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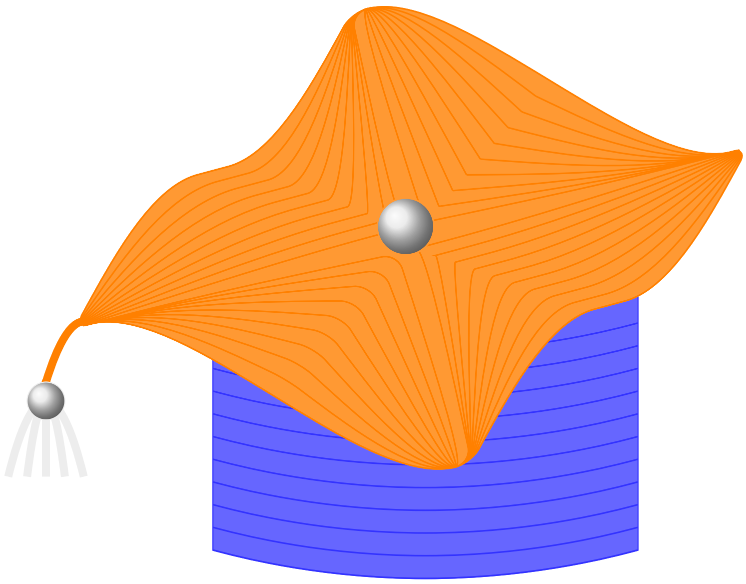
Complex Analysis



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1 Preface

What this will become? I have all kind of ideas and ideals, but if they will ever be ended, will become to what I want? No idea.
It is something that I want to do already many years and this is not the first attempt of me. If it will be used? That question I can better not ask myself. I start without seeing the finish, maybe I will never get over it?
My ideal is to write a novel about Complex Analysis, as basis I use the lecture notes of Joop Boersma. I enjoy these lecture notes.

How I will write my novel? It's also for me still a question.
It would be nice to work at some problem.
Then to see if a certain method works and then to ask ourself, if we can make a theorem of it? And if that is the case, that we put the theorem and proof somewhere else in our novel?
Let's start and I hope that we have fun together.

2 Preliminaries

A short overview will be given of all kind of terms, which are used in the chapters after this one. It is not my intention to give a complete overview of the analysis in this chapter. There is assumed, that you are already familar with the complex numbers: $\mathbb{C}$. The main points will be repeated.

2.1 In the Past

People, in the past, tried to solve quadratic equations like:

$$x^2 - 2x + 6 = 0. \tag{2.1}$$

The easiest way to solve is by rewriting the given equation:

$$(x - 1)^2 + 5 = 0, \tag{2.2}$$

and that gives:

$$(x - 1)^2 = -5, \tag{2.3}$$

and the solutions are given by:

$$x_1 = +1 + \sqrt{-5} \text{ and } x_2 = +1 - \sqrt{-5}. \tag{2.4}$$

That number $\sqrt{-5}$ was called an imaginary number. It works fine, if we use that $(\sqrt{-5})^2 = -5$. With some manipulation you see that $\sqrt{-5} = \sqrt{5}\sqrt{-1}$ and here we have the $\sqrt{-1}$, which is often called i, so we get:

$$i = \sqrt{-1}. \tag{2.5}$$

And the solution in 2.4 can be rewritten as

$$x_1 = +1 + (\sqrt{5})\, i \text{ and } x_2 = +1 - (\sqrt{5})\, i, \tag{2.6}$$

with:

$$i^2 = -1. \tag{2.7}$$

2.2 Complex Numbers

We can also introduce the complex numbers as a couple of two numbers, just like points in the 2-dimensional space $\mathbb{R}^2$. We leave the addition of two points as we are used to. But in the 2-dimensional space $\mathbb{R}^2$ can be introduced the following multiplication

$$(a, b) \times (c, d) = (ac - bd, ad + bc). \tag{2.8}$$

With that operation, we get a system of things (i.e. pairs of numbers), by which we can calculate at the same way as we can do with the real numbers. These pairs of numbers are called complex numbers.

It is useful to identify the complex numbers $(a, 0)$ with the real numbers, so to write a instead of $(a, 0)$. For the number $(0, 1)$ there is introduced the abbreviation i. Due to this, every complex number z can be uniquely written as $x + iy$, the numbers x and y are real numbers, called the real part and imaginary part of z.

Notation 2.1

If $z = x + yi$ then $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

The complex numbers z with $\mathrm{Re}(z) = 0$ lie on the imaginary axis and those with $\mathrm{Im}(z) = 0$ lie on the real axis. The number $x - yi$ is called the complex conjugate of z.

Notation 2.2

If $z = x + yi$ then $\overline{z} = x - yi$.

The complex numbers can also be represented by the argument and the modulus (absolute value), which one is led to if one enters polar coordinates in the complex plane: r = the distance to the origin, ϕ = the angle of the radius vector with the positive $x-$axis, the real axis. The direction of the increasing ϕ is counterclockwise. The angle ϕ is determined by $(x,\, y)$ except for integer multiples of $2\,\pi$ and there holds that

$$x \;=\; r\,\cos\phi,\; y \;=\; r\,\sin\phi. \tag{2.9}$$

By the modulus, or the absolute value, of z is meant the non-negative value:

$$r \;=\; |z| \;=\; \sqrt{x^2 \,+\, y^2} \;\geq\; 0. \tag{2.10}$$

The angle ϕ is called the argument of z:

$$\phi \;=\; \arg(z) \;=\; \arctan\,(\frac{y}{x}) \;+\; k\,\pi \;+\; n\,2\,\pi, \tag{2.11}$$

with $k \,\in\, \{-1,\, 0,\, +1\}$ and $n \,\in\, \mathbb{Z}$.
A complex number can be written as

$$z \;=\; x \,+\, y\,i \;=\; r\,(\cos\phi \,+\, (\sin\phi)\,i) \;=\; r\,\exp\,(i\,\phi). \tag{2.12}$$

