Question 1. Let $f\colon \mathbb{C}\setminus\{z_0\} \to \mathbb{C}$ be a holomorphic function, where z_0 is a pole of order m for f. Show that

$$\mathrm{res}_{z_0}\left(\frac{f'}{f}\right) = -m.$$

Solution. Since f has a pole of order m in z_0, there is an entire function g such that $f(z) = g(z)/(z-z_0)^m$. Hence,

$$f'(z) = -\frac{mg(z)}{(z-z_0)^{m+1}} + \frac{g'(z)}{(z-z_0)^m} = (z-z_0)^{-m}\left(-\frac{mg(z)}{(z-z_0)} + g'(z)\right),$$

and therefore,

$$\frac{f'(z)}{f(z)} = -\frac{m}{(z-z_0)} + \frac{g'(z)}{g(z)}.$$

Notice that $g'(z)/g(z)$ is holomorphic in z_0. Thus the right-hand side is a Laurent series of f'/f around z_0 with principal part $-m/(z-z_0)$. By definition of the residue, we find $\mathrm{res}_{z_0} = -m$.

Question 2. Let $f(z) = \dfrac{1}{\sin^2(z)} - \dfrac{\alpha}{z} - \dfrac{\beta}{z^2}$, $\alpha,\beta \in \mathbb{C}$. Determine the values of α and β such that $z_0 = 0$ is a removable singularity of f. Determine $f(0)$.

Solution. We begin by determining the Laurent series expansion of $1/\sin^2(z)$ at the point $z = 0$. Since

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{1}{3!}z^3 + g(z)$$

$$\sin^2(z) = \left(z - \frac{1}{6}z^3 + g(z)\right)\left(z - \frac{1}{6}z^3 + g(z)\right) = z^2\left(1 - \frac{1}{3}z^2 + h(z)\right),$$

where g and h are holomorphic functions in $z = 0$ with g/z^5, h/z^4 bounded in a ball around $z = 0$, therefore

$$\frac{1}{\sin^2(z)} = \frac{1}{z^2}\frac{1}{1-(\frac{1}{3}z^2 - h(z))} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{3}z^2 - h(z)\right)^n$$

$$= \frac{1}{z^2}\left(1 + \frac{1}{3}z^2 - h(z)\right) = \frac{1}{z^2} + \frac{1}{3} + k(z),$$

where k is a holomorphic function with k/z^2 bounded in a ball around $z = 0$. Choosing $\alpha = 0$ and $\beta = 1$, we obtain

$$f(z) = \frac{1}{\sin^2(z)} - \frac{\alpha}{z} - \frac{\beta}{z^2} = \frac{1}{3} + k(z).$$

Thus, $z = 0$ is a removable singularity of f with $f(0) = 1/3$.

Question 3. Given that $(1+z)^n = \displaystyle\sum_{k=0}^{n}\binom{n}{k}z^k$ for all $z \in \mathbb{C}$, where $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$.

(a) For $k \leq n$, show that $\displaystyle\int_{\partial B_1(0)} \frac{(1+w)^n}{w^{k+1}}\,dw = 2\pi i\binom{n}{k}$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty}\frac{(1+z)^{2n}}{z^{n+1}}\frac{1}{6^n}$ converges absolutely and uniformly for $z \in \partial B_1(0)$, and that

$$\sum_{n=0}^{\infty}\frac{(1+z)^{2n}}{z^{n+1}}\frac{1}{6^n} = -\frac{6}{z^2-4z+1}.$$

(c) Use (a) and (b) to find $\displaystyle\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{6^n}$.

Solution.

(a) Set $f(z) = (1+z)^n$. Then from the generalised Cauchy integral formula, we obtain

$$\frac{1}{2\pi i}\int_{\partial B_1(0)} \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}\frac{d^k}{dz^k}(1+z)^n\Big|_{z=0} = \frac{n(n-1)\cdots(n-k-1)}{k!} = \frac{n!}{(n-k)!k!}.$$

(b) From (a), we see that

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{6^n} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{6^n}\int_{\partial B_1(0)}\frac{(1+z)^{2n}}{z^{n+1}}\,dz.$$

