The first point is to describe vector spaces with topologies arising from \textit{(separating) families of semi-norms.} These all turn out to be locally convex, for straightforward reasons.

The second point is to check that any locally convex topological vectorspace’s topology can be given by a collection of seminorms. These seminorms are made in a natural way from a local basis consisting of balanced convex neighborhoods of 0 in the vectorspace.

We review the notion of \textit{completion}, in which the only novelty is that we might want to allow merely \textit{pseudo-metrics} rather than \textit{metrics}, allowing completion with respect to \textit{semi-norms} that may not be \textit{norms}.

Then we prove that a \textit{complete} topological vectorspace with topology defined by a collection of seminorms is a \textit{projective limit of Banach spaces}. We will assume that the reader is acquainted with the notions of \textit{products}, \textit{coproducts}, \textit{(projective) limits}, and \textit{colimits (inductive limits)} in a general but elementary category-theory framework.

The Banach spaces obtained from vectorspaces with seminorms by \textit{completion} are the building blocks from which many important topological vectorspaces are constructed, via the categorical constructs just mentioned, and by \textit{duality}.

\textit{(Nevertheless, completeness} is too strong a condition for non-metrizable topological vectorspaces. \textit{Quasi-completeness} is the appropriate general condition.\textit{)}

As a special case, dual spaces with weak star topologies are \textit{projective limits of finite dimensional spaces.} Using an explicit construction of projective limits, we can give a complete characterization of pre-compact subsets of weak-star dual spaces, subsuming the usual version of the Banach-Alaoglu theorem.

\begin{itemize}
  \item Families of seminorms
  \item Locally convex spaces and Minkowski functionals
  \item Completeness: a reminder
  \item Completions of spaces with seminorms
\end{itemize}
1. Families of seminorms

Here we’ll define seminorms, describe topological vectorspaces whose topologies are given via seminorms, and note that such things are invariably locally convex, i.e., have a local basis at 0 consisting of convex sets.

Let \( V \) be a vectorspace over scalars \( k \). A seminorm \( \nu \) on \( V \) is a real-valued function on \( V \) so that

\[
\begin{align*}
\nu(x) &\geq 0 \quad \text{for all } x \in V \quad \text{(non-negativity)} \\
\nu(\alpha x) &= |\alpha|\nu(x) \quad \text{for all } \alpha \in k, \ x \in V \quad \text{(homogeneity)} \\
\nu(x + y) &\leq \nu(x) + \nu(y) \quad \text{for all } x, y \in V \quad \text{(triangle inequality)}
\end{align*}
\]

Note that it may happen that \( \nu(x) = 0 \) yet \( x \neq 0 \). If we require, further, that \( \nu(x) = 0 \) implies \( x = 0 \), then the seminorm is a norm.

Recall that a pseudo-metric on a set \( X \) is a real-valued function \( d \) on \( X \times X \) so that

\[
\begin{align*}
d(x, y) &\geq 0 \quad \text{(non-negativity)} \\
d(x, y) &= d(y, x) \quad \text{(symmetry)} \\
d(x, x) &\leq d(x, y) + d(x, z) \quad \text{(triangle inequality)}
\end{align*}
\]

Note that it may happen that \( d(x, y) = 0 \) yet \( x \neq y \). If require, further, that \( d(x, y) = 0 \) implies \( x = y \), then the pseudo-metric is a metric.

The associated pseudo-metric attached to the seminorm \( \nu \) is

\[ d(x, y) = \nu(x - y) = \nu(y - x) \]

This is a metric if and only if the seminorm is actually a norm.

Let \( \{\nu_i : i \in I\} \) be a collection of semi-norms on a vectorspace \( V \), where \( I \) is an index set. This family is a separating family of seminorms if \( 0 \neq x \in V \) implies that there is some \( \nu_i \) so that \( \nu_i(x) \neq 0 \).

