Question 1. Let $f(z) = \dfrac{z\cos(z)}{\sin(z)-1}$.

(a) Determine the isolated singularities of f and their type.

(b) Compute the value of the integral $\displaystyle\int_{|z|=\pi} f(z)\,dz$.

Solution.

(a) The isolated singularities are the zeros of $\sin(z)-1$, i.e., $p=\pi/2+2k\pi$, $k\in\mathbb{Z}$. The zeros of $z\cos(z)$ are $q=0$ and $q=\pi/2+k\pi$, $k\in\mathbb{Z}$.

The zeros of $\sin(z)-1$ are of order 2, since

$$(\sin(z)-1)'|_p = \cos(p) = 0, \quad \text{and} \quad (\sin(z)-1)''|_p = -\sin(p) = -1 \neq 0,$$

while the zeros of $z\cos(z)$ are of order 1 since

$$(z\cos(z))'|_q = \cos(q) - q\sin(q) \neq 0.$$

Hence, the isolated singularities of f in p are poles of order 1.

(b) The value of the integral can be determined by the residue theorem. Since $p=\pi/2$ is the only pole of f within the inner region of $\{|z|=\pi\}$, we have that

$$\begin{aligned}
\int_{|z|=\pi} f(z)\,dz &= 2\pi i\,\mathrm{res}_{\frac{\pi}{2}}(f) = 2\pi i \lim_{z\to\frac{\pi}{2}} \frac{(z-\pi/2)\cos(z)}{\sin(z)-1} z \\
&= 2\pi i \left(\lim_{z\to\frac{\pi}{2}} \frac{(z-\pi/2)\cos(z)}{\sin(z)-1}\right)\left(\lim_{z\to\frac{\pi}{2}} z\right) \\
&= \pi^2 i \left(\lim_{z\to\frac{\pi}{2}} \frac{\cos(z)-(z-\pi/2)\sin(z)}{\cos(z)}\right) \\
&= \pi^2 i \left(\lim_{z\to\frac{\pi}{2}} \frac{-\sin(z)-\sin(z)-(z-\pi/2)\cos(z)}{-\sin(z)}\right) = 2\pi^2 i.
\end{aligned}$$

Question 2. Determine the integral $\displaystyle\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx$.

Solution. Set $f(x) = \frac{\sin(x)}{x(x^2+1)}$. We first notice that $f(-x) = f(x)$, i.e., f is even. Therefore,

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{\mathbb{R}} f(x)\,dx.$$

We further observe that $f(z)$ has an isolated singularity at $z=0$ that is removable. To avoid the singularity, we will consider a deformed axis γ that loops over the point $z=0$. More specifically, we consider $\gamma = (-\infty,1]\cup K\cup[1,\infty)$, where $K=\{\frac{1}{2}e^{-i(\theta-\pi)}\in\mathbb{C}\mid\theta\in(0,\pi)\}$. Since f is holomorphic in the strip between$-i$ and i, and $\mathbb{R}$ and γ are homotopic, we have due to homotopic invariance

$$\begin{aligned}
\int_{\mathbb{R}} f(z)\,dz = \int_\gamma f(z)\,dz = \int_\gamma \frac{\sin(z)}{z(z^2+1)}\,dz &= \frac{1}{2i}\int_\gamma \frac{e^{iz}}{z(z^2+1)}\,dz - \frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z(z^2+1)}\,dz \\
&= \frac{1}{2i}\int_\gamma \frac{e^{iz}}{z(z-i)(z+i)}\,dz - \frac{1}{2i}\int_\gamma \frac{e^{-iz}}{z(z-i)(z+i)}\,dz.
\end{aligned}$$

For the first integral, we obtain

$$\int_\gamma \frac{e^{iz}}{z(z-i)(z+i)}\,dz = 2\pi i\,\mathrm{res}_i \frac{e^{iz}}{z(z-i)(z+i)} = 2\pi i \lim_{z\to i}\frac{e^{iz}}{z(z+i)} = -\pi i e^{-1}.$$