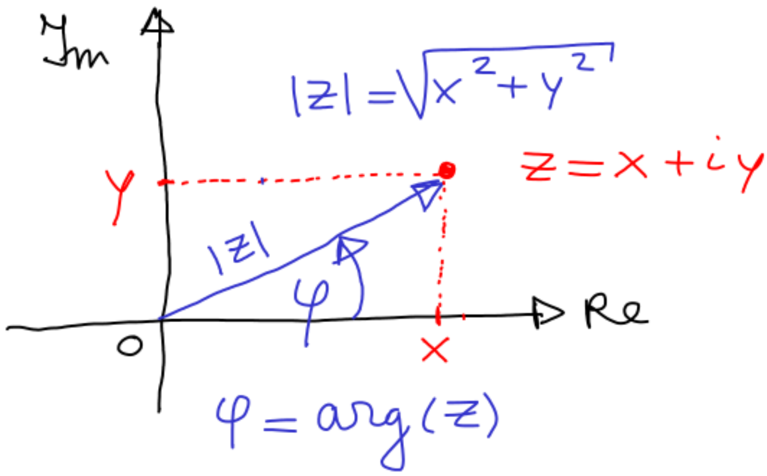


Figure 2.1 Sketch coordinates

My hope is that you are already familar with these ways of writing (as in **2.12**).
A new concept for you can be the principal value of z, written by $\mathrm{Arg}(z)$. As

already noted, the argument of z is determined except for integer multipes of 2π. There is spoken about the principal value of z if the $\arg(z)$ is limited to an interval of length 2π. Very often there will be taken the following interval (left open, right closed):

$$-\pi < \arg(z) \leq \pi. \tag{2.13}$$

If nothing is said, there will be meant by $\arg(z)$ this principal value, see **2.13**. The importance of this will be discussed later.

Remark 2.1

In $\mathbb{C}$ we can do the same calculations as with real numbers.
But you can not compare complex numbers with each other!
Inside the complex system you don't have a less or greater than.

2.2.1 Exponential function

A lot of people take the expression:

$$\exp(i\,\phi) = \cos\phi + (\sin\phi)\,i \tag{2.14}$$

as a definition, with $\phi \in \mathbb{R}$. But if a calculator calculates $\exp(x)$ then the following series is used:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2.15}$$

filling in $x = i\,\phi$ that gives:

$$\exp(i\,\phi) = \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{(2k)}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{\phi^{(2k+1)}}{(2k+1)!}. \tag{2.16}$$

The real part of this series gives the series of the cos-function and the imaginary part the series of the sin-function, so we have found the expression **2.14**.

Important is the fact that:

$$|\exp(i\,\phi)| = 1, \tag{2.17}$$

if ϕ is a real parameter then exp $(i\,\phi)$ describes the unit circle in the complex plane $\mathbb{C}$.
We can also calculate the exponential function of a complex number:

$$\exp(z) = \exp(x + i\,y) = \exp(x)\,\exp(i\,y), \tag{2.18}$$

the radius of $\exp(z)$ becomes $\exp(x)$ and its argument y.
A short overview will be given of all kind of terms, which are used in the chapters after this one. It is not my intention to give a complete overview of the analysis in this chapter. There is assumed, that you are already familar with the complex numbers: $\mathbb{C}$. The main points will be repeated.

2.3 Sets of Complex Numbers

First all kind of notations, used in these lecture notes, will be defined (explained). After that there will be given two important theorems with their proofs.

Notation 2.3

The notations are:

$V := \{z\,|\,.....\}$, $a \in V$, $a \notin V$, $A \subset B$, $B \supset A$, $A \cap B$, $A \cup B$, $A^c = \mathbb{C} \setminus A$, $\overline{A}$.

The set of all complex numbers we have given already some name: $\mathbb{C}$. A subset A out of $\mathbb{C}$ can often be described by some prescription or there can be mentioned properties of the elements z out of that subset. A frequently used notation is $A := \{z\,|\,........\}$ at which the place of the dots the prescription is written where z has to satisfy. So is $\{z\,|\,|z| < 1\}$ the inner part of the unit circle in $\mathbb{C}$. The imaginary axis is $\{z\,|\,z = i\,y,\, y\,\text{real}\}$. In the remainder of these lecture notes we make no distinction between the following three expressions "z is a complex number", "z is an element out of the set $\mathbb{C}$" and "z is a point in the complex plane".

Let V be some subset of complex numbers. To express that a certain complex number a belongs to V, we use the notation $a \in V$; we also say: "a is an element of V". If we want to express that a does not belong to V then we write $a \notin V$.

Finite set : is a set with a finite number of elements.

Infinite set : is a set with an infinite number of elements.

Bounded set : a set V is called bounded if we can find some real number M such that $|z| < M$ for all $z \in V$. Such a set we can cover with a circle around the origin and a finite radius; it is also possible to cover it with some square with finite dimensions.

Subset : A is a subset of B if every element of A is an element of B. The notation for this is $A \subset B$ (A is contained in B) or $B \supset A$ (B includes A). The possiblity that $A = B$, is not excluded by this notation $A \subset B$. If $A \neq B$ then A is a proper subset of B.

Intersection of A and B is the set of elements which belong to both A and B. The notation is $A \cap B$. If $A \cap B = \emptyset$ ($\emptyset$ = empty set), then A and B are called disjoint .

Union of A and B is the set of elements which belong to A, to B, or both. The notation is $A \cup B$.

The concepts of intersection and union can be applied to an arbitrary collection of sets.

Complement of A: set of complex numbers that not belongs to A. Notation for complement of A is A^c (also $\mathbb{C} \setminus A$).

Next there will be given several descriptions of sets, which will often be used by proofs of theorems and so on.

Neighbourhood

Let a be a complex number and ρ positive (so $\rho \in \mathbb{R}$). The numbers z with the property $|z - a| < \rho$ lie inside the circle with midpoint a and radius ρ. If ρ is small, then they say that z lies in the neigbourhood of a.

Neighbourhood of a is the set of numbers z which satisfy $|z - a| < \rho$.

The neighbourhood depends on a and ρ.

Therefore: ρ-neighbourhood of a. The point a belongs to every of its neighbourhoods. One speaks of a reduced neigbourhood if a is explicitly excluded. So: reduced ρ-neighbourhood of a is the set of numbers z with $0 < |z - a| < \rho$.

Accumulation point

The number a is called an accumulation point of V if in every reduced neigbourhood of a infinitely many elements are of V.

Remark 2.2

An accumulation point of V is not necessarily a point of V.

Closed set

A set V is called closed if every accumulation point of V is also a point of V. A closed set contains all its accumulation points.

Interior point

The number a is an interior point of V if there exist some neighbourhood of a which belongs entirely to V. An interior point of V belongs to V.

Open set

A set consisting solely out of interior points, is called an open set.

Connected set

A set V is called connected if every pair of points P, Q of V can be connected by a curve, which lies inside V (definition of curve, see page 15).

Region

This is for us an important concept. A region G in the complex plane is a set of complex numbers which

(1) is not empty,
(2) is open,
(3) is connected.

