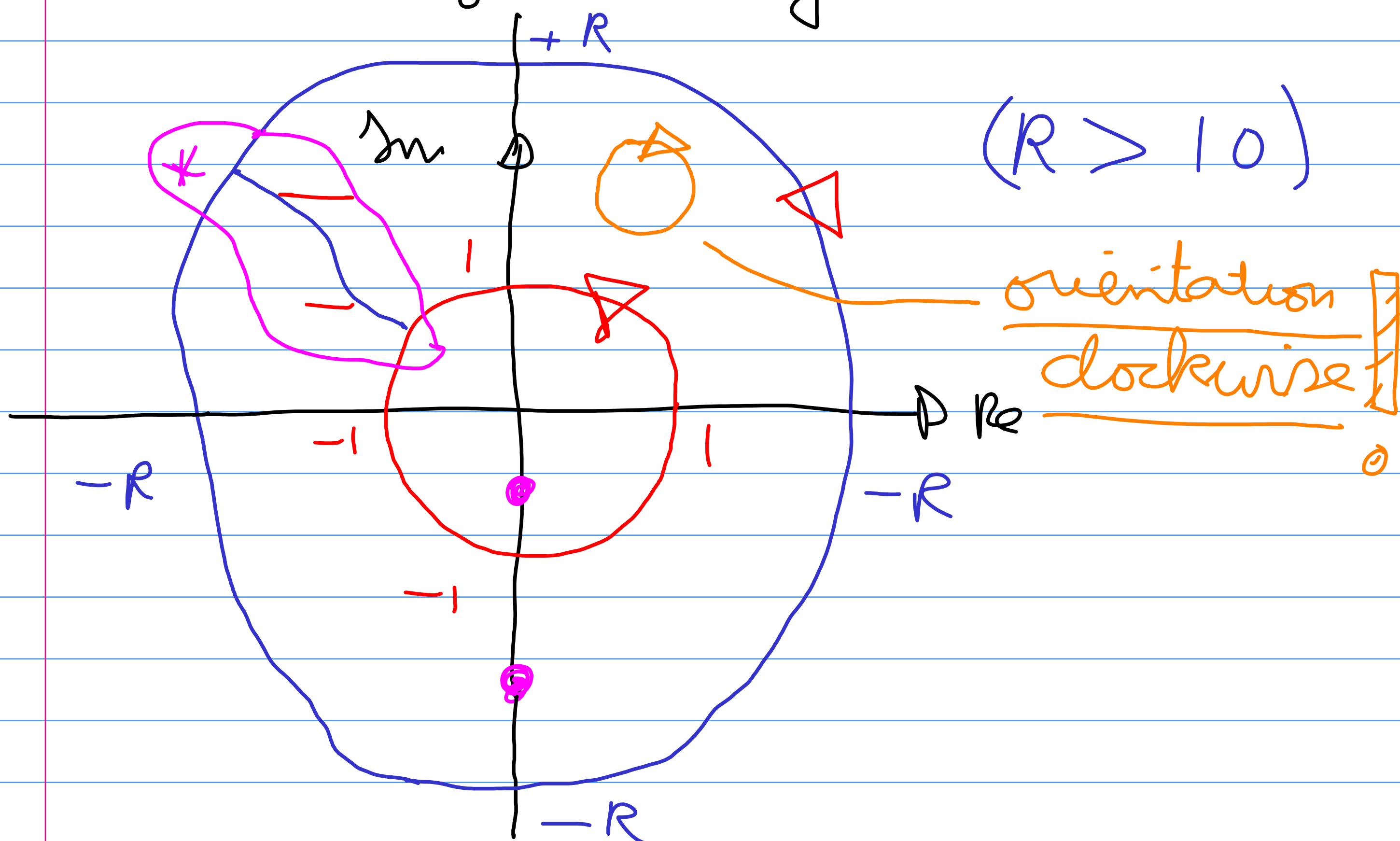
almost $\frac{\pi}{4 \cdot (1.4)} = \frac{\pi}{5.6}$ → good

We have done it well!

Be careful the unit circle with a positive orientation

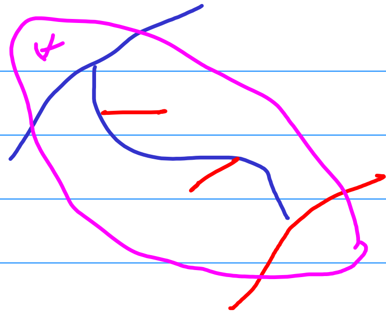
Just for fun:

We could have also used that other singularity!



$$\left| \int_{|z|=R} \frac{1}{z^2 + i6z - 1}\, dz \right| \leq$$

(i) $$\frac{2\pi R}{R^2 - 6R - 1} \to 0 \text{ if } R \to \infty$$

(ii) 

integrals fall away to each other, see orientation

So if $R \to \infty$:

$$\oint_{|z|=1} \frac{1}{z^2 + i\cdot 6\, z - 1}\, dz =$$

$$-\frac{2\pi i}{2} \cdot \frac{1}{\left((\cancel{-3} - \sqrt{8})i - (\cancel{-3} + \sqrt{8})i\right)}$$

|| see orientation, inside the closed contour, that is clockwise !! ||

$$= \frac{-\pi i}{-2\sqrt{8}\, i} = \frac{\pi}{4\cdot\sqrt{2}}$$

gives the same result as the other one, only more to do and be careful with the orientation.

