Lecture Notes about Complex analysis

The nicest thing would be if this could be read like a novel.

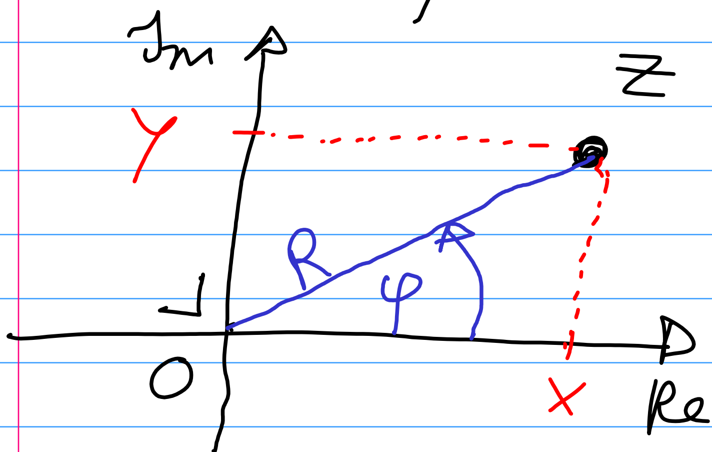
Every time I say to people "hi, hi", I hear "-1" in mind, so "$i * i = -1$".
The same with "hoj", "hoj" then "$j * j = -1$".

Internet and search:
www.win.tue.nl/~sjoerdr/

(I) Complex numbers:

1) Start with: $i^2 = -1$ (or $j^2 = -1$)

$z = x + iy = R \cdot \exp(i\varphi)$



$\begin{cases} R\text{: radius} \\ \varphi\text{: argument} \end{cases}$ $(R, \varphi \in \mathbb{R})$

$(z \in \mathbb{C})$

$x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$ both $\in \mathbb{R}$

real part, imaginary part

Adding and multiplying almost the same as in $\mathbb{R}$, only be careful with the $i^2 = -1$.

2) (R, φ) polar coordinates

$x = R \cdot \cos(\varphi)$, $y = R \cdot \sin(\varphi)$

- $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$

$|\exp(i\varphi)| = 1$ (unit circle of $\mathbb{C}$)

(can be read as a kind of definition)

3) Complex conjugate of z, written by $\bar{z}$

$$\bar{z} = \overline{(x + iy)} = x - iy = R \cdot \exp(-i\varphi)$$

$$z \cdot \bar{z} = |z|^2 = R^2$$

$$Re(z) = \frac{z + \bar{z}}{2}, \quad Im(z) = \frac{z - \bar{z}}{2 \cdot i}$$

$$z = \bar{z} \Rightarrow Im(z) = 0 \qquad Re\text{-axis}$$

$$z = -\bar{z} \Rightarrow Re(z) = 0 \qquad Im\text{-axis}$$

4) $Arg(z)$ versus $arg(z)$

There is made an appointment about the values of $Arg(z)$

$0 \leq Arg(z) < 2\pi$ or

$-\pi < Arg(z) \leq \pi$ or

something else, important is the length of interval of $Arg(z)$ is: 2π.

So $arg(z) = Arg(z) + k \cdot 2\pi, \; k \in \mathbb{Z}$.

5) Searching of zeros of polynomial
$p(z)$

order of $p \Rightarrow$ the maximum
number of possible
different solutions.

coëfficients of p are real then
if $p(z_0) = 0$ then also $p(\bar{z}_0) = 0$

With the zeros, you can
factorise p. So searching
zeros is also a method to find
factors of p.

If $p(z_0) = 0 \Rightarrow$

$$p(z) = (z - z_0) \cdot q(z)$$

order of q, one lower than of p.

6) $(w)^n = (a + ib) \quad a, b \in \mathbb{R}$

$w \in \mathbb{C}$

Use of polar coordinates useful?

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α) $|(w)^n| = |a+ib| \Rightarrow |w|$

β) $\arg(w^n) = \arg(a+ib) =$
$\text{Arg}(a+ib) + k\cdot 2\pi$ with
$k \in \mathbb{Z}$: $0 \leq k \leq (n-1)$, or
$1 \leq k \leq n$ or ...

important: n successive k's.
(That gives the n solutions.)

$$\begin{cases} w_k = |w| \cdot \exp(i\varphi_k) \\ \varphi_k = \dfrac{\text{Arg}(a+ib)}{n} + \left(\dfrac{k}{n}\right)\cdot 2\pi. \\ \quad 0 \leq k \leq (n-1) \end{cases}$$

If $w = (z - z_0) \Rightarrow$

$z_k = z_0 + w_k,\ k \in \mathbb{Z},\ 0 \leq k \leq (n-1)$

7) To prove something?

α) Busy and you get some stupid equations, maybe useful to do something else.

β) Try to do something with the given information.

Play with it.

γ) Maybe $z = x + iy$ or

$z = R \cdot \exp(i \cdot \varphi)$ useful to use?

δ) Real and Imaginary part are often important of certain expressions?

ε) What part of the complex plane is used?

But once and for all:

Try to do something.

Too long busy, try another escercise, and then before the traffic light of the Bijenkorf you think: "Oh yes, I see ....!"

(s.n.p.)

8) Triangle inequalities maybe also good to mention:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$$

9) In $\mathbb{C}$ you can not compare numbers with each other so there is no $>$ or $<$!

$\boxtimes$

(II) Series

Sunday morning I always run and always cross the "Snelle Loop", some little river between Aarle-Rixtel and Gemert. There is a bench. During the Corona period, you heard only birds and the leaves of the trees, no noice of traffic and the sky was clear of white lines.