Every point of G is an interior point of G, because a region is open by defintion. Two arbitrary point of G can be connected by a polygoon draw, which ly completely in G, because G is connectes. An accumnulation point of G belongs to G or doesn't belong to G. In the first case it is an accumulation point of G and so an interior point of G. In the second case the accumulation point is called a boundary point of G. The set of accumulation points of G, which not belong to G together form an edge of G. (The edge of a set V is defined by the set of points where is every neighbourhood ly a point of V and V^c.) The union of G with its edge is a closed set, which is notated by $\overline{G}$. $\overline{G}$ is called a closed region. For this we will choose the word domain, so domain = region + edge.

2.3.1 Two important theorems.

Two important theorems are:
the covering theorem of Heine and Borel (theorem 2.1) and
the theorem of Bolzano and Weierstrass (theorem 2.2).

Theorem 2.1

(H.B.) If a bounded and closed set A of complex numbers is contained in the union of a collection of open sets, the is A can be covered by a finite number of these open sets.

Proof of Theorem 2.1

If the collection itself is finite, there is nothing to prove.
Further by contradiction. A is bounded, so A can be covered by a square V_0 (boundary included) entirely located in the finite z-plane.
Suppose the statement was incorrect. Divide the square V_0 into quarters. Then one of these quarters (say V_1, boundary included) has the property that the part of A located in V_1 can not be covered by finite number out of the collection of open sets referred to.
Repeat:
V_1 is divide into four quarters. At least one of these quarters (V_2, boundary included) has the property that $A \cap V_2$ can not be covered by a finite of these conscious open sets. And so on.
We get so a sequence of squares: V_0, V_1, V_2, ..., with $V_0 \supset V_1 \supset$ ...., which shrinks to a point P. This point P belongs to every V_n, and is a accumulation point of A. Because A is closed, belongs P to A.
In the given collection of open sets there is at least one which contains P, and thus internally contained. So there is a ρ-neigbourhood of P (if ρ is taken small enough) with the property that the intersection with A is completely covered by this only open set.

This leads to a contradiction. In the lang run (n great enough) all the V_n ly inside that ρ-neighbourhood. On the one hand, V_n could not be covered by a finite number, on the other hand it is covered (from a certain number n) by one of the open sets. Out of this contradiction follows that the theorem is correct.
□

Theorem 2.2

(B.W.) A bounded infinite set has at least one accumulation point.

Proof of Theorem 2.2

Hint: Use the square-method, used in the proof of theorem (2.1) once more.
□

2.3.2 Convergence criteria.

Definition 2.1

A sequence of numbers $\{z_n\}$ ($n = 1, 2, 3, ...$; there may be equals among them) has the limit a (converges to a) if for every $\epsilon > 0$ there can be found a rank number $N(\epsilon)$ such that $|z - a_n| < \epsilon$ for every $n > N(\epsilon)$.

Definition 2.2

A convergence criterium, where the limit a does not occur, reads:
(Cauchy) A sequence $\{z_n\}$ has a limit if to every $\epsilon > 0$ there exists a $N(\epsilon)$ such that for every $n, m > N(\epsilon)$ holds that $|z_n - z_m| < \epsilon$.

Proof of Theorem 2.2

There is some M such that for all $m > M$ $|z_m - z_{M+1}| < 1$. Let a be an accumulation point of the z_m (use B.W.!). Then is $|z_n - a| \leq |z_n - z_m| + |z_m - a| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$.
□

Definition 2.3

If $\{a_n\}$ is a sequence of real numbers, then we define:

$$\limsup_{n\to\infty} a_n = \begin{cases} L & \text{if for every } \epsilon > 0 \text{ only finitely elements of the sequence} \\ & \text{are } > L + \epsilon\text{, while infinitly elements are } > L - \epsilon, \\ \infty & \text{if the sequence is not bounded from above,} \\ -\infty & \text{if } \{a_n \to -\infty\} \text{ for } n \to \infty \end{cases}$$

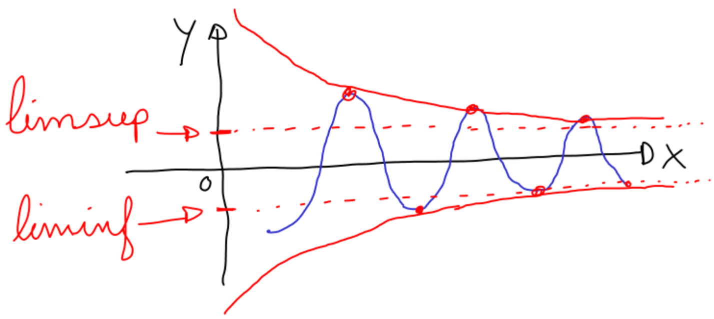


Figure 2.2 Limsup and liminf

Just in words, the $\limsup$ of a sequence is the limit of the lowest upper bound.
(And so $\liminf$, the limit is of the greatest lower bound.)

Remark 2.3

If a sequence $\{a_n\}$ has a limit L then $\limsup_{n\to\infty} a_n = L$.

Theorem 2.3

Every real sequence $\{a_n\}$ has a $\limsup$.

Proof of Theorem 2.3

Hint: If $\limsup \neq \pm\infty$ then all $a_n < C$ and ∞ may $> D$; divide $[D, C]$.....
□

2.4 Arc, curve, path in the complex plane

By a smooth arc $a \leq t \leq b$ the functions $x(t)$ and $y(t)$ have continuous derivatives then we call the set of point $z = f(t) = x(t) + i\,y(t)\,(a \leq t \leq b)$ a smooth arc. The point $z_1 = f(a)$ is called the beginpoint of the arc, $z_2 = f(b)$ endpoint. The variable t is called the parameter and $z = f(t) = x(t) + i\,y(t)$ a parameter representation of the arc. The same arc can have different parameter representations.
(Usually we still demand that $\{x'(t)\}^2 + \{y'(t)\}^2 > 0$.)
If we write arc in the sequel, we still mean: a smooth arc.

A curve is a concatenation of a finite number of arcs; so it has a begin and an endpoint. A simple curve is a curve without double points. If the beginpoint and endpoint fall together we speak about a closed curve.
A Jordan curve is a simple closed curve.

Properties 2.1

A Jordan curve divides the complex plane into two disjoint pieces, the inner area and the outer area. Both pieces are regions, which have the curve as boundary. The Jordan curve is called positive oriented if we moving in the direction of the arrow have the inner area on our left (counterclockwise).

