Question 1. Let $f\colon \mathbb{C}\setminus\{z_0\} \to \mathbb{C}$ be a holomorphic function, where z_0 is a pole of order m for f. Show that

$$\operatorname{res}_{z_0}\left(\frac{f'}{f}\right) = -m.$$

Question 2. Let $f(z) = \dfrac{1}{\sin^2(z)} - \dfrac{\alpha}{z} - \dfrac{\beta}{z^2}$, $\alpha, \beta \in \mathbb{C}$. Determine the values of α and β such that $z_0 = 0$ is a removable singularity of f. Determine $f(0)$.

Question 3. Given that $(1+z)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} z^k$ for all $z \in \mathbb{C}$, where $\dbinom{n}{k} = \dfrac{n!}{(n-k)!k!}$.

(a) For $k \leq n$, show that $\displaystyle\int_{\partial B_1(0)} \frac{(1+w)^n}{w^{k+1}}\, dw = 2\pi i \binom{n}{k}$.

(b) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{z^{n+1}} \frac{1}{6^n}$ converges absolutely and uniformly for $z \in \partial B_1(0)$, and that

$$\sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{z^{n+1}} \frac{1}{6^n} = -\frac{6}{z^2 - 4z + 1}.$$

(c) Use (a) and (b) to find $\displaystyle\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{6^n}$.

Question 4. Consider the complex function $f(z) = \dfrac{z^3}{(z-1)^2(z+1)^2}$.

(a) Find the Laurent expansion of f around $z = 0$ with the convergence ring $\{z \in \mathbb{C} : |z| > 1\}$.
Hint: Begin by writing f as $f(z) = z^3/(z^2-1)^2$.

(b) Determine the integral $\displaystyle\int_{\partial B_2(0)} f(z)\, dz$. **Hint:** Use (a).