So sitting at that bench, I asked myself: "What do I do with series and why?". So somebody comes with $\sum_{n=1}^{\infty} c_n$, "what is asked?", "what we have to do?".

First of all: $\sum_{n=1}^{\infty} c_n = \lim_{N\to\infty} \sum_{n=1}^{N} c_n$,

so series: a limit of finite sums.
Maybe we can first look if the series converges absolutely? Take series of $|c_n|$, $\left|\sum_{n=1}^{\infty} c_n\right| \leq \sum_{n=1}^{\infty} |c_n|$

Let $S_N = \sum_{n=1}^{N} |c_n|$ then $S_{N+1} \geq S_N$ so the sequence $(S_N)_{N\in\mathbb{N}}$ increases.

If we can prove that the sequence $(S_N)_{N\in\mathbb{N}}$ is bounded, we have a bounded increasing sequence, so there excists a limit. We can tell to the people that the series: $\sum_{n=1}^{\infty} c_n$ escists. What comes out of it, that is a completely other question. Most of the time: prove first the escistence and then calculte its value.

$\sum_{n=1}^{\infty} |c_n|$: Criteria to prove excistence:

$*_1$ $\lim_{n\to\infty} |c_n| = 0$, otherwise you get problems.

Ex: $\underbrace{1-1}_{0} + \underbrace{1-1}_{0} + \underbrace{1-1}_{0} \cdots\cdots = ?$ $= 0 ?$

or $1 + (-1+1) + (-1+1) \cdots = 1 ?$

or $1 - 1 + 1 - 1 + 1 - 1 + 1 = 2 ?$

So we assume that $*_1$ holds.

After some time you get experience by seeing some series of what criterium can be used, or tried.

But in my mind I have a certain order of using those criteria.

See also at internet:

wikipedia convergence tests

Sitting at that bench near the little

river and during the run, I asked myself, shall I let the people see, why those criteria work? Not nice proofs, but just simple examples, that the people get some feeling why? I can not resist to do it. But then there is some series, which you will see very often: $\sum\limits_{n=0}^{\infty} \alpha^n$

$$(1-\alpha)\left(\sum_{n=0}^{N} \alpha^n\right) = 1 + \cancel{\alpha} + \cancel{\alpha^2} + \cdots + \cancel{\alpha^N} - \cancel{\alpha} - \cancel{\alpha^2} - \cancel{\alpha^N} - \alpha^{N+1}$$

$$\sum_{n=0}^{N} \alpha^n = \frac{(1-\alpha^{N+1})}{1-\alpha}$$

Look if $|\alpha| < 1$ then series converges

$$\sum_{n=0}^{\infty} \alpha^n = \left(\frac{1}{1-\alpha}\right)$$ geometric series

$|\alpha| > 1$ then the series diverges (not convergent)

$$|\alpha| = 1 \rightarrow \begin{cases} \alpha = 1, & \lim_{n\to\infty} \alpha^n = 1 \neq 0 \\ \alpha = -1, & \text{inconclusive} \end{cases}$$

$$1 - 1 + 1 - 1 + 1 \; \cdots = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdots$$
$$= \frac{1}{2}$$

Read and see what agreement is made about $\alpha = -1$. I should say not convergent, because you can get out of it, what you want.

With $\alpha = \frac{1}{3}$, you get the Corona: $\frac{3}{2} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{1 - \left(\frac{1}{3}\right)}\right)$

the "Grabbenhaus constant" α.

Now those criteria, in the order as I should use them:

$*_2$ Ratio test: if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| < 1$

then series converges.

Why? For great values of n then

let $n \geq N$

N fixed

$$\frac{|c_{n+1}|}{|c_n|} \approx \alpha\ (<1) \Rightarrow$$

$$|c_{n+1}| \approx \alpha |c_n|$$

$$\sum_{n=0}^{\infty} |c_n| = \sum_{n=0}^{(N-1)} |c_n| + \sum_{n=N}^{\infty} |c_n| =$$

finite sum so exists

?

$$\sum_{n=N}^{\infty} |c_n| \leq |c_N| + \alpha |c_N| + \alpha^2 |c_N| + \cdots$$

$$= |c_N| (1 + \alpha + \alpha^2 + \cdots) = |c_N| \left(\frac{1}{1-\alpha}\right)$$

as small as you want because $\lim_{n \to \infty} |c_n| = 0$, see $*_1$

finite

So if N great enough, the second part can be get as small as you want, so series exists.

When the Ratio Test doesn't work nice I often try to use the

$*_3$ Integral test:

So $|c_n| = f(n)$ $(f : \mathbb{R} \to \mathbb{R})$

If $\int_1^\infty f(x)\,dx < \infty$

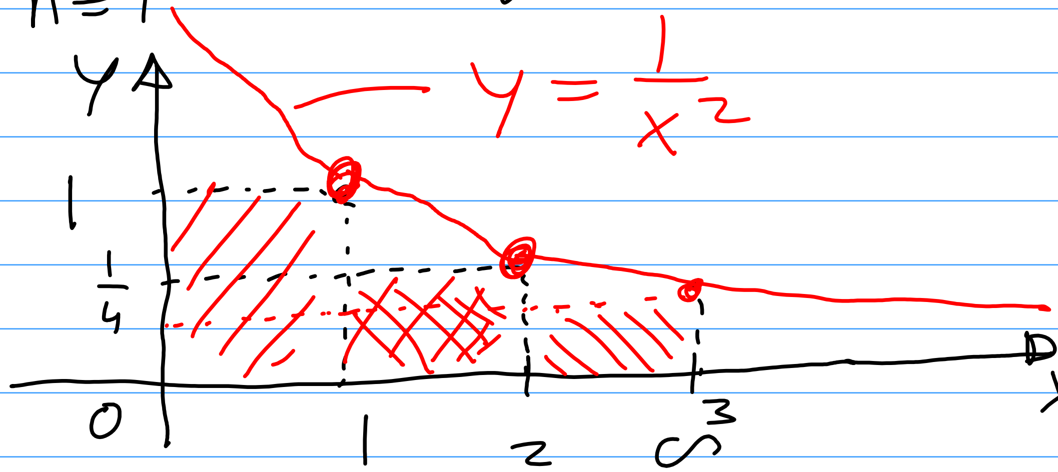
the series converges

(and in certain sense if $\int^\infty f(x)\,dx$ not bounded then the series diverges)

Be careful with the lower boundary of the integral, of importance is most of the time the last part of the series.

To keep in mind:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sim f: x \to \frac{1}{x^2}, \quad \frac{1}{n^2} = f(n)$$



So don't take $\int f(x)\,dx$, then it $\text{↯} \to 0$ goes wrong, but

$$1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

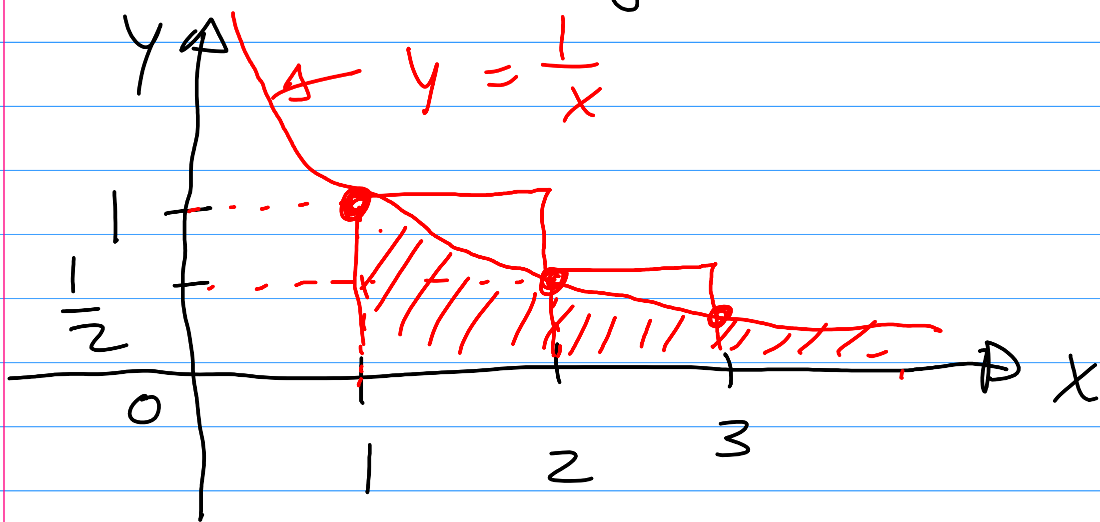
$$= 1 + \left[-\frac{1}{x}\right]_1^{\infty} = 2 < \infty$$

So the series converges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \sim f: x \to \frac{1}{x} \sim \frac{1}{n} = f(n)$$

$\frac{1}{x}$ something to do with $\ln|x|$

that function is not bounded, series diverges?



So: $\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\,dx = \left[\ln|x|\right]_1^{\infty} \to \infty$

integral is not bounded, so the series diverges!

When these two criteria doesn't work, I try the root test, but it is not my favourite.

This because you have sometimes terrible limits,

$$\lim_{n\to\infty}\left(n^{\frac{1}{n}}\right) = \lim_{n\to\infty} \exp^{\left(\frac{1}{n}\ln n\right)} = e^{0} = 1,$$

so $\exp(\ln(x)) = x$ is often convenient to use.

*4 Root test:

If $\lim_{n\to\infty}\left(|c_n|\right)^{\frac{1}{n}} < 1$

then series converges.

Why? If for great values of n:

$$|c_n|^{1/n} \sim \alpha \ (<1) \text{ then}$$

$|c_n| \simeq \alpha^n$, and we know that

$$\sum_{n=1}^{\infty} \alpha^n \text{ converges if } |\alpha| < 1.$$

The reason is quite simple, but calculating that limit not always.

*5 Look at wikipedia, comparison test also nice in use.
But now, I'm not running anymore, we have to be aware that we spoke about absolute convergence the whole time. What to do with alternating series, that are series you can write as

$$\sum_{n=1}^{\infty} (-1)^n \cdot c_n, \text{ with } c_n > 0 \quad \forall n \in \mathbb{N}.$$

*6 Alternating series test

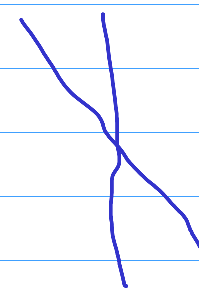
- 1 $\lim_{n\to\infty} c_n = 0$
- 2 $c_{n+1} < c_n \quad \forall n \in \mathbb{N}.$

then the series $\sum_{n=1}^{\infty} (-1)^n c_n$

converges.

(2 simple conditions $\bullet_1$ and $\bullet_2$)

Why? I look to the sky and
I see just two white lines,
which cross each other
and ask myself, shall
I write down the idea, I had
running near the nunnery?