Later on we need the term path in the complex plane. A path is a concatenation of a sequence of arcs in a not necessarily finite number. A path can run to infinity, or come from there (for instance out of a certain direction), etc..

Theorem 2.4

Let K be some arc with parameter representation $z = x(t) + i\,y(t)$, $(a \leq t \leq b)$.

The length L of K is equal to: $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt = \int_a^b |\frac{dz}{dt}| dt.$

A region G is called connected if two different points P and Q in G can be connected to each other by a curve in G.
A region G is called simply connected if with every Jordan curve J which belongs to G also the inner region of J belongs to G. So every Jordan curve J which belongs to G can be subtracted, in a continuous way, to a point of G.
If G is connected but not simply connected, then G is called multiply connected.

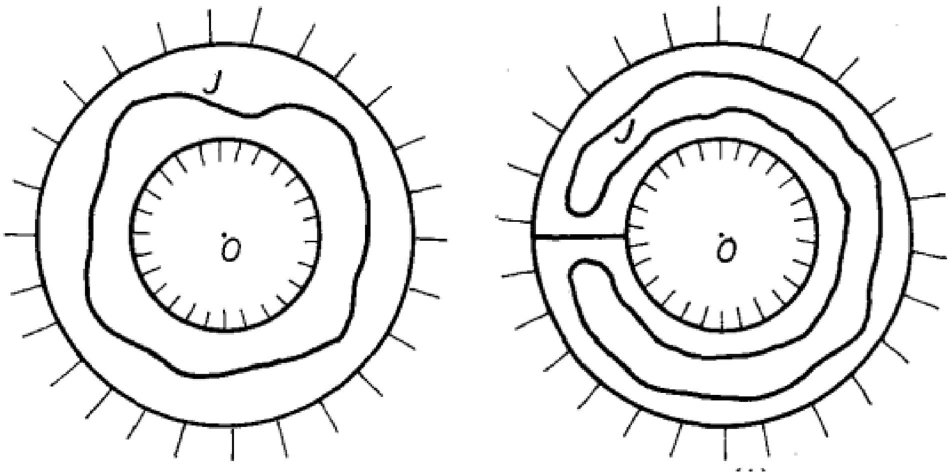
Example 2.1

(1) $\{z \mid |z| < 1\}$ is simply connected.

(2) $\{z \mid |z| > 1\}$ is multiply connected.

(3) $\{z \mid 1 < |z| < 2\}$ is multiply connected.

(4) Leave the line segment $\{z \mid \mathrm{Re}(z) < 0 \,\mathrm{en}\, \mathrm{Im}(z) = 0\}$ out of the area described in (3), then arises a simply connected region.



(3) multiply connected (4) simple connected

Figure 2.3 Connectedness

Below are some more examples with corresponding drawing.

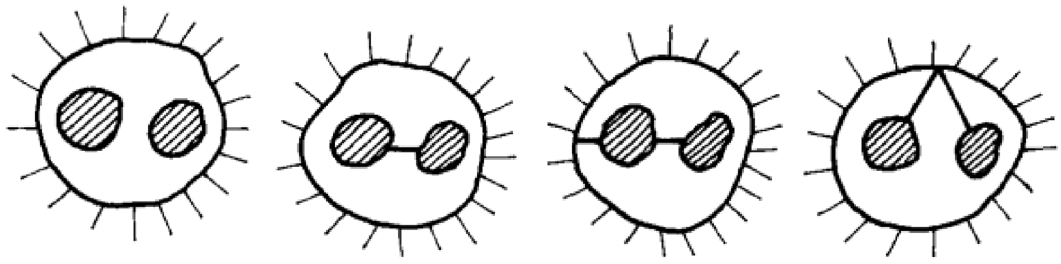


Figure 2.4 Connectedness, other examples

The "islands" that not belong to G can also be points: just as in (3) is for instance $\{z \mid 0 < |z| < 1\}$ multiply connected.

3 Function Concept

3.1 Functions of a complex variable

Let be given:

(1) a set A of complex numbers,
(2) a prescription with which to every $z \in A$ some complex number w is added.

We write $w = f(z)$ and represent the with: f (sometimes is written $f(z)$) is an (one-valued) complex function of z defined on A. It is essential to mention the set at which the function is defined. Compare with the former term of "permissible" values of the independent variable.
If we propose $z = x + i\,y$ and $w = u + i\,v$, we can splits $f(z)$ into a real and an imaginary part: $f(z) = u(x, y) + i\,v(x, y)$.The functions u and v are real functions of two real variables, defined for $(x, y) \in A$. A complex functions is nothing more than an ordered pair of two real functions of two real variables. It is clear that we have to narrow the functions concept to get something of interest; otherwise we would just do real function theory.

Concept of Limit:

The definition of limit is formal the same as given in the real analysis.

Definition 3.1

A function f defined on A has a limit L for $z \to a$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $0 < |z - a| < \delta$ and $z \in A$ holds that $|f(z) - L| < \epsilon$.
We write:

$$\lim_{z \to a} f(z) = L, \text{ or also } f(z) \to L\ (z \to a).$$

Another definition of limit is given by:

Definition 3.2

The function f has a limit L for z to a if

$$\lim_{n\to\infty} f(z_n) = L, \text{ for every sequence } \{z_k\} \text{ with } z_k \in A \text{ and } z_k \to a \ (k \to \infty).$$

The definitions in **3.1** and **3.2** are equivalent.

Theorems about the sum, difference, product and quotient of limits are the same as in the real analysis.

Concept of Continuity:

Definition 3.3

The function f is called continuous in the point $a \in A$ if

$$f(a) = \lim_{z\to a} f(z).$$

An equivalent definition of continuity is given by:

Definition 3.4

f is continuous in $a \in A$ if to every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for $|z - a| < \delta$ and $z \in A$.

The function f is called continuous on (or in) A if $f(z)$ is continuous in every point of A. Most of the time is A a region, a domain, or a curve.
Out of the real analysis we know the concepts of left- or right- continuity:

$$\lim_{x\uparrow a} f(x) = f(a) \text{ resp. } \lim_{x\downarrow a} f(x) = f(a).$$

In the complex plane there are far more possibilities to approach a point a.
Let f be defined on A and let B a subset of A. Then is called f continuous in the point a with respect to $B \subset A$ if

(1) $a \in B$,

(2) to every $\epsilon > 0$ there exists a $\delta > 0$ such that
$|f(z) - f(a)| < \epsilon$ for $|z - a| < \delta$ and $z \in B$.

