On Holomorphic Functions
Complex Analysis (2WA80)

Question 1. Which of the following statements are true about the function $f(z) = z \cdot \overline{z}$?

(a) f is analytic.

(b) f is nowhere analytic.

(c) f is differentiable in $z = 0$ and thus also in a neighbourhood of $z = 0$.

(d) f is differentiable in $z = 0$ and nowhere else.

(e) f is nowhere differentiable.

Question 2. Let $v\colon \mathbb{R}^2 \to \mathbb{R}$, $v(x, y) = e^x(x\sin(y) + y\cos(y))$. Determine a holomorphic function f satisfying $\mathrm{Im}(f)(x + iy) = v(x, y)$ with $f(0) = 0$. Express the function f in terms of $z \in \mathbb{C}$.

Question 3. Find the points $z \in \mathbb{C}$ for which the function $f(z) = z\mathrm{Im}(z) - \mathrm{Re}(z)$ is complex differentiable and determine its derivative $f'(z)$. Where is f holomorphic?

Question 4. Let f be holomorphic on $B_1(0)$. Which of the following statements is/are true:

(a) If $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$, then $f(z) = z^2$ on $B_1(0)$.

(b) If $f\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^2$ for all $n \in \mathbb{N}$, then $f(z) = z^2$ on $B_1(0)$.

Question 5. Let $f(z) = \mathrm{Re}(z) + 2\mathrm{Im}(z) + i(2\mathrm{Re}(z) - \mathrm{Im}(z))^2$.

(a) Determine the points $z \in \mathbb{C}$ for which f is complex differentiable.

(b) Is f in the points found in (a) holomorphic?

(c) Find entire functions g that satisfy $\mathrm{Re}(g(z)) = \mathrm{Re}(f(z))$ for all $z \in \mathbb{C}$.

Question 6. Determine if the following statements are **true** or **false**. In each part, give a brief justification of your answer.

(a) $|\cos(z)| \leq 1$ for all z in the upper half plane, i.e., $z \in \{x + iy : x \in \mathbb{R},\ y > 0\}$.

(b) The function $f(x + iy) = (x^2 - y^2 + y) + i(2xy - x)$ is entire.

(c) If f is holomorphic on $B_1(0)$, then $f\left(\frac{1}{n}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ cannot be satisfied.

(d) If f is entire, then $f\left(\frac{1}{n^2}\right) = \frac{1}{n}$ for all $n \in \mathbb{N}$ cannot be satisfied.

Hint: Use the identity theorem for (c) and (d).