Let \( \{\nu_i : i \in I\} \) be a separating family of seminorms on a vectorspace \( V \). For \( \varepsilon > 0 \) and \( i \in I \) let

\[ U_{i, \varepsilon} = \{x \in V : \nu_i(x) < \varepsilon\} \]

Let \( \mathcal{B}_0 \) be the collection of all finite intersections of such sets. The assumption that the family is separating implies that

\[ \cap_{i, \varepsilon} U_{i, \varepsilon} = \{0\} \]

Define a topology on \( V \) by saying that a set \( U \) is open if and only if for every \( x \in U \) there is some \( N \in \mathcal{B} \) so that

\[ x + N \subset U \]

This is the induced topology associated to the family of seminorms.

**Proposition:** The topology induced by a separating family of seminorms on a vectorspace \( V \) really is a topology, and, further, gives \( V \) the structure of topological vectorspace. The sets \( U_{i, \varepsilon} \) are a local basis at 0 for the topology. Such topologies are locally convex.

**Proof:** There is nothing surprising in this proof, which can be done as an exercise. This exercise is carried out as follows:

First, we can check that we have a topology. This itself does not make use of the hypothesis that the family of seminorms is separating. Certainly arbitrary unions of sets containing ‘neighborhoods’ of the form \( x + N \) around each point \( x \) have the same property. And the empty set and the whole space \( V \) are visibly ‘open’.
The least trivial issue is to check that finite intersections of ‘opens’ are ‘open’. Looking at each point \( x \in \) in a given finite intersection, this amounts to checking that finite intersections of sets in \( \mathcal{B}_0 \) are again in \( \mathcal{B}_0 \). But \( \mathcal{B}_0 \) is defined to be the collection of all finite intersections of sets \( U_{i,\varepsilon} \), so this works: we have closure under finite intersections, and we really have a topology on \( V \).

To verify that this topology makes \( V \) into a topological vector space, we must verify the continuity of vector addition, joint continuity of scalar multiplication, and closed-ness of points. (We know from more elementary discussions of topological vector spaces that these properties imply Hausdorff-ness). None of these issues are terribly difficult, but nevertheless we will carry out the verifications.

It is here that we use the separating property: it implies that the intersection of all the sets \( x + N \) with \( N \in \mathcal{B}_0 \) is just \( x \). Thus, given a point \( y \in V \), for each \( x \neq y \) let \( U_x \) be an open set containing \( x \) but not \( y \). Then

\[
U = \bigcup_{x \neq y} U_x
\]

is open and has complement \( \{ y \} \), so the singleton set \( \{ y \} \) is indeed closed.

To prove continuity of vector addition, it suffices to prove that, given \( N \in \mathcal{B}_0 \) and given \( x, y \in V \) there are \( U, U' \in \mathcal{B}_0 \) so that

\[
(x + U) + (y + U') \subset x + y + N
\]

Now the triangle inequality for semi-norms implies that (in the notation above) for a fixed index \( i \) and for \( \varepsilon_1, \varepsilon_2 > 0 \)

\[
U_{i,\varepsilon_1} + U_{i,\varepsilon_2} \subset U_{i,\varepsilon_1 + \varepsilon_2}
\]

Then

\[
(x + U_{i,\varepsilon_1}) + (y + U_{i,\varepsilon_2}) \subset (x + y) + U_{i,\varepsilon_1 + \varepsilon_2}
\]

Thus, given

\[
N = U_{i_1,\varepsilon_1} \cap \ldots \cap U_{i_n,\varepsilon_n}
\]

take

\[
U = U' = U_{i_1,\varepsilon_1/2} \cap \ldots \cap U_{i_n,\varepsilon_n/2}
\]

This proves continuity of vector addition.