Example 3.1

Define $f(z) = \frac{1}{z} - \frac{1}{\bar{z}}$ $(z \neq 0)$, $f(0) = 0$ is defined on $A =$ the whole complex plane; f is continuous in 0 with respect to the real axis.

Remark 3.1

If there is spoken about "continuity in a" without question, then there is assumed that the function is defined in a full neighbourhood of a.

Remark 3.2

Sum, difference, product and quotient of continuous funktions are continuous functions (with the known exception not dividing by zero), just as in the real function theory.

If $f(z)$ is a continuous function of z, then the real functions $u(x, y)$ and $v(x, y)$, defined by $f(z) = u(x, y) + i\,v(x, y)$, contnuous functions of x and y. The reverse holds also. To get really interesting problems in the complex function theory we have to require more then only continuity.

Remark 3.3

If f is continuous, then also $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ and $|f|$ continuous functions of z. The principal value $\arg(z)$ is not continuous in the neighbourhood of the negative real axis; it is continuous on that axis with respect to that axis (there is $\arg(z)$ equal to π).

Theorem 3.1

Let f be defined and continuous on a bounded and closed set A, then is f bounded on A, and $|f(z)|$ has both a maximum and a mimimum on A.

3.2 Differentiable functions

We have already noticed that we don't get far with continuous functions. Therefore we will require differentiability. A priori it is not clear if there will come something of interest. Because in the real analysis we have also functions which depend on two variables x and y, which can be differentiated to these variables.
We shall define the derivative f' of f define at a way completely analoguous as the derivative in the real analysis is defined of a function of one real variable. The results will be very surprising. It turns out that if a function is once differentiable, it is automatically arbitrary often differentiable.
We shall define the derivative only for an interior point of the domain A of the function. The set of all interior point of A is an open set. One can prove that an open set is the union of a collection of disjoint regions. Therefore it is enough, to define differentiability of a function, which is defined in een region (open + connected).

Definition 3.5

A function f, defined in a region G, is called differentiable in the point a of G if the differential quotient

$$\frac{f(a+h)-f(a)}{h} \quad (h \neq 0)$$

has a limit for $h \to 0$.

For $|h|$ small enough lies $a+h$ also in G. The above mentioned limit $L(a)$ we indicate with $f'(a)$, and $f'(a)$ is called the derivative of $f(z)$ in the point a. This is completely analogous to the case in the real analysis. (remember, however, that h is complex here!)

Equivalent definitions:

(1) $f(z)$ defined in G has a derivative $L(a)$ in the point $a \in G$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$\left|\frac{f(z)-f(a)}{z-a} - L(a)\right| < \epsilon$$

for every z with $0 < |z-a| < \delta$.

(2) If in the neighbourhood of a holds:

$$f(z) = f(a) + (z-a)L(a) + (z-a)\eta(z,a),$$

with $\eta(z,a) \to 0$ for $z \to a$, then we can $L(a)$ define as the derivative of $f(z)$ in the point a.

Theorem 3.2

If f is differentiable in the point a, then is f continuous in a.

Proof of Theorem 3.2

f is differentiable in a, so f is defined in a full neighbourhood of a.
We can find numbers $L = f'(a)$ and δ_1 such that (take $\epsilon = 1$

$$\left|\frac{f(z)-f(a)}{z-a} - L\right| < 1 \text{ for all } z \text{ with } 0 < |z-a| < \delta_1].$$

So also

$$|f(z) - f(a) - (z-a)L| \leq |z-a| \text{ for } 0 < |z-a| < \delta_1].$$

Under these conditions

$$\begin{aligned} |f(z)-f(a)| &= |f(z)-f(a) - L(z-a) + L(z-a)| \\ &\leq |f(z)-f(a)-L(z-a)| + |L|\,|z-a| \\ &\leq (1+|L|)|z-a| \ \leq \ (1+|L|)\delta_1. \end{aligned}$$

Let $\epsilon > 0$ be given. Choose $\delta = \min(\delta_1, \frac{\epsilon}{(1+|L|)\delta_1})$. Then holds that

$|f(z) - f(a)| < \epsilon$ provided that $|z-a| < \delta$. So f is continuous in a.

□

We have written here the proof in all accuracy. The insight that the statement is correct, we get faster: if $z \to a$ has $\frac{f(z)-f(a)}{z-a}$ a finite limit. Because the denominator approaches 0 the numerator has also to approach 0.

Remark 3.4

Of course, the reversal of the above statement does not hold.

Example 3.2

Re(z) is a continuous function, but it has no point where it is differentiable. The function $f(z) = \overline{z} = z - i\,y$ is not differentiable.

Remark 3.5

The known rules for differtiation of the sum, product and quotient (numerator not zero) of functions just go on.

Theorem 3.3

The chain rule:
If $\phi(z)$ is differentiable in z_0, $f(\phi)$ differentiable in $\phi_0 = \phi(z_0)$, then is $F(z) = f(\phi(z))$ differentiable in z_0 and

$$F'(z_0) = f'(\phi_0)\phi'(z_0).$$

Theorem 3.4

Typically u and v are the real and the imaginary parts of a complex-valued function of a single complex variable $z = x + i\,y$, so $f(z) = u(x, y) + i\,v(x, y)$. Suppose that u and v are real differentiable in an open subet of $\mathbb{C}$, considered as functions of $\mathbb{R}^2$ to $\mathbb{R}$. Then is f complex differentiable in that point if and only if the partial derivatives lf u and v satisfy the Cauchy-Riemann equations:

(a) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$,

(b) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Proof of Theorem 3.4

Not the whole proof is given, but an essential part out of it. Let $h \in \mathbb{R}$, if f is complex differentiable then the limit of the differential quotient will always give the same value:

$$f'(z) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{(x+h+i\,y) - (x+i\,y)} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x},$$
$$= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{(x+i\,(y+h)) - (x+i\,y)} = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

□

Example 3.3

Consider $f(z) = x^2\,y + i\,y^2$, this function is differentiable in $z = 0$, but $f(z)$ is not analytic in $z = 0$.

Definition 3.6

If f is defined in a region G, and f is differentiable in every point of G, then f is called differentiable in G. Then f is called an (one-valued) analytic function, or also holomorphic function, in G. Also the term regular function is often used in stead of holomorphic.

Analytic in a point, on a curve:

Analtyic in an point = differentiable in that point and differentiable in a full neighbourhood of that point.

Analytic on a curve = analytic in every point of that curve.

Remark 3.6

By analytic in a point or on a curve, the point or the curve has to be imbedded in a region and require that the function is differentiable in that region.