As for the second integral, we obtain

$$\int_\gamma \frac{e^{-iz}}{z(z^2+1)}\,dz = -2\pi i\left(\operatorname{res}_0 \frac{e^{-iz}}{z(z-i)(z+i)} + \operatorname{res}_{-i}\frac{e^{-iz}}{z(z-i)(z+i)}\right)$$
$$= -2\pi i\left(\lim_{z\to 0}\frac{e^{-iz}}{(z^2+1)} + \lim_{z\to -i}\frac{e^{-iz}}{z(z-i)}\right) = -2\pi i\left(1-\frac{e^{-1}}{2}\right).$$

Altogether, we obtain

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{\mathbb{R}} f(x)\,dx = \frac{1}{2}\left(\frac{1}{2i}(-\pi i e^{-1}) - \frac{1}{2i}(-2\pi i)\left(1-\frac{e^{-1}}{2}\right)\right) = \frac{\pi}{2}(1-e^{-1}).$$

Question 3. Determine the integral $\displaystyle\int_0^{2\pi}\frac{\sin\theta}{2-\cos\theta}\,d\theta$.

Solution. We simply apply the transformation $\cos(\theta) = (e^{i\theta}+e^{-i\theta})/2$, $\sin(\theta) = (e^{i\theta}-e^{-i\theta})/(2i)$ to obtain

$$\int_0^{2\pi}\frac{\sin\theta}{2-\cos\theta}\,d\theta = \int_{\partial B_1(0)}\frac{1}{i}\frac{z-z^{-1}}{4-(z+z^{-1})}\frac{dz}{iz} = \int_{\partial B_1(0)}\frac{z^2-1}{z(z^2-4z+1)}\,dz$$
$$= \int_{\partial B_1(0)}\frac{z^2-1}{z(z-\lambda_1)(z-\lambda_2)}\,dz =: \int_{\partial B_1(0)} f(z)\,dz,$$

where $\lambda_1 = 2+\sqrt{3}\notin B_1(0)$, $\lambda_2 = 2-\sqrt{3}\in B_1(0)$. Using the residue theorem, we get

$$\int_{\partial B_1(0)} f(z)\,dz = 2\pi i\,(\operatorname{res}_0(f)+\operatorname{res}_{\lambda_2}(f)) = 2\pi i\left(\lim_{z\to 0}\frac{z^2-1}{z^2-4z+1} + \lim_{z\to\lambda_2}\frac{z^2-1}{z(z-\lambda_1)}\right)$$
$$= 2\pi i\left(-1+\frac{\lambda_2^2-1}{\lambda_2(\lambda_2-\lambda_1)}\right) = 2\pi i\left(-1-\frac{2\sqrt{3}(\sqrt{3}-2)}{2\sqrt{3}(2-\sqrt{3})}\right) = 0.$$

Question 4. (a) Show that the function $f(z) = \dfrac{\sin(z)-z\cos(z)}{z^3}$ is entire.

(b) Use (a) to determine the integral $\displaystyle\int_{\mathbb{R}}\frac{\sin(x)-x\cos(x)}{x^3}\,dx$.

Solution.

(a) From the definition of $\sin(z)$ and $\cos(z)$, we find

$$\sin(z)-z\cos(z) = \sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)!}z^{2n+1} - \sum_{n=0}^\infty\frac{(-1)^n}{(2n)!}z^{2n+1} = \sum_{n=0}^\infty\left(\frac{(-1)^n}{(2n+1)!}-\frac{(-1)^n}{(2n)!}\right)z^{2n+1}$$
$$= \left(\frac{(-1)}{6}-\frac{(-1)}{2}\right)z^3 + \sum_{n=2}^\infty\left(\frac{(-1)^n}{(2n+1)!}-\frac{(-1)^n}{(2n)!}\right)z^{2n+1}$$
$$= \frac{1}{3}z^3 + \sum_{n=2}^\infty\left(\frac{(-1)^n}{(2n+1)!}-\frac{(-1)^n}{(2n)!}\right)z^{2n+1}.$$

Hence, $f(z) = \dfrac{1}{3}+\displaystyle\sum_{n=2}^\infty\left(\frac{(-1)^n}{(2n+1)!}-\frac{(-1)^n}{(2n)!}\right)z^{2(n-1)}$ is a power series with convergence radius ∞, and is therefore entire.