Let's do, but keep in mind it
is not a proof of this test *6

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\cdots$$

$$\frac{1}{n} - \left(\frac{1}{n+1}\right) + \cdots\cdots$$

You know this series is not
absolute convergent

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges!}$$

Look: $c_n = \frac{1}{n}$

$\lim_{n\to\infty} c_n = 0$, $c_{n+1} = \frac{1}{n+1} < \frac{1}{n} = c_n$

Look to: $\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{(n+1)-n}{n(n+1)}$

$= \frac{1}{n(n+1)} \leq \frac{1}{n^2}$ and the

series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, so

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

and $(1-\frac{1}{2})>0$, $(\frac{1}{3}-\frac{1}{4})>0$ ....

So $\sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ is increasing. (increase and bounded)

Let well, in mathematical sense, is not completely well, but I hope, you see that to this series the test works.

Conclusion:

These tests I have in mind and they are also useful working with complex numbers.

But see (I) g)!

If $c_n \in \mathbb{C}$, you have to work with $|c_n|$, use the absolute value.

Later on those tests will be applied to "power series"

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{then take}$$

$$c_n = |a_n (z - z_0)^n|$$

$\boxtimes$

(III) Example of those "power series"

$$\frac{1}{2-z} \overset{*}{=} \frac{1}{2} \cdot \frac{1}{\left(1-\left(\frac{z}{2}\right)\right)} =$$

$\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$, when do you have convergence?

"geometrical series" so:

$$\left|\frac{z}{2}\right| < 1 \quad \to \quad \underline{\underline{|z| < 2}}$$

But what to do, if $z = 4$?

$\frac{1}{2-4} = -\frac{1}{2}$ exists, but

$|4| = 4 > 2$, so given series diverges!! See $*$, why divided by: 2? We can also divide by z, if $z \neq 0$. Let's do:

$$\frac{1}{2-z} = \frac{1}{\left(\frac{2}{z}-1\right)z} =$$

$$-\frac{1}{z}\cdot\left(\frac{1}{1-\left(\frac{2}{z}\right)}\right) = -\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

we have convergence if

$$\left|\frac{2}{z}\right| < 1 \rightarrow |z| > 2$$

$$\frac{1}{2-z} = \begin{cases} \frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n & \text{if } |z|<2 \quad (i) \\ -\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n & \text{if } |z|>2 \quad (ii) \end{cases}$$

So, if $z=4 \rightarrow |z| = 4 > 2$

$$-\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{4}\right)^n = -\frac{1}{4}\cdot\frac{1}{\left(1-\frac{1}{2}\right)} =$$

$$-\frac{1}{2} \quad \left(= \frac{1}{2-4}\ \text{!}\right)$$

But see the differences, (i) has positive powers of z and

(ii) has negative powers of z

So you see that you can play with those series.

Let's limit to positive powers of z.

What can we do more? Somebody wants to have powers of $(z+1)$?

Let's do the following,

let's $w = z+1 \Rightarrow z = (w-1)$

We know: $\frac{1}{2-z} = \frac{1}{2-(w-1)} =$

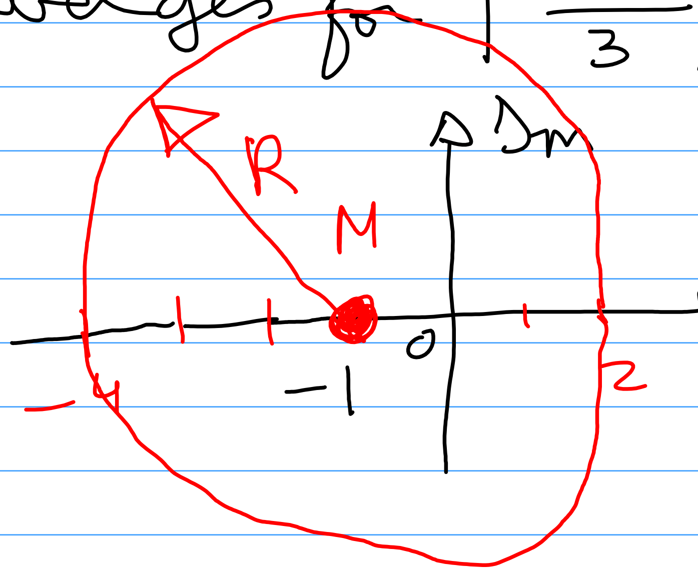
$$\frac{1}{3-w} = \frac{1}{3} \cdot \frac{1}{\left(1-\frac{w}{3}\right)} =$$

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{w}{3}\right)^n = \frac{1}{3} \cdot \sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

which converges for $\left|\frac{z+1}{3}\right| < 1$

$|z+1| < 3$

circle in $\mathbb{C}$-plane.



$M = -1$

$R = 3$

The series of $\frac{1}{(2-z)^2}$?

We know: $\frac{1}{2-z} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$

and

$$\frac{1}{(2-z)^2} = +\frac{d}{dz}\left(\frac{1}{(2-z)}\right) =$$

$$\frac{d}{dz}\left(\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n\right) = \frac{1}{2} \cdot \sum_{n=0}^{\infty} n \cdot \left(\frac{z}{2}\right)^{n-1} \cdot \frac{1}{2}$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{n}{2^{n-1}}\right) \cdot z^n$$

It looks nice, but can it be

done without any problems?

– differentiate to z $(z \in \mathbb{C})$

what does it mean?

– $\frac{d}{dz}\left(\sum\right) = \sum\left(\frac{d}{dz}\right)$

can that be done?