The real function defined by

$$f(x) = \begin{cases} 0 & \text{if x} = 0, \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if x} \neq 0. \end{cases}$$

is for all x differentiable, but f' is in $x = 0$ not continuous, therefore not differentiable. Something like that is not possible in the complex function theory. This is shown by the following:

Properties 3.1

Fundamental Property:

If $f(z)$ holomorphic in G, then is also $f'(z)$ holomorf in G.

The proof is postponed. We refer to this result by calling it: property-(**3.1**).
Due to property-(**3.1**): if (z) holomorf in G, then all of its derivatives exist $f'(z)$, $f''(z)$, ..., in G and they are all holomorphic in G.
Although we don't use it yet, it is good to understand that differentiability of a function is such a heavy requirement, that the function has a lot more beautiful properties.

Example 3.4

Examples of analytic functions:

$$\begin{aligned}
f(z) &= \text{constant}, & f'(z) &= 0. \\
f(z) &= z, & f'(z) &= 1. \\
f(z) &= z^n, & f'(z) &= n\, z^{(n-1)} \text{ (n whole number)}. \\
f(z) &= \sum_{n=0}^{m} a_n\, z^n, & f'(z) &= \sum_{n=1}^{m} n\, a_n\, z^{(n-1)} \text{ (polynomial)}.
\end{aligned}$$

A polynomial is an analytic function in $G =$ whole $z-$plane.

3.3 Function defined by power series

Power series are extremely important in the complex analysis.

A power series around the point z_0 is of the form

$$\sum_{n=0}^{\infty} a_n\, (z - z_0)^n.$$

A power series around the origin is

$$\sum_{n=0}^{\infty} a_n\, z^n.$$

Theorem 3.5

Root test:
The power series $\sum_{n=0}^{\infty} a_n\, z^n$ is absolute and uniform convergent if $0 < \rho < R$ and divergent for $|z| > R$. Here is R a number that is determined by the sequence $\{a_n\}$, according to Cauchy-Hadamard:

$$\frac{1}{R} = \limsup_{n\to\infty}(\sqrt[n]{|a_n|}).$$

(Thereby is $R = 0$ if $\limsup_{n\to\infty}(\sqrt[n]{|a_n|}) = \infty$ and
$R = \infty$ if this lim sup is equal to 0.)

Proof of Theorem 3.5

First the rival case that $R = 0$:
For $z = 0$ the power series always converges, with the sum equal to a_0. That is in de case $R = 0$ also the only point of convergence, because under these circumstances the general term of the power series doesn't even go to zero (what is needed for convergfence). After all: $(z \neq 0)$

$$\limsup_{n\to\infty}(\sqrt[n]{|a_n\, z^n|}) \;=\; |z| \,\limsup_{n\to\infty}(\sqrt[n]{|a_n|}) \;=\; \infty.$$

So $a_n\, z^n$ does not approach to zero if $n \to \infty$.
Second case, $R > 0$:
Take $0 < \rho < R$ and $\rho < r < R$. Then is $\frac{1}{R} = \limsup_{n\to\infty}(\sqrt[n]{|a_n|}) < \frac{1}{r}$. So lie to the right of $\frac{1}{r}$ only a finite number of numbers $\sqrt[n]{|a_n|}$. In other words: there is a N such that $\sqrt[n]{|a_n|} < \frac{1}{r}$ for $n > N$. So if $n > N$ and for all z with $|z| < \rho$

$$|a_n\, z^n| < \left(\frac{\rho}{r}\right)^n.$$

Because the term $\left(\frac{\rho}{r}\right)^n$ does not depend on z and $\sum_{n=0}^{\infty}\left(\frac{\rho}{r}\right)^n$ converges is $\sum_{n=0}^{\infty} a_n\, z^n$ absolute and uniform convergent for $|z| \leq \rho$.
If $|z| > R$ then is $\limsup(\sqrt[n]{|a_n\, z^n|}) = \frac{|z|}{R} > 1$, such that the general term of the series goes not to zero if $n \to \infty$. That means that the series diverges.

Herewith the theorem has been fully proven.

□

Remark 3.7

The value R is called the radius of convergence; the points z with $|z| = R$ the circle of convergence.

Example 3.5

$$R = 0: \quad \sum_{n=0}^{\infty} n! z^n.$$

$$R = \infty: \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

$$R = 1: \quad \sum_{n=0}^{\infty} z^n, \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

The behaviour of the power series <u>on</u> the circle of convergence can be anything.

Example 3.6

$$\sum_{n=0}^{\infty} z^n \text{ divergent for } |z| = R = 1.$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ absolute and uniform convergent } |z| = R = 1.$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \text{ convergent for } z = -1, \text{ divergent for } z = 1.$$

Theorem 3.6

The function $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ is inside the circle of convergence $|z| < R$ a holomorphic function of z, and its derivative can be found by termswise differentiation: $f'(z) = \sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$.The last series has the same radius of convergence as the first series.

Proof of Theorem 3.6

$\limsup_{n\to\infty}(\sqrt[n]{n|a_n|}) = \limsup_{n\to\infty}(\sqrt[n]{|a_n|})$, because $\sqrt[n]{n} \to 1$ for $n \to \infty$. The series $\sum_{n=0}^{\infty} n\, a_n\, z^n$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n\, z^n$. So also $\sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$ has this radius of convergence.
Is the just obtained function $g(z) = \sum_{n=0}^{\infty} n\, a_n\, z^{(n-1)}$ really the derivative of $f(z)$? Take z inside the circle of convergence of those two series. Choose $\rho > 0$ such that all points ζ of the circle $|\zeta - z| \leq \rho$ lie inside the circle of convergence. Let $\zeta = z + h$, with $|h| \leq \rho$, and $h \neq 0$, then has the following sense:

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=0}^{\infty} a_n \left[\frac{(z+h)^n - z^n}{h} - n\, z^{(n-1)} \right].$$

