Question 1. Which of the following statements are true about the function $f(z) = z \cdot \overline{z}$?

(a) f is analytic.

(b) f is nowhere analytic.

(c) f is differentiable in $z = 0$ and thus also in a neighbourhood of $z = 0$.

(d) f is differentiable in $z = 0$ and nowhere else.

(e) f is nowhere differentiable.

Solution. Only (b) and (d) are true. $f(z) = z \cdot \bar{z} = |z|^2$. Therefore, $f(x + iy) = x^2 + y^2$ is real-valued with component functions $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. $F = (u, v)$ is totally differentiable, but satisfies the Cauchy–Riemann equations only at the points $(0, 0)$ and nowhere else.

Question 2. Let $v\colon \mathbb{R}^2 \to \mathbb{R}$, $v(x, y) = e^x(x \sin(y) + y \cos(y))$. Determine a holomorphic function f satisfying $\mathrm{Im}(f)(x + iy) = v(x, y)$ with $f(0) = 0$. Express the function f in terms of $z \in \mathbb{C}$.

Solution. We begin by checking if v is harmonic. Since

$$\partial_x v = e^x(x \sin(y) + y \cos(y) + \sin(y)), \qquad \partial_y v = e^x(x \cos(y) + \cos(y) - y \sin(y))$$
$$\partial_x^2 v = e^x(x \sin(y) + y \cos(y) + 2 \sin(y)), \qquad \partial_y^2 v = e^x(-x \sin(y) - 2 \sin(y) - y \cos(y)),$$

we have that $\partial_x^2 v + \partial_y^2 v = 0$, i.e., v is harmonic. Hence, there is a corresponding holomorphic function f with $\mathrm{Im}(f)(x + iy) = v(x, y)$. From the Cauchy–Riemann equations, we have that

$$\partial_x u = \partial_y v = e^x(x \cos(y) + \cos(y) - y \sin(y)).$$

Consider the function $U(x, y) = e^x(x \cos(y) - y \sin(y)) + \Phi(y)$, for some function Φ, depending only on y. Then $\partial_x U = \partial_y v$. We now solve for

$$e^x(-x \sin(y) - \sin(y) - y \cos(y)) + \Phi'(y) = \partial_y U = -e^x(x \sin(y) + y \cos(y) + \sin(y)),$$

which gives $\Phi'(y) = 0$ and therefore, $\Phi(y)$ is a constant function. Hence $u(x, y) = e^x(x \cos(y) - y \sin(y)) + c$ for some $c \in \mathbb{R}$ is a candidate for the corresponding component function of $f = u + iv$. To determine c, we insert $z = 0$ to obtain $0 = f(0) = c$, i.e., $c = 0$. To conclude, we have

$$\begin{aligned} f(x + iy) &= e^x(x \cos(y) - y \sin(y)) + i(e^x(x \sin(y) + y \cos(y))) \\ &= e^x x(\cos(y) + i \sin(y)) + ie^x y(\cos(y) + i \sin(y)) \\ &= (x + iy)e^{x+iy} = z \exp(z). \end{aligned}$$

Question 3. Find the points $z \in \mathbb{C}$ for which the function $f(z) = z\mathrm{Im}(z) - \mathrm{Re}(z)$ is complex differentiable and determine its derivative $f'(z)$. Where is f holomorphic?

Solution. We write $f(x + iy) = (x + iy)y - x = (y - 1)x + iy^2 =: u(x, y) + iv(x, y)$. Clearly, the real function $F = (u, v)$ is totally differentiable u and v are polynomials in x and y. The Cauchy–Riemann equations read

$$y - 1 = \partial_x u = \partial_y v = 2y, \qquad x\partial_y u = -\partial_x v = 0,$$

and therefore $x = 0$ and $y = -1$, i.e., $z = -i$. The derivative is given by

$$f'(-i) = \partial_x u(0, -1) + i\partial_x v(0, -1) = -2.$$

The function is nowhere holomorphic.

Question 4. Let f be holomorphic on $B_1(0)$. Which of the following statements is/are true:

(a) If $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$, then $f(z) = z^2$ on $B_1(0)$.

(b) If $f\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ for all $n \in \mathbb{N}$, then $f(z) = z^2$ on $B_1(0)$.

Solution.

(a) We set $g(z) := f(z) - z^2$. Then $g(1/n) = 0$ for all $n \in \mathbb{N}$ where the accumulation point of $(1/n)$ is $0 \in B_1(0)$. The identity theorem then gives $g \equiv 0$ on $B_1(0)$, i.e., $f(z) = z^2$. Hence, the statement is true.

(b) The function $f(z) = z^2 + \sin\left(\frac{\pi}{1-z}\right)$ is holomorphic in $B_1(0)$ and satisfies the condition. Hence, the statement is false.

Question 5. Let $f(z) = \mathrm{Re}(z) + 2\mathrm{Im}(z) + i(2\mathrm{Re}(z) - \mathrm{Im}(z))^2$.