!! Let now: $z \in \mathbb{R}$: (also $w \in \mathbb{R}$)

$$\int_{?}^{z} \frac{1}{2-w}\,dw = -\ln|2-z| + C$$

?

$$\int_{0}^{z} \frac{1}{(2-w)}\,dw = -\ln|2-z| + \ln 2$$

!

$$\| $$

$$\int_{0}^{z}\left(\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{w}{2}\right)^{n}\right)dw =$$

$$\frac{1}{2}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right)\left(\frac{z}{2}\right)^{n+1}\cdot 2 = \sum_{n=0}^{\infty}\frac{1}{(n+1)}\left(\frac{z}{2}\right)^{n+1}$$

So we have:

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$$\ln|2-z| \overset{*}{=} \ln(2) - \sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(\frac{z}{2}\right)^{n+1}$$

if $z \in \mathbb{R}$ and $|z| < 2$ then

$2 - z > 0$ and $|2-z| = (2-z)$

So $\ln|2-z| = \ln(2-z)$.

I'm asking myself, if somebody gives me that series $*$, I can fill in $z \in \mathbb{C}$, with $|z|<2$, what I'm calculating?

Logarithm of complex numbers??

Be aware that we have then

differentiated to z,

integrated to z.

Can we do that in $\mathbb{C}$?

It looks so easy, but......?

That we have to study!

(IV) Differentiation

Given a function $f: \mathbb{C} \to \mathbb{C}$, what can we do with it?

A graph is not possible to construct, $\mathbb{C} \to 2$ dimensions, for a graph you need 4 dimensions. To imagine the behaviour of $f: \mathbb{C} \to \mathbb{C}$ is difficult.

If $f: \mathbb{R} \to \mathbb{R}$ then the derivative is defined by

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{(x+h) - x} \right)$$

$$= \lim_{w \to x} \frac{f(w) - f(x)}{(w - x)},$$

provided that the derivative exists!

At the same way we can define a derivative for $f : \mathbb{C} \to \mathbb{C}$,

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{(w - z)}$$

$$= \lim_{h \to 0} \frac{f(z+h) - f(z)}{(z+h) - z},$$

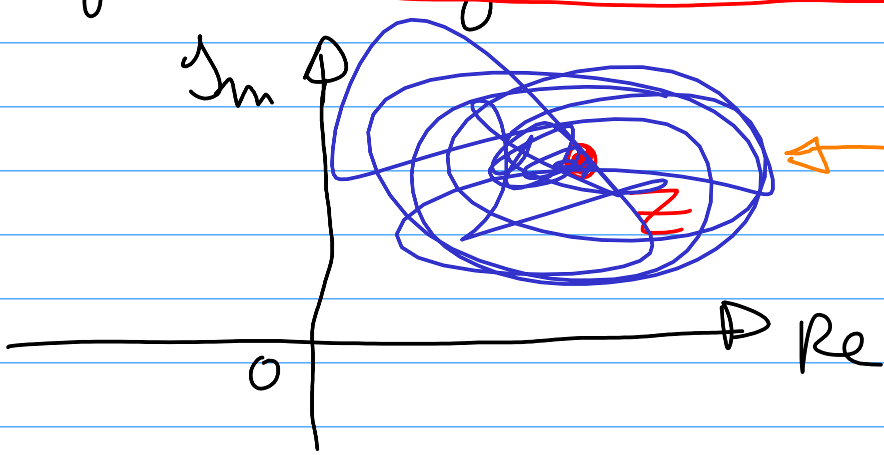
be aware $z, w, h \in \mathbb{C}$, and we assume that the derivative exists.

Questions:

"When everything goes well?"

"What does $\lim\limits_{w \to z}$ or $\lim\limits_{h \to 0}$ mean?"

Be aware of the fact that when there is written $w \to z$, there is not given some way of how to go with w to z.



Here is given some way.

To repeat it? I can't, because I walked in $\mathbb{C}$ and at a certain moment I came into a very neighbourhood of z, to do a little step and I was at z.

w

This remembers me to the sentence:
"You can not step twice into the same river".
Keep in mind, that calculating some limit, the next time you are doing it, you maybe have taken quite another way.
It is better to take another way and see if you get the same value out of it, if not then the limit not excists.
Maybe it is better to do it path-independent.

<u>Example</u>: $f: z \to \dfrac{z^2}{z \cdot \bar{z}}$

$\lim_{z \to 0} f(z) = ?$ (<u>in $\mathbb{C}$</u>)

Let $z \in \mathbb{R}$, then $z = x \in \mathbb{R}$

and $x \to 0$ so

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2}{x^2} = 1$$

Have we calculated the limit?

Let $z = iy$, with $y \in \mathbb{R}$ and

$y \to 0$, then

$$\lim_{y \to 0} f(iy) = \lim_{y \to 0} \frac{(iy) \cdot (iy)}{(y)^2} = -1.$$

<u>Compare the results!</u>

Here some other way, $(R \in \mathbb{R})$

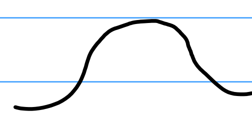
$$\lim_{R \to 0} f(R.\exp(i\varphi)) =$$

$$\lim_{R \to 0} \frac{R^2 \cdot e^{2i\varphi}}{R^2} = e^{2i\varphi}$$

so the limit depends on the angle, the arg(z).

Keep in mind:

"Walking from the station to the university."

There is not told to you, how you have to walk! Some people cross the traffic way near Chemical Engineering; other people use the traffic lights; other people like to walk their morning walk; I don't like to cross that bridge  over the Aa, incredible smooth if it has frozen!