Now is

$$\left|\frac{(z+h)^n - z^n}{h} - n\,z^{(n-1)}\right| = |h|\left|\binom{n}{2} z^{(n-2)} + \binom{n}{3} h\,z^{(n-3)} + \ldots + \binom{n}{n} h^{(n-2)}\right|$$
$$\leq |h|\left(\binom{n}{2} |z|^{(n-2)} + \binom{n}{3} \rho\,|z|^{(n-3)} + \ldots + \binom{n}{n} \rho^{(n-2)}\right)$$
$$\leq \frac{|h|}{\rho^2}\left(\binom{n}{2} \rho^2\,|z|^{(n-2)} + \binom{n}{3} \rho^3\,|z|^{(n-3)} + \ldots + \binom{n}{n} \rho^{n}\right)$$
$$\leq \frac{|h|}{\rho^2}\left(|z|^n + \binom{n}{1} \rho\,|z|^{(n-1)} + \binom{n}{2} \rho^2\,|z|^{(n-2)} + \ldots + \binom{n}{n} \rho^{n}\right)$$
$$= \frac{|h|}{\rho^2}(|z| + \rho)^n$$

Therefore

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \leq \frac{|h|}{\rho^2}\sum_{n=0}^{\infty} |a_n|\,(|z| + \rho)^n$$

Since $|z| + \rho < R$ the series at the right site converges and its value is indepedent of h. The right hand side goes zero for $h \to 0$. So the limit of $\frac{f(z+h) - f(z)}{h}$ exist and is equal to $g(z)$. So f is differentiable and its derivative is g. With this, the proof is ready.

□

Remark 3.8

This proof and many that follow all have the following form. From a function f we have the suspicion that $f' = g$. Take a point z. Write then

$$\frac{f(z+h) - f(z)}{h} - g(z) = h\,\phi(z, h).$$

Show that $|\phi(z, h)|$ is bounded for $|h|$ small enough, such that the righthand site $\to 0$ if $|h| \to 0$. Then is, with the definition of the derivative, proved that $f'(z) = g(z)$. Because z was arbitrary chosen the proof is ready.

Remark 3.9

A power series defines a function which is holomorphic in the circular region of convergence. For these special holomorphic functions the existence of all higher order derivative clear. These derivatives are obtainded by termswise differentiation; all these series have the same radius of convergence. For power series is with this the fundamental property proven, see property: (**3.1**) (at page: **26**).
Later on we shall see that a function, which is holomorphic in a neighbourhood of z_0 the sum is of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ which converges in a circle around z_0.

3.4 Exponential Function

Definition 3.7

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The radius of convergence is infinitely large.
So the $\exp(z)$ is holomorphic on the whole $z-$plane.

Theorem 3.7

$$\lim_{n\to\infty}(1+\frac{z}{n})^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{n}\right)^k = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{z^k}{k!} = \exp(z).$$

The usual notation is e^z instead of $\exp(z)$.

Proof of Theorem 3.7

If k fixed and n increasing: $\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{1}{k!}\frac{(n-k+1)\ldots n}{n\ldots n} \uparrow \frac{1}{k!}$. Take $|z| < R$ (arbitrary). Determine K such that $|\sum_{k=K}^{\infty}(\binom{n}{k}\left(\frac{z}{n}\right)^k - \frac{z^k}{k!})| < 2\sum_{k=K}^{\infty}\frac{R^k}{k!} < \epsilon$.
Subsequently $\sum_{k=0}^{K-1}(\binom{n}{k}\left(\frac{z}{n}\right)^k \to \sum_{k=0}^{K-1}\frac{z^k}{k!}$ if $n\to\infty$.

□

Properties 3.2

Properties of $\exp(z)$:

(1) $(\exp(x))' = \exp(x)$.
(2) addition theorem: $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ (follows out of (1)).
(3) $\exp(z)$ has no zeros (follows out of (2)).

Definition 3.8

The trigoniometric functions are defined by

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2n+1)}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2n)}}{(2n)!}$$

They were everywhere analytic. They have the following properties everywhere in the z-plane:

Properties 3.3

$$\sin(z) = \frac{e^{(i\,z)} - e^{(-i\,z)}}{2\,i}, \quad \cos(z) = \frac{e^{(i\,z)} + e^{(-i\,z)}}{2},$$
$$(\sin(z))' = \cos(z), \qquad (\cos(z))' = -\sin(z),$$
$$e^{(i\,z)} = \cos(z) + i\,\sin(z); |e^{i\,z}| = e^{-\mathrm{Im}(z)} \neq 1, \text{unless } z \text{ is real.}$$

This is fully consistent with the elementary theory. We say: the exponential and the trigoniometric functions of a real variable we have analytical continued to the complex plane. (See Chapter (5)).

Relation with the hyperbolic functions.
For all z: $\cos(i\,z) = \cosh(z)$, $\sin(i\,z) = i\,\sinh(z)$, $(\cosh(z))^2 - (\cosh(z))^2 = 1$.
Pay attention: For a lot of values of z holds $|\cos(z)| > 1$ and $|\sin(z)| > 1$!
For instance is $\cos(i) = \dfrac{e^{-1} + e}{2} > 1$.
From

$$\begin{aligned}\sin(z) = \sin(x + i\,y) &= \sin(x)\,\cos(i\,y) + \cos(x)\,\sin(i\,y)\\ &= \sin(x)\,\cosh(y) + i\,\cos(x)\,\sinh(y)\end{aligned}$$

follows that
$|\sin(z)| = ((\sin(x))^2\,(\cosh(y))^2 + (\cos(x))^2\,(\sinh(y))^2)^{\frac{1}{2}} = ((\sin(x))^2 + (\sinh(y))^2)^{\frac{1}{2}}$.

So $|\sin(z)| = 1$ if $\sinh(y) = \cos(x)$ or $\sinh(y) = -\cos(x)$.

This leads to two cures in the $z-$plane

$$y_1 = \log(\sqrt{1 + (\cos(x))^2} + \cos(x)) \text{ and}$$
$$y_2 = \log(\sqrt{1 + (\cos(x))^2} - \cos(x)) = -y_1.$$

There is $|\sin(x)| = 1$; in between $|\sin(x)| < 1$ and everywhere else $|\sin(x)| > 1$.
Analogous for $|\cos(x)|$, curves shifted over $\frac{\pi}{2}$.