(a) Determine the points $z \in \mathbb{C}$ for which f is complex differentiable.

(b) Is f in the points found in (a) holomorphic?

(c) Find entire functions g that satisfy $\mathrm{Re}(g(z)) = \mathrm{Re}(f(z))$ for all $z \in \mathbb{C}$.

Solution.

(a) Let $u(x, y) = x + 2y$ and $v(x, y) = (2x - y)^2$. Both u and v are polynomials in (x, y) and therefore continuously differentiable on $\mathbb{R} \times \mathbb{R}$. In particular, $f = (u, v) \times \mathbb{R}^2 \to \mathbb{R}^2$ is totally differentiable (1p) We now check the Cauchy–Riemann equations: Since $\partial_x u = 1$, $\partial_y u = 2$, $\partial_x v = 4(2x - y)$ and $\partial_y v = -2(2x - y)$, this yields

$$1 = \partial_x u = \partial_y v = -2(2x - y), \qquad 2 = \partial_y u = -\partial_x v = -4(2x - y), \qquad (1p)$$

i.e., the Cauchy–Riemann equations are satisfied in the points z satisfying

$$4\mathrm{Re}(z) - 2\mathrm{Im}(z) = -1. \qquad (1p)$$

Therefore, f is complex differentiable in these points.

(b) Since the set $\{z \in \mathbb{C} : 4\mathrm{Re}(z) - 2\mathrm{Im}(z) = -1\}$ is a line and is closed in $\mathbb{C}$ (1p) f is not holomorphic in any of these points (1p)

(c) Since $\partial_x u = 1$, $\partial_y u = 2$, we choose $V(x, y) = y + \Phi(x)$ (1p) for some function Φ satisfying

$$2 = \partial_y u = -\partial_x V = -\Phi'(x), \qquad (1p)$$

i.e., $\Phi(x) = -2x + c$ for some constant $c \in \mathbb{R}$ (1p) Hence, $V(x, y) = -(2x - y) + c$. Therefore, entire functions g satisfying $\mathrm{Re}(g(z)) = \mathrm{Re}(f(z))$ for all $z \in \mathbb{C}$ are of the form

$$\begin{aligned} g(z) &= \mathrm{Re}(z) + 2\mathrm{Im}(z) - i(2\mathrm{Re}(z) - \mathrm{Im}(z)) + ic \\ &= \mathrm{Re}(z) + i\mathrm{Im}(z) - 2i(\mathrm{Re}(z) + i\mathrm{Im}(z)) + ic = (1 - 2i)z + ic. \qquad (2p) \end{aligned}$$

Question 6. Determine if the following statements are **true** or **false**. In each part, give a brief justification of your answer.

(a) $|\cos(z)| \le 1$ for all z in the upper half plane, i.e., $z \in \{x + iy : x \in \mathbb{R},\ y > 0\}$.

(b) The function $f(x + iy) = (x^2 - y^2 + y) + i(2xy - x)$ is entire.

(c) If f is holomorphic on $B_1(0)$, then $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ cannot be satisfied.

(d) If f is entire, then $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for all $n \in \mathbb{N}$ cannot be satisfied.

Hint: Use the identity theorem for (c) and (d).

Solution.

(a) **False**. Take $x = 0$, $y = 2$, which gives

$$\cos(2i) = \frac{1}{2}(e^{-2} + e^2) = \frac{e^2}{2}(1 + e^{-4}) > \frac{e^2}{2} > 1.$$

(b) **True**. Set $u(x, y) = x^2 - y^2 + y$ and $v(x, y) = 2xy - x$. It is easy to see that $F = (u, v)$ is totally differentiable since u and v are polynomials in x and y. Furthermore,

$$\partial_x u = 2x = \partial_y v, \qquad \partial_y u = -2y + 1 = -(2y - 1) = -\partial_x v,$$

i.e., the Cauchy–Riemann equations are satisfied for $(x, y) \in \mathbb{R}^2$. Hence, f is entire.

(c) **False**. Set $g(z) = f(z) - \frac{z}{1+z}$. g is holomorphic on $B_1(0)$ since f and $\frac{z}{1+z}$ are holomorphic on $B_1(0)$. Furthermore, $g(1/n) = 0$ for all $n \in \mathbb{N}$ and $\frac{1}{n}$ has the accumulation point $0 \in B_1(0)$. Therefore, $g \equiv 0$ and $f(z) = z/(1 + z)$, which is holomorphic on $B_1(0)$, i.e., (c) is false.

(d) **True**. Suppose there is an entire function f satisfying the property above. Then $g(z) = f(z^2) - z$ is entire due to the chain rule, and g satisfies $g(1/n) = 0$ for all $n \in \mathbb{N}$ with accumulation point $0 \in \mathbb{C}$. Hence, $g \equiv 0$ due to the Identity theorem, i.e., $f(z^2) = z$. However, for $z = -1$, $f(1) = -1$ and for $z = 1$, $f(1) = 1$, which contradicts the assumption that f is a well-defined function. Hence, (d) is true.