But all we meet each other at the same point. And now during Corona time, think about all those different connections with the server, there is just one place of which the information is taken from. Maybe the signals have just gone around the earth?

Taking limits, be careful,

path-independent is of importance.

If you know that a limit exsist, and you have to calculate it, take an easy path. But read well, first you have to know that the limit excists.

Try yourself:

$$\lim_{z\to 0}\left(\frac{1}{2i}\left(\frac{z}{\bar z}-\frac{\bar z}{z}\right)\right)$$ exists or not?

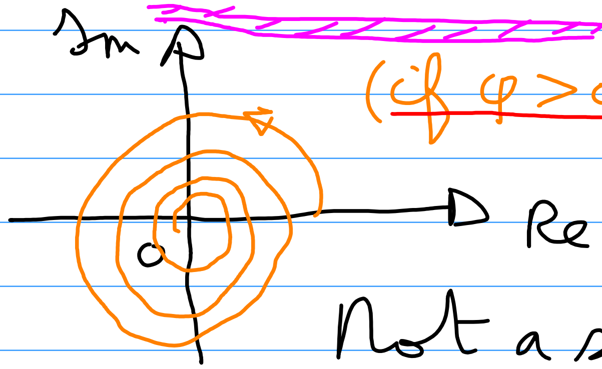
Let's write out and look to it,

$$\frac{1}{2i}\left(\frac{z}{\bar z}-\frac{\bar z}{z}\right)=\frac{1}{2i}\left(\frac{z}{\bar z}-\overline{\left(\frac{z}{\bar z}\right)}\right)\in\mathbb{R}.$$

$z = x+iy$, $\quad Re(z)=0$, $Im(z)=0$,
$Re(z) = Im(z)$, $\quad z = R\cdot\exp\left(i\frac{\varphi}{R}\right)$

something to do with: $$\frac{2\cdot x\cdot y}{x^2+y^2}=\frac{2\cdot R\cos\left(\frac{\varphi}{R}\right)\cdot R\cdot\sin\left(\frac{\varphi}{R}\right)}{R^2}=2\cdot\cos\left(\frac{\varphi}{R}\right)\sin\left(\frac{\varphi}{R}\right)$$
$$=\sin\left(2\cdot\frac{\varphi}{R}\right)\to\ ??\quad \text{if } R\to 0$$

(if $\varphi>0$ and constant)



Not a straight line but a spiral and $|z| = R\to 0$, if $R\to 0$, so $z\to 0$.

Why soo much attention to limits?

We want to differentiate complex functions $f: \mathbb{C} \to \mathbb{C}$.

If $g: \mathbb{R} \to \mathbb{R}$ then derivative in $x = x_0$ defined by $\lim_{x \to 0} \frac{g(x) - g(x_0)}{(x - x_0)} =$

$$\lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = g'(x_0)$$

if the limit exists.

if $h > 0$: $\frac{h}{h} = 1$
if $h < 0$, $\frac{-h}{h} = -1$

Exc. $\mathbb{R}$: path dependent!

i) $g: x \to |x|$, $x_0 = 0$, $\lim_{h \to 0} \frac{|h| - 0}{h}$ does not exist, g not differentiable in $x_0 = 0$.

ii) $g: x \to x^2$ $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} =$

$$\lim_{h \to 0} \frac{(x^2 + 2h \cdot x + h^2) - x^2}{h} = 2 \cdot x$$

limit exists, for every $x \in \mathbb{R}$.

Let's do the same for $f: \mathbb{C} \to \mathbb{C}$

The derivative is defined by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} =$$

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

if those limits excist and $z_0, z, h \in \mathbb{C}$.
Let's look if those limits always excist? If they are path-indepent?

<u>Exe. $\mathbb{C}$</u>:

i) $f: z \to \bar{z}$ differentiable?

<u>Look if</u>: $z \in \mathbb{R} \leadsto f(z) = z \leadsto f'(z) = 1$

if $z \in i\mathbb{R} \leadsto z = iy,\ y \in \mathbb{R}$ then

$f(z) = \overline{(iy)} = -iy = -z \leadsto f'(z) = -1$

Maybe not good calculated but I see some difference, so I want to calculate those limits with different paths. For instance (α) parallel to the real-axis, so if $z = x + iy$, then x-direction; (β) parallel to the imaginary-axis, or y-direction.

Let's try and see!

Arbitrary $z \in \mathbb{C}$ is given, to calculate

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{(z+h) - z}$$

(α) ~ (x-direction), let $h \in \mathbb{R}$:

$$\frac{f(z+h) - f(z)}{(x+h+iy) - (x+iy)} = \frac{\overline{(x+h+iy)} - \overline{(x+iy)}}{h}$$

$$= \frac{((x+h) - iy) - (x - iy)}{h} = \frac{h}{h} = 1$$

(β) ~ (y-direction), let $h \in \mathbb{R}$:

$$\frac{f(z+ih) - f(z)}{(z+ih) - (z)} = \frac{\overline{(x+i(y+h))} - \overline{(x+iy)}}{ih} =$$

!

$$\frac{(x-i(y+h))-(x-iy)}{ih}=\frac{-ih}{ih}=-1$$

Compare results: so $f: z \to \bar{z}$ is not differentiable to z.

ii) $f: z \to z^2$, let $z \in \mathbb{C}$ be fixed,

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=$$

$$\lim_{h\to 0}\frac{(z^2+2z\cdot h+h^2)-z^2}{h}=2\cdot z=f'(z)$$

This limit is not path-dependent, if $z \in \mathbb{C}$ is fixed, you will always get the same value out of that limit, so

$f': z \to 2z$, or $f'(z)=2\cdot z$, or $\frac{\partial f}{\partial z}(z)=2\cdot z$.