Let's see, what we can discover ourselves? Such that reading the lecture notes, we can say: "That we have seen already!"

Somewhere I saw some integral, which might be interesting. Defined is

$$f(X) = \frac{\sin(X) - X\cdot\cos(X)}{X^3}$$ and asked is to calculate $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$.

The first thing I would do that is to find out what happens near $X = 0$ and for $x \to \pm\infty$?

I would look if the integral exists? (Maybe: if an odd function?)

i) $x \to \pm\infty$ then $|f(x)| \le \dfrac{1+|X|}{|X^3|} = \dfrac{1}{|x^3|} + \dfrac{1}{|x^2|}$

so $|f(x)| \le \dfrac{2}{|x^2|}$ if $|X| > R >> 1$.

So function is integrable for $X \to \pm\infty$.

ii) What happens near $x = 0$?

$$f(x) = \frac{\left(x - \frac{x^3}{3!} + O(x^5)\right) - x\left(1 - \frac{x^2}{2!} + O(x^4)\right)}{x^3}$$

$$= \frac{x^3\left(-\frac{1}{6} + \frac{1}{2}\right) + O(x^5)}{x^3} =$$

$$\frac{1}{3} + O(x^5)$$

So let's define: $f(0) = \frac{1}{3}$, then no discussions anymore about what value $f(x)$ takes in $x = 0$.

(iii) And I see that $f(x)$ can be written out in a power series with a convergence radius ∞. The function f is entire.

$$f(z) = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{(2n+1)} - z \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}}{z^3}$$

$$= \frac{1}{3} + \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(2n+1)!} - \frac{(-1)^n}{(2n)!} \right) \cdot z^{2(n-1)}$$

$$= \frac{1}{3} + \sum_{n=2}^{\infty} (-1)^n \cdot \frac{(-2n)}{(2n+1)!} \cdot z^{2(n-1)}$$

(iv) By filling in by the original f, I see $f(-x) = f(x)$, so f is even.

In the power series you see also only

even powers of z. This is also a little bit of control if the power series is correct or not.

I think we don't need the power series itself. The fact that $f(0) = 1/3$ and you don't have a singularity in $z=0$, together with the fact that $\sin(z)$ and $z \cdot \cos(z)$ are entire functions is enough to conclude that f is entire.

But now?

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx \text{, let's}$$

R be fixed and let's see what we can do with the finite integral:

$$\int_{-R}^{R} f(x)\,dx \quad (\text{and then: let } R\to\infty)$$

We can go into the complex plane, but how?

$$f(z) = \frac{\sin(z) - z\cdot\cos(z)}{z^3} =$$

$$\frac{(e^{iz} - e^{-iz})}{2i\cdot z^3} - \frac{1}{2}\frac{1}{z^2}\left(e^{iz} + e^{-iz}\right) =$$

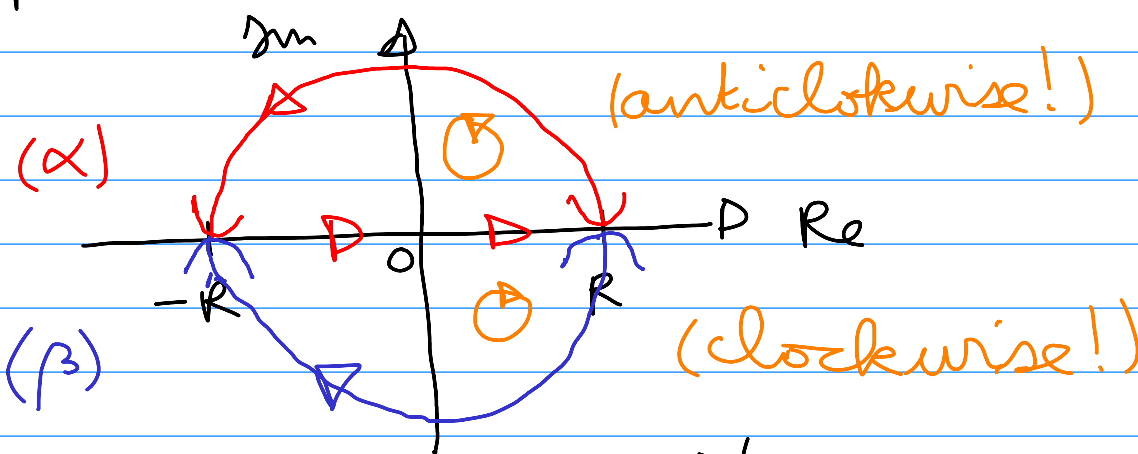
$$\left(\frac{1}{i} = \frac{-i}{-i\cdot i} = -i\right)$$

