Question 1. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 3 in $z=0$, simple poles in $z=i$ and $z=-i$, and holomorphic otherwise.

(ii) $zf(z) \to 1$ as $|z| \to \infty$.

(iii) f is an odd function, i.e., $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

(iv) The Laurent series corresponding to f with convergence ring $K = \{z \in \mathbb{C} \mid |z| > 1\}$ has coefficients (c_n) satisfying $c_{-3} = -1$ and $c_{-5} = 2$.

Solution. From (i) we know that

$$f(z) = \frac{g(z)}{z^3(z-i)(z+i)},$$

for an entire function g. From (ii), we see that $zf(z) = \frac{g(z)}{z^2(z-i)(z+i)} \to 1$ as $|z| \to \infty$. In particular, there is some $R_0 > 0$ for which

$$\left|\frac{g(z)}{z^2(z-i)(z+i)}\right| \leq 2 \qquad \text{for all } |z| \geq R_0,$$

and consequently, $|g(z)| \leq 4|z|^4$ for all $|z| \geq R_1$ for some $R_1 > 0$. Liouville's theorem then says that $g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$ with $a_i \in \mathbb{C}$. From the limit, we further deduce $a_4 = 1$. From (iii), we obtain

$$\frac{g(-z)}{-z^3(z+i)(z-i)} = \frac{g(-z)}{-z^3(-z-i)(-z+i)} = f(-z) = -f(z) = -\frac{g(z)}{z^3(z-i)(z+i)},$$

i.e., $g(-z) = g(z)$. Therefore, $a_1 = 0$, $a_3 = 0$. We are now left to determine a_0 and a_2.

The obtain the appropriate Laurent series, we write

$$\begin{aligned} f(z) &= \frac{g(z)}{z^3(z^2+1)} = \frac{g(z)}{z^5(1+1/z^2)} = \frac{g(z)}{z^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} \\ &= \frac{a_0 + a_2 z^2 + z^4}{z^5}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \cdots\right) \\ &= \frac{a_0}{z^5} + \frac{a_2}{z^3} + \frac{1}{z} - \frac{a_0}{z^7} - \frac{a_2}{z^5} - \frac{1}{z^3} + \frac{1}{z^5}\cdots = \frac{a_0 - a_2 + 1}{z^5} + \frac{a_2 - 1}{z^3} + \cdots \end{aligned}$$

Therefore, $a_2 = 0$ and $a_0 + 1 = 2$, i.e., $a_0 = 1$. To conclude, we obtain

$$f(z) = \frac{1 + z^4}{z^3(z-i)(z+i)}.$$

Question 2. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a simple pole in $z=1$, and a pole of order 2 in $z=0$, and holomorphic otherwise.

(ii) $\mathrm{res}_0(f) = 0$.

(iii) $f(z) \to -2$ as $|z| \to \infty$.

(iv) $f(-1) = 0$.

(v) $\displaystyle\int_{|z|=2} zf(z)\,dz = 0$.

Solution. From (i) we obtain an entire function g such that $f(z) = \dfrac{g(z)}{z^2(z-1)}$.

From (ii) we find that

$$0 = \mathrm{res}_0(f) = \frac{1}{1!}\frac{d}{dz}\frac{g(z)}{z-1}\bigg|_{z=0} = \frac{g'(z)(z-1)-g(z)}{(z-1)^2}\bigg|_{z=0} = -g'(0)-g(0) \implies g'(0) = -g(0).$$

From (iii) we obtain from Liouville's theorem that g is a polynomial of degree at most 3, i.e., $g(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ with $a_3 = -2$. Together with (ii), we get

$$a_1 = g'(0) = -g(0) = -a_0 \implies g(z) = a_0(1-z) + a_2 z^2 - 2z^3.$$

From (iv) we get

$$0 = f(-1) = -\frac{g(-1)}{2} = -\frac{2a_0 + a_2 + 2}{2} \implies 2a_0 + a_2 = -2,$$

while (v) gives

$$\begin{aligned} 0 &= \int_{|z|=2} z f(z)\, dz = \int_{|z|=2} \frac{g(z)}{z(z-1)}\, dz = 2\pi i \left(\mathrm{res}_0 \frac{g(z)}{z(z-1)} + \mathrm{res}_1 \frac{g(z)}{z(z-1)}\right) \\ &= 2\pi i\left(-g(0) + g(1)\right) = 2\pi i\left(-a_0 + a_2 - 2\right) \implies -a_0 + a_2 = 2. \end{aligned}$$

Solving the linear system

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \implies \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

gives $a_0 = -4/3$ and $a_2 = 2/3$. To conclude, we obtain $f(z) = \dfrac{2}{3}\dfrac{-2(1-z) + z^2 - 3z^3}{z^2(1-z)}$.