May be we can treat all the real function, we know, with some $z \in \mathbb{C}$ in it, and may be the derivatives at

the same way?
But what we have to think by
$\ln(z)$, $\sin(z)$, $\cos(z)$, $\arctan(z)$?
At this moment an exciting business!
For instance:
? $\ln(z)$? if $z = R \cdot e^{i\varphi} \leadsto f(z) = \ln(R \cdot e^{i\varphi})$
$= \ln(R) + i\varphi = \ln|z| + i\arg(z)$ and
$\exp(\ln|z| + i\arg(z)) =$
$\exp(\ln(|z|)) \circ e^{i\arg(z)} = |z| \cdot e^{i\arg(z)} = z$
So: $\exp(\ln(z)) = z$, just as in the
real case! But are the exp- and ln-
function really its inverse functions?
Look to: $\arg(z)$, a lot values can be
given to that! If you take $\arg(z) = \gamma$
then I take: $(\gamma + 2\pi)$, also good!

Maybe Arg(z) can help us? We shall see later on.

Sin (z) no idea how to calculate?

We know: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$;

why not $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots$

and then $\frac{d}{dz}(\sin(z)) = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots$

$= \cos(z)$

At these pages we did a lot, just playing with mathematics. In the past they did the same and then comes the problem of proving those things.

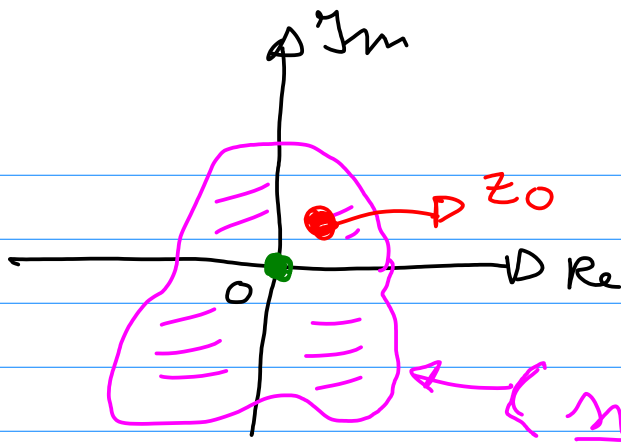
Can you do it always? Or only in certain cases? Let's ask ourself that question in the case of taking the derivative.

To get some feeling for possible problems, let's look to $f : z \to |z|^2$.
Maybe of importance for later on, in this specific case: $f : \mathbb{C} \to \mathbb{R}$ $(\neq \mathbb{C})$.
First I want to see if f is differentiable in $z = 0$? We need to calculate $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$, $h \in \mathbb{C}$

$$\frac{f(0+h) - f(0)}{h} = \frac{|0+h|^2 - |0|}{h} = \frac{h \cdot \bar{h}}{h} = \bar{h}$$

So $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \bar{h} = 0$,

the derivative in $z = 0$ exists!
Can we do the same in a small neighbourhood of $z = 0$? See figure on next page,



$$f: z \to \frac{|z|^2}{2}$$

$$(f'(0) = 0)$$

(small neighbourhood: V)

Maybe I have taken some magnifying glass (1000x or 10^{18}x??)

$0 \neq z_0 \in V$, and now:

$$(*)\quad \frac{f(z_0+h) - f(z_0)}{h} =$$

$$\frac{1}{2}\cdot\frac{(z_0+h)\overline{(z_0+h)} - z_0\cdot\overline{z_0}}{h} =$$

$$\frac{1}{2}\cdot\frac{(z_0\cdot\overline{z_0} + z_0\cdot\overline{h} + h\cdot\overline{z_0} + h\cdot\overline{h}) - z_0\cdot\overline{z_0}}{h}$$

$$= \frac{1}{2}\left(z_0\cdot\left(\frac{\overline{h}}{h}\right) + \overline{z_0} + \overline{h}\right).$$

$$\frac{-ik}{ik} = -1$$

What happens if $h \to 0$? See pg. 37-38?

(α) if $h \in \mathbb{R} \leadsto \lim_{h \to 0} (*) = \frac{z_0 + \overline{z_0}}{2} = \mathrm{Re}(z_0)$

(β) if $h = ik \in i\mathbb{R}$, with $k \in \mathbb{R}$, $\leadsto$

$$\lim_{h \to 0} (*) = \lim_{k \to 0} \frac{1}{2}\left(z_0\cdot(-1) + \overline{z_0}\right) = -i\,\mathrm{Im}(z_0)$$

(!) $z_0 \in V$ is fixed, so we see that $f: z \to |z|^2$ is not differentiable in z_0, different paths give different values!

Conclusion:

$f: z \to |z|^2$ is differentiable in $z = 0$, but not differentiable in a (small) neighbourhood of $z = 0$.

We have already seen (pg. 38), that $f: z \to z^2$ is differentiable for every $z \in \mathbb{C}$. So also in $z = 0$ and so in a neighbourhood of $z = 0$.

Let's give them some name.

If $f: \mathbb{C} \to \mathbb{C}$ differentiable in $z = z_0$ and also in a neighbourhood of $z = z_0$ then f is called holomorphic.

For the fun: google on internet with: differentiable holomorphic.