$$= \frac{e^{iz}}{z}\cdot\left(\frac{1}{iz^3} - \frac{1}{z^2}\right) + \frac{e^{-iz}}{z}\left(\frac{-1}{i\cdot z^3} - \frac{1}{z^2}\right)$$

$$= -\frac{1}{z}\cdot\left(\frac{z+i}{z^3}\right)e^{iz} + \frac{1}{z}\frac{(i-z)}{z^3}\cdot e^{-iz}$$

Most of the time you go to the upper of the lower half plane of $\mathbb{C}$.
But be careful, which one you take!

Why to be careful? Let's examine
$\exp(iz)$, we can do: $(R>0!)$



First (α): $z = R\cdot e^{it}$, $0 \le t \le \pi$

$$|\exp(iz)| = |\exp(iR\cos t - R\cdot\sin t)|$$

$$= \exp(-R\sin(t)) \to 0, \text{ if } R\to\infty.$$

$$|\exp(-iz)| = |\exp(-iR\cos(t) + R\cdot\sin(t))|$$

$$= \exp(+R\cdot\sin(t)) \to \infty \quad \text{if } R\to\infty$$

Let's now look to (β): $z = R\cdot e^{it}$, $-\pi \le t \le 0$

$|\exp(iz)| = |\exp(iR\cos(t) - R\cdot\sin(t))| =$

$\exp(-R\cdot\sin(t)) \to \infty$ if $R \to \infty$

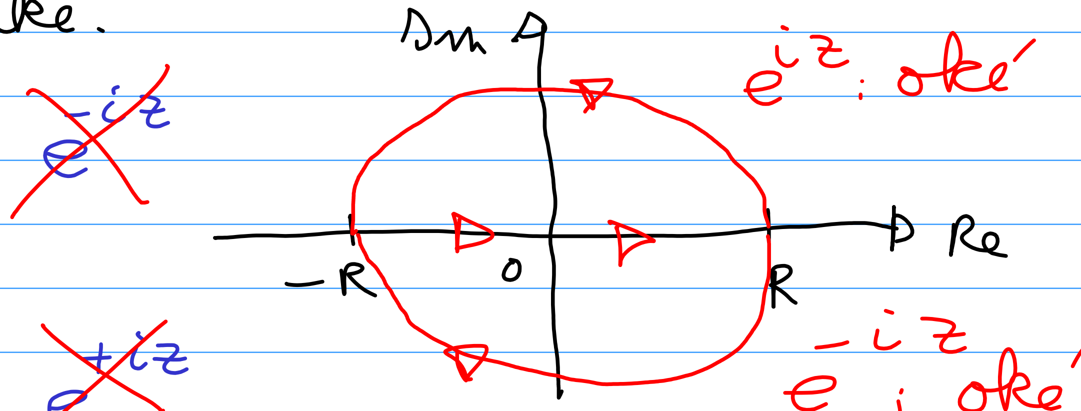
If $-\pi < t < 0$ then $\sin(t) < 0$ so

$-R\cdot\sin(t) > 0$!

$|\exp(-iz)| = |\exp(-iR\cos t + R\cdot\sin(t))| =$

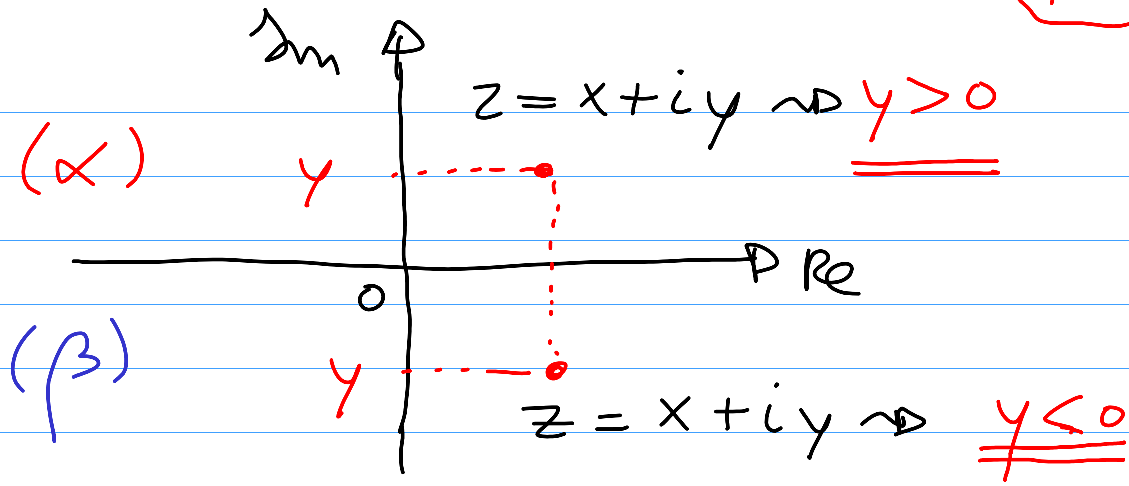
$\exp(R\sin(t)) \to 0$ if $R \to \infty$

