On Entire Functions
Complex Analysis (2WA80)

Question 1. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 3 in $z = 0$, simple poles in $z = i$ and $z = -i$, and holomorphic otherwise.

(ii) $zf(z) \to 1$ as $|z| \to \infty$.

(iii) f is an odd function, i.e., $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

(iv) The Laurent series corresponding to f with convergence ring $K = \{z \in \mathbb{C} \mid |z| > 1\}$ has coefficients (c_n) satisfying $c_{-3} = -1$ and $c_{-5} = 2$.

Question 2. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a simple pole in $z = 1$, and a pole of order 2 in $z = 0$, and holomorphic otherwise.

(ii) $\text{res}_0(f) = 0$.

(iii) $f(z) \to -2$ as $|z| \to \infty$.

(iv) $f(-1) = 0$.

(v) $\displaystyle\int_{|z|=2} zf(z)\,dz = 0$.

Question 3. Use Liouville's theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 3 in $z = 0$ and simple poles in $z = -1$ and $z = 1$ respectively, and holomorphic otherwise.

(ii) $zf(z)$ is bounded for $|z| > 1$.

(iii) f is odd, i.e., $f(-z) = -f(z)$.

(iv) $\text{res}_0(z^2 f(z)) = -1$.

(v) The Laurent series corresponding to f with convergence ring $K = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ has coefficients (c_n) satisfying $c_{-1} = 0$ and $c_1 = -2$.

Question 4. Use the (generalised) Liouville theorem to determine the meromorphic function f satisfying

(i) f has a pole of order 2 in $z = 0$ and simple poles in $z = -i$ and $z = i$ respectively, and holomorphic otherwise.

(ii) $z^2 f(z) \longrightarrow a$ as $|z| \to \infty$ for some real constant $a > 0$.

(iii) $\text{res}_0(f(z)) = -1$.

(iv) f has a zero of order 2 in $z = 1$.

Explain clearly where and to which function you apply the (generalised) Liouville theorem.