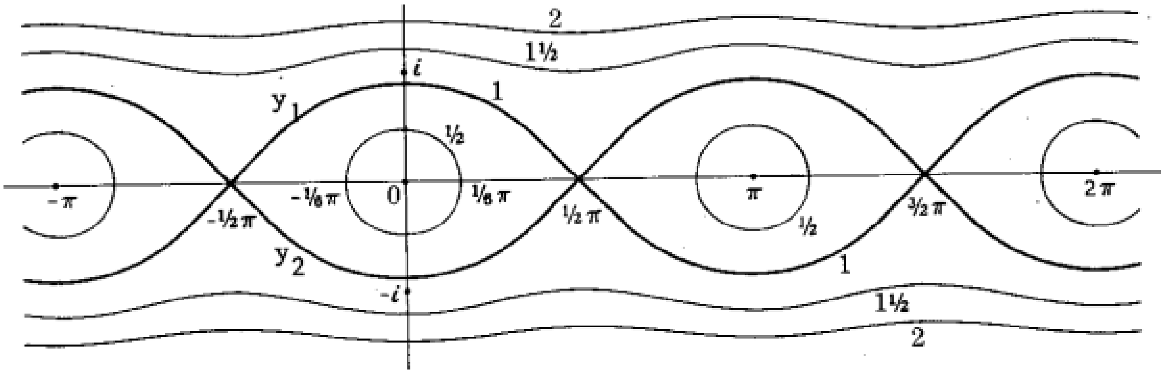


Figure 3.1 $|sin(x)|$

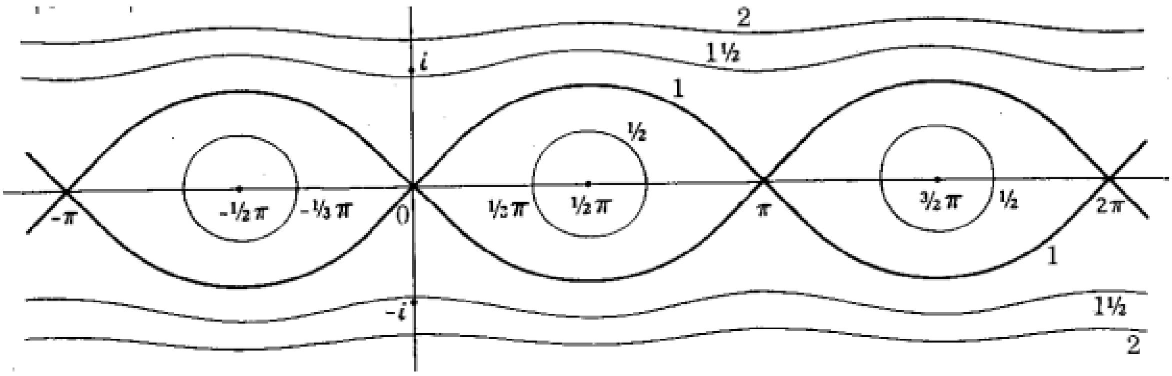


Figure 3.2 $|cos(x)|$

4 Integration Complex Plane

4.1 Complex Integration

We start from the definition of a definite integral of real functions, which we assume known out of other lectures and the following theorem:

Theorem 4.1

If $g(x)$ is continuous for $a \leq x \leq b$ then exists $\int_a^b g(x)\, dx$.

Let K be a smooth arc with parameterization $z = \phi(t)$ $(a \leq t \leq b)$. We remind that in section (2.4) there was required that $\phi'(t)$ is continuous. Let further $f(t)$ be a contnuous function of z at the arc K. Then we define

Definition 4.1

$$\int_K f(z)\, dz = \int_a^b f(\phi(t))\, \phi'(t) dt.$$

(Notice: the integral is a complex function of the real variable t.)
Because of theorem (4.1) exist the integral at the right hand side. (You see that it had reason to demand that arcs are always continuous differentiable.)
We define the integral of $f(z)$ over the arc K as the sum of the integrals over the arcs, which makes up K. The above given definition is independent of the choice of the parametertrization.

Example 4.1

(a) $f(z) = 1$, K given by $\phi(t)$, $a \leq t \leq b$ with beginpoint $A = \phi(a)$, endpoint $B = \phi(b)$.
$\int_K f(z)\,dz = \int_a^b 1\,\phi'(t)\,dt = \phi(b) - \phi(B) = B - A.$

(b) $f(z) = z$, K as in (a).
$\int_K f(z)\,dz = \int_a^b \phi(t)\,\phi'(t)\,dt = \left[\frac{\phi^2(t)}{2}\right]_a^b = \frac{B^2}{2} - \frac{A^2}{2}.$

In the examples 4.1(a) and 4.1(b) the value of the integral is dependent of the begin- and end-point of K but independent of the chosen path from A to B. This is not always the case, see the examples 4.2(a) and 4.2(b) (and this will become one of the main points of these lectures).

Example 4.2

(a) $f(z) = |z|$ is nowhere analytic. Let K be the curve consisting of the line segment $[0,1]$ and the circular arc from 1 tot i. Let K' be the line segment from 0 to i. It is easy to calculate that
$\int_K |z|dz = \frac{1}{2} + (i-1)$ and $\int_{K'} |z|dz = \frac{1}{2}i.$

(b) Choose $f(z) = \frac{1}{z}$ and $A = 1$ and $B = -1$ and for the curves K and K' we take $|z| = 1,\ \mathrm{Im}(z) \geq 0$, respectively $|z| = 1,\ \mathrm{Im}(z) \leq 0$. After some calculation follows that:
$\int_K \frac{1}{z}dz = \pi\,i$ and $\int_K \frac{1}{z}dz = -\pi\,i.$

The next property is of great importance to memorize.

Properties 4.1

Let K be some circle with midpoint z_0 and radius R. For K we can choose the following parametrization $z = z_0 + R\exp(i\,t)$ $(0 \le t \le 2\,\pi)$. Let m be a whole number. Then is

$$\frac{1}{2\pi\,i}\int_K (z - z_0)^m dz = \begin{cases} 0 \text{ if } m \neq (-1) \\ 1 \text{ if } m = -1. \end{cases}$$

Just use the definition (4.1).

Remark 4.1

The value of the integral, given in property (4.1) is independent of the radius R of the circle.

The known theorems about integrals of real functions can be directly transfered to the complex integrals. We call just one example

$$\int_K (\lambda\, f(z) + \mu\, g(z))dz = \lambda \int_K f(z)dz + \mu \int_K g(z)dz$$

It is common in the sequel that rough estimates are required for the values of an integral. Therefore we use the ML-lemma .

Lemma 4.1

If $|f(z)| \le M$ for $z \in K$ and if L is the length of the curve K then holds (assumed that the integral exists)

$$|\int_K f(z)dz| \le M\,L.$$

Proof of Lemma 4.1

□

5 Holomorphic Functions; Series

A short overview will be given of all kind of terms (***)

6 Application of the Residues

A short overview will be given of all kind of terms Use-Residues.

7 Analytic Continuation

A short overview will be given of all kind of terms Analytic-Continuation.

8 Miscellaneous

A short overview will be given of all kind of terms Miscellaneous-v0.

9 List of figures

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11 References