(b) Since f is entire, we can choose a deformed axis γ that loops over the top of $z = 0$ to avoid the point $z = 0$, without changing the integral. Then we use the formulas $\cos(z) = (e^{iz} + e^{-iz})/2$ and $\sin(z) = (e^{iz} - e^{-iz})/(2i)$ to obtain

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,dx &= \int_\gamma f(z)\,dz = \frac{1}{2}\int_\gamma \frac{e^{iz} - e^{-iz}}{iz^3} - \frac{e^{iz} + e^{-iz}}{z^2}\,dz \\ &= \frac{1}{2}\int_\gamma \left(\frac{1}{iz^3} - \frac{1}{z^2}\right) e^{iz}\,dz - \frac{1}{2}\int_\gamma \left(\frac{1}{iz^3} + \frac{1}{z^2}\right) e^{-iz}\,dz \\ &= -\frac{1}{2}\int_\gamma \frac{z+i}{z^3} e^{iz}\,dz + \frac{1}{2}\int_\gamma \frac{i-z}{z^3} e^{-iz}\,dz.\end{aligned}$$

Notice that $\frac{z+i}{z^3}e^{iz}$ is holomorphic on the upper side of γ with $\deg(\frac{z+i}{z^3}) \le -2$, and therefore the first integral vanishes. Since $\deg(\frac{i-z}{z^3}) \le -2$ and the only isolated singularity on the lower side of γ is $z = 0$, which is a pole of order 3, we obtain from a theorem from the lecture:

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,dx &= \frac{1}{2}\int_\gamma \frac{i-z}{z^3} e^{-iz}\,dz = -\pi i \operatorname{res}_0 \frac{(i-z)e^{-iz}}{z^3} \\ &= -\pi i \frac{1}{2!}\frac{d^2}{dz^2}(i-z)e^{-iz}\bigg|_{z=0} = -\frac{\pi i}{2}(i+z)e^{-iz}|_{z=0} = \frac{\pi}{2}.\end{aligned}$$

Question 5. Evaluate the integral $\displaystyle\int_0^\infty \frac{1-\cos(x)}{x^2(x^2+1)}\,dx$.

Solution. We begin by noticing that the integrand is even. Hence,

$$\int_0^\infty \frac{1-\cos(x)}{x^2(x^2+1)}\,dx = \frac{1}{2}\int_{\mathbb{R}} \frac{1-\cos(x)}{x^2(x^2+1)}\,dx.$$

Now set $f(z) = (1-\cos(z))/(z^2(z^2+1))$. Then

$$\frac{1-\cos(z)}{z^2} = \frac{1}{z}\left(1 - \sum_{n=0} \frac{(-1)^n}{(2n)!} z^{2n}\right) = \frac{1}{z}\left(1 - (1 - z^2/2 + O(z^4))\right) = \frac{1}{2} + O(z^2),$$

and f is therefore meromorphic with simple poles $z = \pm i$. Choosing the deformed axis $\gamma = (-\infty, 1/2] \cup K \cup [1/2, \infty)$ that avoids $z = 0$ from above, where K is the semi-circle with radius $1/2$ and centred in $z = 0$, and noticing that

$$f(z) = \frac{1}{z^2(z^2+1)}\left(1 - \frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{z^2(z^2+1)} - \frac{1}{2}\frac{e^{iz}}{z^2(z^2+1)} - \frac{1}{2}\frac{e^{-iz}}{z^2(z^2+1)},$$

we have that

$$\int_{\mathbb{R}} f(z)\,dz = \int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z^2(z^2+1)}\,dz - \frac{1}{2}\int_\gamma \frac{e^{iz}}{z^2(z^2+1)}\,dz - \frac{1}{2}\int_\gamma \frac{e^{-iz}}{z^2(z^2+1)}\,dz$$