Since $|1+z|^{2n} \le (1+|z|)^{2n} = 2^{2n}$ for $z \in B_1(0)$, we have

$$\left|\binom{2n}{n}\right| \le \frac{1}{2\pi}\int_{B_1(0)}\left|\frac{(1+z)^{2n}}{z^{n+1}}\right| dz \le \frac{1}{2\pi}2\pi\, 2^{2n} = 4^n,$$

and therefore the series $\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{6^n}$ converges absolutely and uniformly.
Furthermore,

$$\sum_{n=0}^{\infty}\frac{(1+z)^{2n}}{z^{n+1}}\frac{1}{6^n} = \sum_{n=0}^{\infty}\left(\frac{(1+z)^2}{(6z)}\right)^n\frac{1}{z} = \frac{1}{1-\frac{(1+z)^2}{(6z)}}\frac{1}{z} = -\frac{6}{z^2-4z+1}.$$

(c) Due to (a) and (b), we can interchange the summation with the integral to obtain

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{6^n} = \sum_{n=0}^{\infty}\frac{1}{6^n}\int_{\partial B_1(0)}\frac{(1+z)^{2n}}{z^{n+1}}\,dz = -6\int_{\partial B_1(0)}\frac{1}{(z-\lambda_1)(z-\lambda_2)}\,dz,$$

where $\lambda_1 = 2+\sqrt{3}$, $\lambda_2 = 2-\sqrt{3}$. Only $\lambda_2 \in B_1(0)$, and therefore,

$$\int_{\partial B_1(0)}\frac{1}{(z-\lambda_1)(z-\lambda_2)}\,dz = 2\pi i\,\mathrm{res}_{\lambda_2}\frac{1}{(z-\lambda_1)(z-\lambda_2)} = 2\pi i\,\frac{1}{\lambda_2-\lambda_1} = -\frac{\pi i}{\sqrt{3}}.$$

Altogether, we obtain

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{6^n} = \frac{1}{2\pi i}(-6)\left(-\frac{\pi i}{\sqrt{3}}\right) = \sqrt{3}.$$

Question 4. Consider the complex function $f(z) = \dfrac{z^3}{(z-1)^2(z+1)^2}$.

(a) Find the Laurent expansion of f around $z=0$ with the convergence ring $\{z \in \mathbb{C} : |z| > 1\}$.
Hint: Begin by writing f as $f(z) = z^3/(z^2-1)^2$.

(b) Determine the integral $\displaystyle\int_{\partial B_2(0)} f(z)\,dz$. **Hint:** Use (a).

Solution.

(a) We write $f(z) = \dfrac{z^3}{(z^2-1)^2} = \dfrac{z}{(1-1/z^2)^2} = \dfrac{g(z)}{(h(z))^2}$ with $g(z) = z$ and $h(z) = 1-1/z^2$.
Since,

$$\frac{1}{(1-\xi)^2} = \frac{d}{d\xi}\frac{1}{1-\xi} = \frac{d}{d\xi}\sum_{n=0}^{\infty}\xi^n = \sum_{n=1}^{\infty}n\xi^{n-1} = \sum_{n=0}^{\infty}(n+1)\xi^n,$$

we have, with $\xi = 1/z^2$, that

$$\frac{1}{(1-1/z^2)^2} = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty}(n+1)\left(\frac{1}{z^{2n}}\right).$$

Hence,

$$f(z) = \frac{z^3}{(z-1)^2(z+1)^2} = z\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{z^{2n}}\right) = \frac{1}{z} + \sum_{n=1}^{\infty}(n+1)\left(\frac{1}{z^{2n+1}}\right) = f^-(z),$$

is the Laurent expansion with positive part $f^+ \equiv 0$.

(b) From (a), we have the representation

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty}(n+1)\left(\frac{1}{z^{2n+1}}\right) \qquad \text{for all } z \in \partial B_2(0).$$

Since

$$\lim_{n\to\infty}\sqrt[n]{\frac{n+1}{2^{2n+1}}} = \lim_{n\to\infty}\sqrt[n]{\frac{n+1}{2\cdot 4^n}} = \frac{1}{4} < 1,$$

we have that the Laurent series is absolutely and uniformly convergent on $\partial B_2(0)$. Therefore,

$$\int_{\partial B_2(0)} f(z)\,dz = \int_{\partial B_2(0)} \frac{1}{z}\,dz + \sum_{n=1}^{\infty}(n+1)\int_{\partial B_2(0)} \frac{1}{z^{2n+1}}\,dz = 2\pi i,$$

where we used the fact that the terms in the sums have primitives, and their integrals vanish.