So be careful with exp(iz) and exp(-iz), which half plane of $\mathbb{C}$ you take.



This if you take: $z = R\cdot e^{it}$.

Here beneath may be an easier way to look to it:



In (α):

$$|\exp(iz)| = |\exp(-y)| \to 0 \text{ if } y \to \infty.$$

$$|\exp(-iz)| = |\exp(y)| \to \infty \text{ if } y \to \infty.$$

In (β):

$$|\exp(iz)| = |\exp(-y)| \to \infty \text{ if } y \to -\infty$$

$$|\exp(-iz)| = |\exp(y)| \to 0 \text{ if } y \to -\infty$$

See the different behaviours in the upper and lower plane of $\mathbb{C}$.

Back to our problem, that integral.

$$f(z) = f_1(z) + f_2(z), \text{ with}$$

$$f_1(z) = -\frac{1}{2}\cdot\left(\frac{z+i}{z^3}\right)e^{iz} \text{ and}$$

$$f_2(z) = \frac{1}{2}\left(\frac{i-z}{z^3}\right)e^{-iz}.$$

I don't like it! In f_1 and f_2 we have poles of order 3 in $z=0$.

Small circles around $z=0$?

Don't do that, problems with $\frac{1}{z^3}$ and may be $\frac{1}{z^2}$ terms.

Can we do something else?

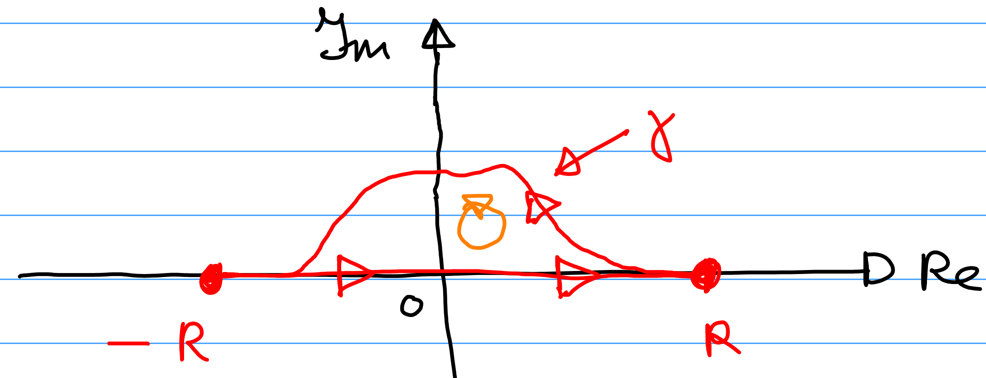
We know that f is entire on $\mathbb{C}$, no singularities or other strange points. Using Cauchy gives us that:

$$\oint_{C_1} f(z)\,dz = 0$$

(Jordan curve)

if C_1 is some nice closed contour.

Let's construct such a C_1, see sketch:



So: $C_1 = [-R, R] \cup \gamma$, then

$$0 = \int_{C_1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\gamma} f(z)\,dz$$

($(-\gamma)$ orientation opposite as given in the sketch)