Also a nice site:

Paul Math Online Notes (not soo much about $\mathbb{C}$), or just google with: Complex Analysis

Don't forget: Z-lib, just google and search inside with: complex analysis, be careful, just 5 books at a day!

But how to look if a function is holomorphic, easier then just done?

Let's first write $f: \mathbb{C} \to \mathbb{C}$ at some other way:

$f(z) = u(z) + i\, v(z)$

with $u: \mathbb{C} \to \mathbb{R}$ and $v: \mathbb{C} \to \mathbb{R}$,

maybe, if we take $z = x + iy$,

$f(z) = u(x, y) + i\, v(x, y)$

with $u: \mathbb{R}^2 \to \mathbb{R}$ and $v: \mathbb{R}^2 \to \mathbb{R}$

also a nice way, $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

In the cases:

$f: z \to |z|^2 \rightsquigarrow \begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$

$f: z \to z^2 \rightsquigarrow \begin{cases} u = x^2 - y^2 \\ v = 2 \cdot x \cdot y \end{cases}$

If $f(z)$ given, fill in $z = (x + iy)$ and search real- and imaginary part as functions of x and y, so $\begin{cases} u(x,y) \\ v(x,y) \end{cases}$

The reverse $g(x,y)$ is given,
fill in $x = \left(\frac{z+\bar{z}}{2}\right)$ and $y = \left(\frac{z-\bar{z}}{2i}\right)$,
try to get expressions with z and $\bar{z}$,
so: $g(z)$.

But now? If f is differentiable
in z, what kind of relations excist
for the functions u and v?
Cauchy and Riemann asked
themselves the same and they
came with the Cauchy-Riemann
equations. Let's do, what they did.
The derivative f' excists if the
limit $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ is
path-independent. So assume that

$f'(z)$ excists, then calculating that limit, in the real direction, has to give the same value as calculating it into the imaginary direction.

Let's just do, assume that $h \in \mathbb{R}$,

$$\frac{f(z+h)-f(z)}{h} = \left[(u(x+h,y)+iv(x+h,y)) - (u(x,y)+iv(x,y)) \right] / h =$$

$$\frac{u(x+h,y)-u(x,y)}{h} + i\,\frac{(v(x+h,y)-v(x,y))}{h}$$

$$\longrightarrow \left(\frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y)\right), \text{ if } h \to 0$$

$$\frac{f(z+ih)-f(z)}{(z+ih)-(z)} = \left[(u(x,y+h)+iv(x,y+h)) - (u(x,y)+iv(x,y)) \right] / ih$$

$\left(\frac{1}{i} = -i\right)$

$$= -i\,\frac{u(x,y+h)-u(x,y)}{h} + \frac{v(x,y+h)-v(x,y)}{h}$$

$$\longrightarrow -i\,\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y), \text{ if } h \to 0.$$

Because $f'(z)$ excists, both limits have to be equal so:

$(z = x+iy)$

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) =$$

$$-i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$$

There follow the C.R. equations:

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \end{cases}$$

applied to: $f: z \to z^2$

$$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases} \Rightarrow \begin{cases} u_x = 2x, \ u_y = -2y \\ v_x = 2y, \ v_y = 2x \end{cases}$$

C.R equations are satisfied for every $z \in \mathbb{C}$, in this case.

Applied to $f: z \to |z|^2$

$$\begin{cases} u(x,y) = x^2 + y^2 \\ v(x,y) = 0 \end{cases} \Rightarrow \begin{cases} u_x = 2x, \ u_y = 2y \\ v_x = 0, \ v_y = 0 \end{cases}$$

See that C.R. equations is only satisfied if $x=0$ and $y=0$, thus only in $z=0$.

So C.R.-equations are very helpful to determine where a function $f: \mathbb{C} \to \mathbb{C}$ is differentiable.

See pg. 41, there I made some remark about $f: \mathbb{C} \to \mathbb{R}$.
Assume that f is differentiable for every $z \in \mathbb{C}$, then C.R. eqn's are satisfied! What do we know: $f(z) = u(x,y) + i \cdot 0$,
so $v(x,y) = 0 \Rightarrow u_x = 0, u_y = 0$
that means that $u(x,y) = C \in \mathbb{R}$,

is constant, so $f(z) = C$ is constant!

Conclusion:

If $f: \mathbb{C} \to \mathbb{R}$, it can be differentiable at certain points but it can not be holomorphic, unless $f(z) = C \ (\in \mathbb{R})$

In mind I have always:

$$f = (u, v) = u + iv$$
$$(x, y) = x + iy$$

$\to u_x = v_y$ and there is an equation with a (-1) in it:

already smooth, for u_x so next one: $u_y = (-1) \cdot v_x$

If I don't know anymore then I write down those calculations on pg. 46.

Now we have for $f: \mathbb{C} \to \mathbb{C}$:

If f holomorphic then C.R. eqn's for every $z \in \mathbb{C}$.

Holds the same the other way around?

Search on internet!

See Goursat's theorem and especially the one with weakened hypothesis:

If f continuous in open set Q and partial derivatives of u and

v exist in Q and C.R-eqn's are satisfied in Q ⟹ f is holomorphic in Q.
(Looman-Menchoff theorem)

Search on internet (wikipedia)
- Cauchy-Rieman equations
- Goursat theorem
- Looman-Menchoff theorem

What will we use often??

Given some $f: \mathbb{C} \to \mathbb{C}$,
- determine u and v?
- are u and v continuous diffb.? $\underline{\underline{Y}}$
- C.R. eqn's are satisfied? $\underline{\underline{Y}}$

⟹ f is holomorphic.