Question 3. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 3 in $z = 0$ and simple poles in $z = -1$ and $z = 1$ respectively, and holomorphic otherwise.

(ii) $zf(z)$ is bounded for $|z| > 1$.

(iii) f is odd, i.e., $f(-z) = -f(z)$.

(iv) $\mathrm{res}_0(z^2 f(z)) = -1$.

(v) The Laurent series corresponding to f with convergence ring $K = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ has coefficients (c_n) satisfying $c_{-1} = 0$ and $c_1 = -2$.

Solution. From (i) we find some entire function g such that

$$f(z) = \frac{g(z)}{z^3(z-1)(z+1)}.$$

From (ii), we have that

$$|zf(z)| = \left|\frac{g(z)}{z^2(z-1)(z+1)}\right| \le c \qquad \text{for all } |z| > 1.$$

Hence, we some real positive constants R and M such that

$$|g(z)| \le M|z|^4 \qquad \text{for all } |z| \ge R.$$

The generalized Liouville's theorem then deduces that g is a polynomial of at most degree 4, i.e., $g(z) = az^4 + bz^3 + cz^2 + dz + e$, for constants $a, b, c, d \in \mathbb{C}$.

From (iii), we see that g must be even, and thus $b = d = 0$. Furthermore, (iv) gives

$$-1 = \mathrm{res}_0(z^2 f(z)) = \mathrm{res}_0\left(\frac{g(z)}{z(z-1)(z+1)}\right) = \lim_{z\to 0} \frac{g(z)}{(z-1)(z+1)} = -e,$$

i.e., $e = 1$. The Laurent series expansion of f satisfying (v) is given by

$$\begin{aligned} f(z) &= -\frac{g(z)}{z^3}\frac{1}{1-z^2} = -\frac{az^4+cz^2+1}{z^3}\sum_{n=0}^{\infty} z^{2n} \\ &= -\frac{az^4+cz^2+1}{z^3} - \frac{az^4+cz^2+1}{z} - z + O(z^3) \\ &= -z^{-3} - (1+c)z^{-1} - (a+c+1)z + O(z^3). \end{aligned}$$

Consequently, we have that $c = -1$ and $2 = a + c + 1 = a$, i.e., $a = 2$. To conclude, we obtain

$$f(z) = \frac{2z^4 - z^2 + 1}{z^3(z-1)(z+1)}.$$

Question 4. Use the (generalised) Liouville theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 2 in $z = 0$ and simple poles in $z = -i$ and $z = i$ respectively, and holomorphic otherwise.

(ii) $z^2 f(z) \longrightarrow a$ as $|z| \to \infty$ for some real constant $a > 0$.

(iii) $\mathrm{res}_0(f(z)) = -1$.

(iv) f has a zero of order 2 in $z = 1$.

Explain clearly where and to which function you apply the (generalised) Liouville theorem.

Solution. From (i) we have an entire function g such that $f(z) = \frac{g(z)}{z^2(z^2+1)}$.

From (ii), we obtain

$$|g(z)| \leq M|z|^2 \qquad \text{for all } |z| > 1.$$

Hence, the generalised Liouville's theorem states that g is a polynomial of degree at most 2, i.e., $g(z) = az^2 + bz + c$ for constants $b, c \in \mathbb{C}$.

From (iii), we have that

$$-1 = \mathrm{res}_0(f) = \frac{1}{1!}\frac{d}{dz}\left(\frac{az^2+bz+c}{z^2+1}\right)\bigg|_{z=0} = b$$

From (iv): Rewriting $g(z) = az^2 - z + c$, and comparing with $(z-1)^2 = z^2 - 2z + 1$, we deduce $c = a = 1/2$, since a and c must coincide.