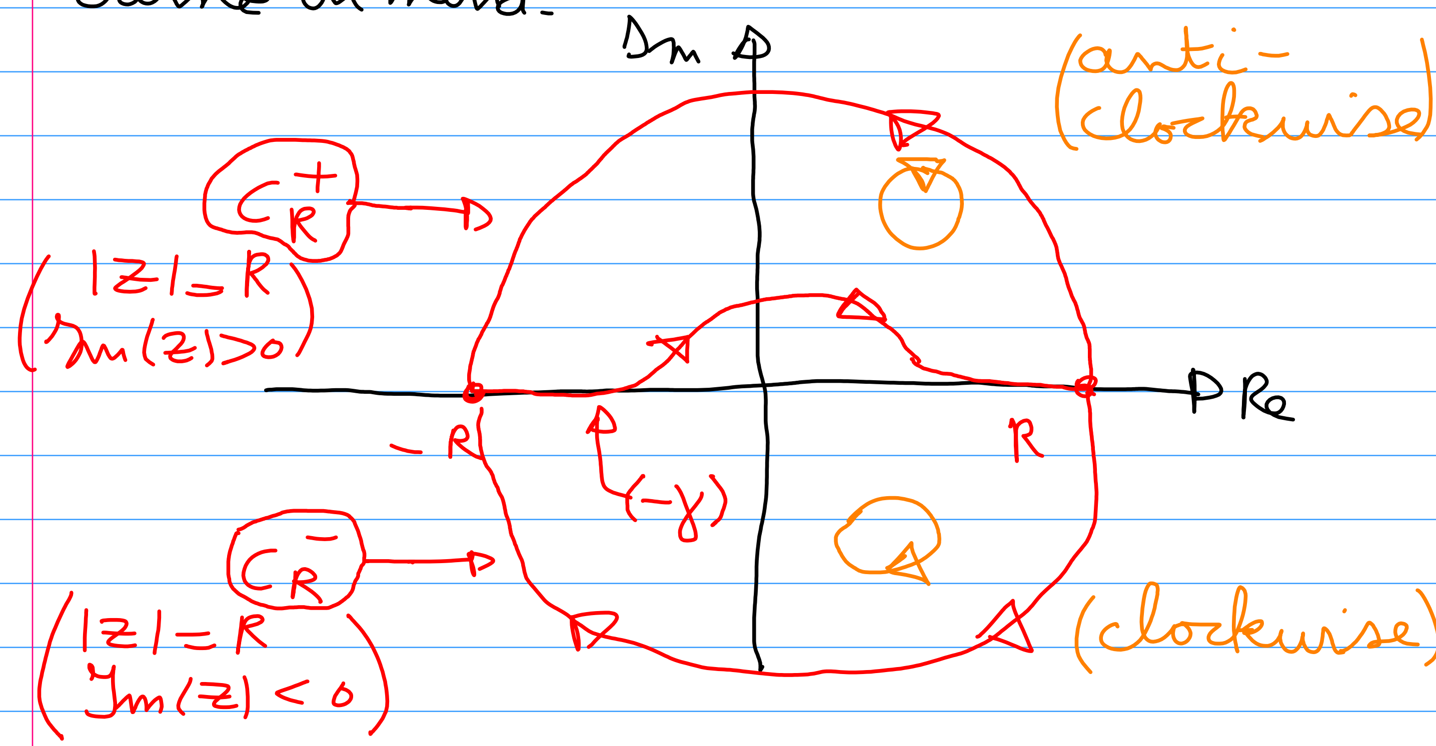
So we have:

$$\int_{-R}^{R} f(x)\,dx = -\int_{\gamma} f(z)\,dz = \int_{(-\gamma)} f(z)\,dz =$$

$$\int_{(-\gamma)} f_1(z)\,dz + \int_{-\gamma} f_2(z)\,dz$$

[f_1 has to do with $\exp(i \cdot z)$,
f_2 has to do with $\exp(-i \cdot z)$

Let's do the following, this just came in mind:



For f_1 and f_2, we are "free" what to do!

Let's take for f_1 the closed contour $C_1' = (-\gamma) \cup C_R^+$, because of the $\exp(+iz)$. Let's take for f_2 the closed contour $C_2 = (-\gamma) \cup C_R^{-1}$, that because of the $\exp(-iz)$.

Be careful with the orientations!

On C_R^+ we have $\operatorname{Im}(z) > 0$, so $|\exp(iz)| = \exp(-\operatorname{Im}(z)) < 1$,

Length of $C_R^+ = \pi \cdot R$ and

$$\left| -\frac{1}{2} \frac{(z+i)}{z^3} \right| \leq \frac{1}{R^2},$$

with the ML-lemma we get:

$$\left| \int_{C_R^+} f_1(z)\,dz \right| \leq \frac{\pi R}{R^2} \to 0, \text{ if } R \to \infty.$$

and that means that $\lim\limits_{R \to \infty} \int_{(-\gamma)} f_1(z)\,dz = 0$

No sing inside C_1.

On C_R^{-} we have $\mathrm{Im}(z) < 0$,

so $|\exp(-iz)| = \exp(\mathrm{Im}(z)) < 1$

Length C_R^{-1}: $\pi \cdot R$ and

$$\left| \frac{1}{2} \frac{(i - z)}{z^3} \right| \leq \frac{1}{R^2},$$

ML-lemma gives us that

$$\left| \int_{C_R^{-1}} f_2(z)\, dz \right| \leq \frac{\pi R}{R^2} \to 0, \text{ if } R \to \infty.$$