Since the rational function $1/(z^2(z^2+1))$ has degree ≤ -2, we can apply the theorems from the lecture to deduce

$$\begin{aligned}\int_\gamma \frac{1}{z^2(z^2+1)}\,dz &= 2\pi i \operatorname{res}_i\left(\frac{1}{z^2(z-i)(z+i)}\right) = 2\pi i \frac{1}{i^2(i+i)} = -\pi \\ \int_\gamma \frac{e^{iz}}{z^2(z^2+1)}\,dz &= 2\pi i \operatorname{res}_i\left(\frac{e^{iz}}{z^2(z-i)(z+i)}\right) = 2\pi i \frac{1}{i^2(i+i)} = -\pi e^{-1} \\ \int_\gamma \frac{e^{-iz}}{z^2(z^2+1)}\,dz &= -2\pi i \left[\operatorname{res}_0\left(\frac{e^{-iz}}{z^2(z^2+1)}\right) + \operatorname{res}_{-i}\left(\frac{e^{-iz}}{z^2(z-i)(z+i)}\right)\right] \\ &= -2\pi i \left[\frac{d}{dz}\frac{e^{-iz}}{(z^2+1)}\bigg|_{z=0} + \frac{e^{-1}}{2i}\right] = -2\pi i\left[-i + \frac{e^{-1}}{2i}\right] = -\pi(2 + e^{-1}).\end{aligned}$$

Altogether, we obtain

$$\int_0^\infty \frac{1-\cos(x)}{x^2(x^2+1)}\,dx = \frac{1}{2}\int_{\mathbb{R}} f(z)\,dz = \frac{1}{2}\left(-\pi + \frac{\pi e^{-1}}{2} + \frac{\pi(2+e^{-1})}{2}\right) = \frac{1}{2}\pi e^{-1}.$$

Question 6. (a) Show that the function $f(z) = \dfrac{1-\cos(z) - \frac{z}{2}\sin(z)}{z^4}$ is entire.

(b) Use (a) to determine the integral $\displaystyle\int_0^\infty \frac{1-\cos(x) - \frac{x}{2}\sin(x)}{x^4}\,dx$.

Solution.

(a) Using the series definition of cos and sin, we have that

$$\begin{aligned}
1-\cos(z) - \frac{z}{2}\sin(z) &= 1 - \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}z^{2n} - \frac{z}{2}\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}z^{2n+1} \\
&= 1 - \left[1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \cdots\right] - \frac{z}{2}\left[z - \frac{1}{3!}z^3 + \cdots\right] \\
&= \frac{1}{4!}z^4 - 2\frac{1}{4!}z^4 - \sum_{n=3}^\infty \frac{(-1)^n}{(2n)!}z^{2n} - \sum_{n=2}^\infty \frac{(-1)^n}{2(2(n+1)-1)!}z^{2(n+1)} \\
&= \frac{1}{4!}z^4 - \sum_{n=3}^\infty (-1)^n\left(\frac{1}{(2n)!} + \frac{1}{2(2n-1)!}\right)z^{2n}.
\end{aligned}$$

Hence,

$$f(z) = \frac{1}{4!} - \sum_{n=3}^\infty \frac{(-1)^n}{2(2n-1)!}\left(1+\frac{1}{n}\right)z^{2n-4} = \frac{1}{4!} - \sum_{n=1}^\infty \frac{(-1)^n}{2(2n+3)!}\left(1+\frac{1}{n+2}\right)z^{2n},$$

which is a power series with the convergence radius of $R=\infty$, and is therefore entire.

(b) We begin by noticing that f is even. Therefore,

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{\mathbb{R}} f(x)\,dx.$$

Since f is entire, we can choose a line γ that goes over $x=0$, without changing the integral. In this case, we have

$$\int_{\mathbb{R}} f(x)\,dx = \int_\gamma f(z)\,dz = \int_\gamma \frac{1}{z^4}\,dz - \frac{1}{2}\int_\gamma \frac{e^{iz}+e^{-iz}}{z^4}\,dz - \frac{1}{4i}\int_\gamma \frac{e^{iz}-e^{-iz}}{z^3}\,dz.$$

Since $1/z^4$ and $1/z^3$ are holomorphic on the upper side of γ with degree ≤ -2, we have from the lecture that

$$\int_\gamma \frac{1}{z^4}\,dz = 0, \qquad \int_\gamma \frac{e^{iz}}{z^4}\,dz = 0, \qquad \int_\gamma \frac{e^{iz}}{z^3}\,dz = 0.$$

