Question 1. Let $f(z) = \dfrac{z\cos(z)}{\sin(z) - 1}$.

(a) Determine the isolated singularities of f and their type.

(b) Compute the value of the integral $\displaystyle\int_{|z|=\pi} f(z)\,dz$.

Question 2. Determine the integral $\displaystyle\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx$.

Question 3. Determine the integral $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{2-\cos\theta}\,d\theta$.

Question 4. (a) Show that the function $f(z) = \dfrac{\sin(z) - z\cos(z)}{z^3}$ is entire.

(b) Use (a) to determine the integral $\displaystyle\int_{\mathbb{R}} \frac{\sin(x) - x\cos(x)}{x^3}\,dx$.

Question 5. Evaluate the integral $\displaystyle\int_0^\infty \frac{1-\cos(x)}{x^2(x^2+1)}\,dx$.

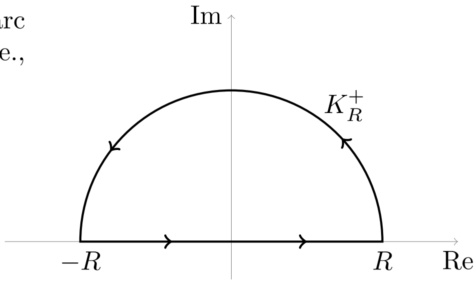
Question 6. (a) Show that the function $f(z) = \dfrac{1-\cos(z) - \frac{z}{2}\sin(z)}{z^4}$ is entire.

(b) Use (a) to determine the integral $\displaystyle\int_0^\infty \frac{1-\cos(x) - \frac{x}{2}\sin(x)}{x^4}\,dx$.

Question 7. Let f be holomorphic on the upper half plane, and on the real axis. Suppose that there exist real positive constants M, R_0 and α such that $|f(w)| \leq M|w|^{-\alpha}$ for all $|z| \geq R_0$.

(a) For a positive real $R > 0$, consider the Jordan curve $\gamma_R^+ = [-R, R] \cup K_R^+$, where K_R^+ is the semi-circular arc in the upper half plane with centre 0 and radius R, i.e.,

$$K_R^+ = \{z \in \mathbb{C} : |z| = R,\ \mathrm{Im}(z) \geq 0\}.$$



Show that

$$\lim_{R\to\infty} \int_{K_R^+} \frac{f(w)}{w-z}dw \longrightarrow 0,$$

for any point z in the upper half plane.

(b) Use (a) to conclude that

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt,$$

for any z in the upper half plane.