<u>But now!</u> Inspect f_2 inside the closed contour C_2 and see that it has a singularity in $z = 0$, so we have to calculate the residue of f_2 in $z = 0$

$$f_2(z) = \frac{1}{2} \frac{(i - z)}{z^3} \cdot \exp(-iz) =$$

$$\frac{1}{2} \cdot \frac{(i - z)}{z^3} \cdot \left(1 - iz - \frac{z^2}{2} + \frac{iz^3}{3!} + O(z^4) \right)$$

$$= \frac{1}{2z^3} \cdot \left(i \cancel{+z} - \frac{iz^2}{2} + O(z^3) \cancel{-z} \right.$$

$$\left. + iz^2 + O(z^3) \right) =$$

$$\frac{i}{2z^3} + \frac{1}{2z^3} \cdot \frac{i}{2} z^2 + O(1) =$$

$$\frac{i}{2} \cdot \frac{1}{z^3} + \frac{i}{4} \cdot \frac{1}{z} + O(1).$$

clockwise (orientation of C_2!)

And so we get:

$$\lim_{R \to \infty} \int_{C_2} f_2(z)\,dz = (-2\pi i)\left(\frac{i}{4}\right) = \frac{\pi}{2}$$

Result:

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx = \lim_{R \to \infty} \int_{(-\gamma)} f_2(z)\,dz = \frac{\pi}{2}$$

Check(s)? The only thing, I'm happy, that is that the result gives me a real number. But further? No ideas.

What next to do? It is a Wednesday, some extra rules about the Corona have to be fulfilled. Corona will keep us busy for a long period of time.

What I'm thinking? Now, I'm thinking to Taylor series. See next page and I will tell why.

If a function f is nice, so enough differentiable and so on then we know that

$$f(x) = f(a) + f'(a).(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + O((x-a)^{(n+1)}) \quad (x \approx a)$$

Of such a function f, we make a polynomial:

$$P_n(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

We transform f to the polynomials.

Keep in mind, to calculate $\sin(\frac{1}{10})$, we use some polynomial around $x=0$ of $f(x) = \sin(x)$. I mean to calculate in floating numbers. The expression $\sin(\frac{1}{10})$, you know it is some number but further? The same with al kind of other functions, like cos, ln, exp, ...
But by transfoming them to the polynomials and we can do something with it.
If such a function f has some periodic behaviour, you can ask

yourself if this f can be expressed by cos and sin function.
If you are busy with waves, it looks me a serious question. Maybe you can see, which wave has the greatest contribution. Maybe you can calculate the energy of such a wave a lot easier, instead of to be busy with polynomials.
Maybe those used sin and cos functions have a nicer behaviour then those polynomials.

Be aware of the fact that if somebody tells to you that he or she uses polynomials, the following set
$\{1, x, x^2, \cdots, x^n, \ldots\}$
that he or she can give you some sequence of numbers, for instance:

$$\left\{ f(0), f'(0), \cdots, \frac{f^{(n)}(0)}{n!}, \cdots \right\}$$

that he or she has given some function to you. Multiply the good powers of $x^{(i)}$ to the corresponding coëfficients and add all things together.

$$f(0).1 + f'(0).x + \frac{f''(0)}{2}.x^2 \cdots\cdots = f(x).$$

So in a computer a lot of these sequences are safed!
Do you ask the computer to differentiate that function, then he will give the sequence

$$\{f'(0), 2\cdot f''(0), \ldots, n\cdot f^{(n)}(0), \ldots\}.$$

Busy with periodic functions, you can try to write such a function f in the following form:

$$f(x) = c_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\cdot\sin(n\cdot x))$$

In the same way as above, if somebody tells you to be busy with the following sequence of functions:

$$\{1, \cos(x), \sin(x), \ldots, \cos(n\cdot x), \sin(n\cdot x), \ldots\}$$

and he gives you the sequence of numbers

$$\{c_0, a_1, b_1, \ldots, a_n, b_n, \ldots\}$$

you know the function. That sequence of numbers is enough. But always know, where to use that sequence of numbers. Otherwise you don't have an idea about what to do with that sequence!

With those polynomials, we know how to calculate those sequences of numbers to express a function f.

How to do that with those summation of sin and cos functions?

That question, we just asked ourselves, let's try to find some solution for it.

Those coëfficients c_0, a_n, b_n are called Fourier coëfficients.

For simplicity we assume that our f has a period 2π, so

$$f(x+2\pi) = f(x) \; , \; \forall x \in \mathbb{R}.$$

Note that $g(x) = f\left(\frac{2\pi x}{L}\right)$ has period L.

We are busy with complex numbers, so assume $f : \mathbb{R} \to \mathbb{C}$.

Maybe we can do something with the functions:

$$\left\{ \cdots e^{-inx}, \cdots\cdots, e^{-i2x}, e^{-ix}, 1, e^{ix}, e^{2ix}, \cdots, e^{inx}, \cdots \right\}$$

and the coëfficients:

$$\{\cdots, c_{(-n)}, \cdots, c_{(-2)}, c_{(-1)}, c_0, c_1, c_2, \cdots, c_n, \cdots\}$$

Let's first look to the reason why is chosen for $\exp(inx)$ $(n \in \mathbb{Z})$?