As for the other terms, we obtain

$$\begin{aligned}
\int_\gamma \frac{e^{-iz}}{z^4}\,dz &= -2\pi i\,\mathrm{res}_0 \frac{e^{-iz}}{z^4} = -2\pi i \frac{1}{3!}\frac{d^3}{dz^3}e^{-iz}|_{z=0} = -\frac{2\pi i}{3!}(-i)^3 = \frac{2\pi}{3!}, \\
\int_\gamma \frac{e^{-iz}}{z^3}\,dz &= -2\pi i\,\mathrm{res}_0 \frac{e^{-iz}}{z^3} = -2\pi i \frac{1}{2!}\frac{d^2}{dz^2}e^{-iz}|_{z=0} = -\frac{2\pi i}{2!}(-i)^2 = \frac{2\pi i}{2!}.
\end{aligned}$$

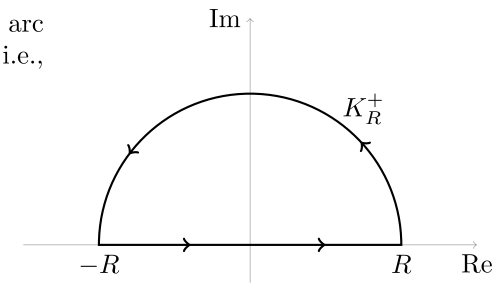
Altogether, we have

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{\mathbb{R}} f(x)\,dx = -\frac{1}{2}\frac{2\pi}{3!} + \frac{1}{4i}\frac{2\pi i}{2!} = \frac{\pi}{12}.$$

Question 7. Let f be holomorphic on the upper half plane, and on the real axis. Suppose that there exist real positive constants M, R_0 and α such that $|f(w)| \leq M|w|^{-\alpha}$ for all $|z| \geq R_0$.

(a) For a positive real $R > 0$, consider the Jordan curve $\gamma_R^+ = [-R, R] \cup K_R^+$, where K_R^+ is the semi-circular arc in the upper half plane with centre 0 and radius R, i.e.,

$$K_R^+ = \{z \in \mathbb{C} : |z| = R, \operatorname{Im}(z) \geq 0\}.$$



Show that

$$\lim_{R\to\infty} \int_{K_R^+} \frac{f(w)}{w-z} dw \longrightarrow 0,$$

for any point z in the upper half plane.

(b) Use (a) to conclude that

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z}\, dt,$$

for any z in the upper half plane.

Solution.

(a) Let z be any point in the upper half plane. Choosing $R \geq R_0$ large such that $|z| \leq R/2$, we have

$$R = |w| = |w - z + z| \leq |z - w| + |z| \leq |z - w| + R/2,$$

for all $w \in K_R^+$, and therefore, $|z - w| \geq R/2$. Hence,

$$\left| \int_{K_R^+} \frac{f(w)}{w-z}\, dw \right| \leq \int_{K_R^+} \frac{|f(w)|}{|w-z|}\, dw \leq \frac{2M}{R} \int_{K_R^+} |w|^{-\alpha}\, dw = \frac{2M}{R} \frac{\pi R}{R^\alpha} \to 0$$

as $R \to \infty$, thereby proving the statement.

(b) Since $\gamma_R^+ = [-R, R] \cup K_R^+$, we can write

$$\int_{[-R,R]} \frac{f(t)}{t-z}\, dt = \int_{\gamma_R^+} \frac{f(w)}{w-z}\, dw - \int_{K_R^+} \frac{f(w)}{w-z}\, dw.$$

For R large such that $|z| \leq R$, we have that $z \in \gamma_R^+$. On the other hand, we know that f is analytic on the upper half plane. Therefore,

$$\int_{\gamma_R^+} \frac{f(w)}{w-z}\, dw = 2\pi i \operatorname{res}_z \frac{f(w)}{w-z} = 2\pi i \lim_{w\to z} f(w) = 2\pi i f(z),$$

due to the residue theorem. Passing to the limit $R \to \infty$ in the equality above gives

$$\int_{\mathbb{R}} \frac{f(t)}{t-z}\, dt = \lim_{R\to\infty} \int_{[-R,R]} \frac{f(t)}{t-z}\, dt = 2\pi i f(z) - \lim_{R\to\infty} \int_{K_R^+} \frac{f(w)}{w-z}\, dw = 2\pi i f(z),$$

which concludes the proof.