First: $(n-m \neq 0$, so $n \neq m)$

$$\int_0^{2\pi} e^{inx} \cdot e^{-imx} dx = \int_0^{2\pi} e^{i(n-m)x} dx =$$

$$\left[\frac{1}{(n-m)} \cdot e^{i(n-m)x}\right]_0^{2\pi} =$$

$$\left(e^{i(n-m)\cdot 2\pi} - 1\right)/(n-m) = 0.$$

Secondly: $(n-m=0$, so $n=m)$

$$\int_0^{2\pi} e^{inx} \cdot e^{-inx} dx = \int_0^{2\pi} 1 \, dx = 2\pi$$

If we have

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

how to calculate c_n?

Let's try, $m \in \mathbb{Z}$ is given,

$$\int_0^{2\pi} f(x)\cdot e^{-imx}\,dx = \int_0^{2\pi} \sum_{n=-\infty}^{+\infty} c_n e^{i(n-m)x}\,dx$$

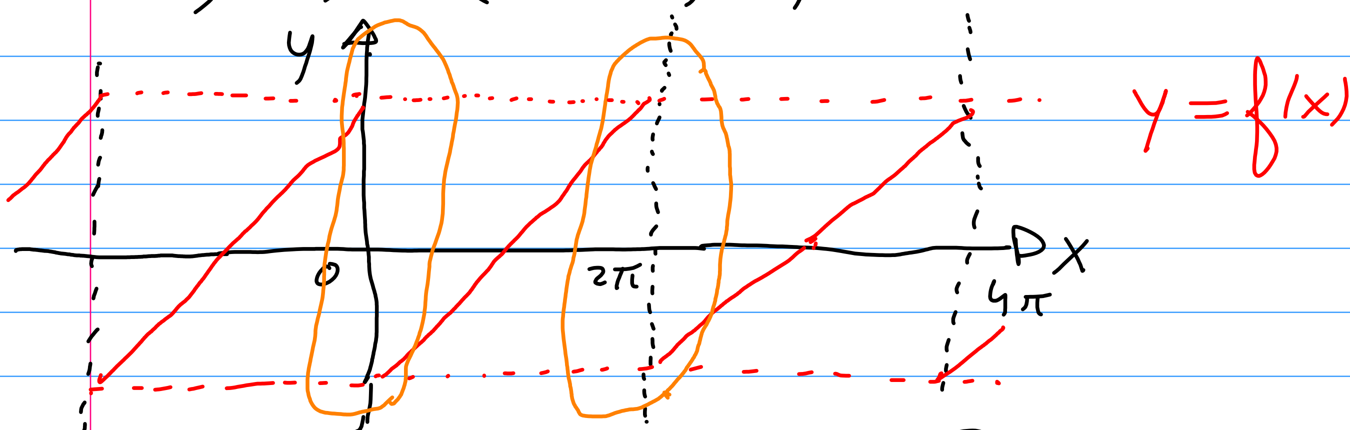
$$= c_m \cdot 2\pi \quad \begin{cases} n \neq m \text{ then } \int_0^{2\pi} e^{i(n-m)x}\,dx = 0 \\ n = m \text{ then } \int_0^{2\pi} 1\,dx = 2\pi \end{cases}$$

We see that:

$$c_m = \frac{1}{2\pi} \cdot \int_0^{2\pi} f(x)\cdot e^{-imx}\,dx$$

Remarks:

- We have taken the interval $[0, 2\pi]$. In other books they often take $[-\pi, \pi]$
- It is maybe better to take open intervals $(0, 2\pi)$ or $(-\pi, \pi)$



$\lim_{x \to 0} f(x)$? or $\lim_{x \to 2\pi} f(x)$?

- Look at pg. 163 and see that

$a_n = (c_n + c_{-n})$ and $b_n = i(c_n - c_{(-n)})$

- Further if $f: \mathbb{R} \to \mathbb{R}$ then $c_n = \overline{c_{(-n)}}$
- With Criterium of Weierstrass have that:

if $\sum_{n=-\infty}^{+\infty} |c_n| < \infty$ then the series at pg. 165 converges uniformly.

Shall I tell the following or not? That is an question I often ask myself.

Oké, let's do. If you have functions, you can define inner products at certain spaces. Which spaces is another problem, just I let you see what you can do with it.

If you have functions f, g at $[0, 2\pi]$ an inner product can be defined by

$$\langle f, g \rangle = \int_0^{2\pi} f(x) \cdot \overline{g(x)}\, dx,$$

f and g can give complex values, therefore that complex conjugate of g. What did we do on page 165?

We just calculated the inner product between e^{inx} and e^{imx}, so:

$$< e^{inx}, e^{imx} > = \int_0^{2\pi} e^{inx} \cdot e^{-imx} dx$$

therefore the minus sign of e^{-imx}.
So, if $n \neq m$ the functions are perpendicular to each other.

If we look to:

$$< \dots, \frac{e^{-i2x}}{\sqrt{2\pi}}, \frac{e^{-ix}}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{e^{ix}}{\sqrt{2\pi}} \dots >$$

we see that we have an orthonormal basis.
Now, we can project some function f at that orthonormal basis, and a coëfficient in the direction of

$\frac{e^{inx}}{\sqrt{2\pi}}$ is given by $< f, \frac{e^{inx}}{\sqrt{2}} >$.

This gives the coëfficient

$$\int_0^{2\pi} f(x)\cdot \frac{e^{-inx}}{\sqrt{2\pi}}\,dx$$ in the direction

$$\frac{e^{inx}}{\sqrt{2\pi}}$$ and the result becomes

$$\left(\frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot e^{-inx}\,dx\right)\cdot e^{inx} =$$

$$c_m \cdot e^{inx}$$ (c_m: see page 166).

If a basis is orthonormal, the length of an element is easily calculated by adding the squares of the coëfficients and taking the square root, just use Pythagoras.

So the "length" of f squared becomes:

$$\int_0^{2\pi} f(x)\cdot \overline{f(x)}\,dx = \int_0^{2\pi} |f(x)|^2\,dx = 2\pi \cdot \sum_{m=-\infty}^{+\infty} |c_m|^2.$$

Be careful with the (2π), at page 166, the basis was only orthogonal, and you have to summate the squares of the absolute value of c_m!
You have to take

$$2\pi \cdot \sum_{m=-\infty}^{+\infty} c_m \cdot \overline{c_m} = 2\pi \cdot \sum_{m=-\infty}^{+\infty} |c_m|^2,$$

because $c_m \in \mathbb{C}$.

You see, those Fourier coefficient have some advantage with respect to the polynomials.

No idea how to calculate

$$\int_0^{2\pi} |f(x)|^2 dx,$$ if f should be given as a Taylor series.

Think to all those cross products!

Here beneath took some time. Calculating those series mean that you have a discrete process and now they want to make it continuous. From a summation to some integral.

Let take some T-periodic function $f: \mathbb{R} \to \mathbb{C}$ on the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$; and then

$$f(x) = \lim_{N \to \infty} \sum_{n=-N}^{N} c_n \cdot e^{\left(i x \cdot n \cdot \left(\frac{2\pi}{T}\right)\right)}$$

and $$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} f(t) \cdot e^{\left(-i \cdot t \cdot n \left(\frac{2\pi}{T}\right)\right)} dt$$

It should be nice to let $T \to \infty$, but keep in mind, the index n becomes also very big. What happens with the $\left(\frac{n}{T}\right)$?

Maybe the best way by writing something down and don't make it difficult.

as idea we can do;

$\Delta w = \frac{2\pi}{T}$, and $w_n = n \cdot \Delta w$, so if $T \to \infty$ then $\Delta\omega \to d\omega$ and $\omega_n \to \omega$. Here the discrete points become a continuum.

Let's try to write $f(x)$ in $\Delta\omega$ and ω_n.

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-it\omega_n}\, dt \cdot e^{ix\omega_n} =$$

$$\sum_{n=-\infty}^{+\infty} \left(\frac{\Delta\omega}{2\pi}\right) \cdot e^{ix\omega_n} \cdot \frac{2\pi}{\Delta\omega} \cdot \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-it\omega_n}\, dt \quad *$$

$$* = \frac{2\pi}{(2\pi/T)} \cdot \frac{1}{T} \cdot \int_{-T/2}^{T/2} f(t)\, e^{-it\omega_n}\, dt =$$

$$\int_{-T/2}^{T/2} f(t) \cdot e^{-it\omega_n}\, dt = F(\omega_n) \Rightarrow$$

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} \cdot e^{ix\omega_n} \cdot F(\omega_n)$$

$$f(x) = \lim_{T \to \infty} \left(\sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} \cdot e^{ix\omega_n} \cdot F(\omega_n) \right)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{ix\omega} \cdot F(\omega)\, d\omega$$ with

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-it\omega}\, dt$$

$$\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-it\omega} \, dt$$

is called the Fourier transform of f.

$$f(x) = \int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot \mathcal{F}(\omega) \cdot e^{+i\omega x} \, d\omega$$

is called the Inverse Fourier transform.

It is not easy to prove! Keep in mind, that somewhere limits have to be changed and that is tricky.

The Fourier transform is one of a lot of possible transformations.
Search on internet:
integral transformations
at Wikipedia.