To prove continuity of scalar multiplication, we want to prove that for given \( \alpha \in k, x \in V \), and \( N \in \mathcal{B}_0 \) there are \( \delta > 0 \) and \( U \in \mathcal{B}_0 \) so that

\[
(\alpha + B_{\delta}) \times (x + U) \subset \alpha x + N
\]

where

\[
B_{\varepsilon} = \{ \beta \in k : |\alpha - \beta| < \varepsilon \}
\]

Since \( N \) is an intersection of the special sub-basis sets \( U_{i,\varepsilon} \), it suffices to consider just the case that \( N \) is such a set. Given \( \alpha \) and \( x \), take \( \delta > 0 \) small enough so that

\[
\delta(|\delta + \nu_i(x) + |\alpha |) < \varepsilon
\]

Then for \( |\alpha' - \alpha| < \delta \), for \( U = U_{i,\delta} \), and for \( x' - x \in U \), we have

\[
\nu_i(\alpha x - \alpha' x') = \nu_i((\alpha - \alpha') x + (\alpha' (x - x'))) \leq \nu_i((\alpha - \alpha') x) + \nu_i(\alpha' (x - x'))
\]

\[
= |\alpha - \alpha'| \nu_i(x) + |\alpha'| \nu_i(x - x') \leq |\alpha - \alpha'| \nu_i(x) + (|\alpha| + \delta) \nu_i(x - x')
\]

\[
\leq \delta(\nu_i(x) + |\alpha| + \delta)
\]

Here we use the homogeneity of the seminorm \( \nu_i \).

Taking finite intersections then presents no further difficulty, since we can take the corresponding finite intersections of the sets \( B_{\delta} \) and \( U = U_{i,\delta} \). This finishes the proof that separating families of seminorms give a vector space a structure of topological vector space.
Last, we must check that finite intersections of the sets $U_{i,\varepsilon}$ are convex. But since intersections of convex sets are surely convex, it suffices to check that the sets $U_{i,\varepsilon}$ themselves are convex. This will follow easily from the homogeneity and the triangle inequality: let $0 \leq t \leq 1$ and take $x, y \in U_{i,\varepsilon}$. Then

$$\nu_i(tx + (1-t)y) \leq \nu_i(tx) + \nu_i((1-t)y) = t\nu_i(x) + (1-t)\nu_i(y) \leq t\varepsilon + (1-t)\varepsilon = \varepsilon$$

Thus, the set $U_{i,\varepsilon}$ is convex. Done.

**2. Locally convex spaces and Minkowski functionals**

Let $U$ be a convex open set containing 0 in a topological vectorspace $V$. Further, suppose that $U$ is balanced; that is, suppose that for scalar $\alpha$ with $|\alpha| = 1$ and for $v \in U$ we have $\alpha v \in U$. Then define the Minkowski functional or gauge $p_U$ associated to $U$ by

$$p_U(v) = \inf \{ t \geq 0 : v \in tU \}$$

**Proposition:** The Minkowski functional $p_U$ associated to a balanced convex open neighborhood $U$ of 0 in a topological vectorspace $V$ is a seminorm on $V$, and is continuous in the topology on $V$.

**Proof:** This is another quasi-exercise, which we execute:

First, recall that in any topological vectorspace every neighborhood $U$ of 0 is absorbing, in that every $v \in V$ lies inside some $tU$ for $t > 0$. Thus, the set over which we take infimum to define the Minkowski functional is non-empty, so the infimum exists. Thus, the thing is defined in the first place.

Let $\alpha$ be a scalar, and let $\alpha = s\mu$ with $s = |\alpha|$ and $|\mu| = 1$. The balanced-ness of $U$ implies the balanced-ness of $tU$ for any $t \geq 0$, so if $v \in tU$ then also

$$\alpha v \in tU = s\mu tU = stU$$

From this we have

$$\{ t \geq 0 : \alpha v \in tU \} = |\alpha|\{ t \geq 0 : \alpha v \in tU \}$$

from which follows the homogeneity property required of a seminorm:

$$p_U(\alpha v) = p_U(v)$$

Next, to prove the triangle inequality we use the convexity.

Let $V$ be a locally convex topological vectorspace. That is, we assume that there is a local basis at 0 consisting of convex open sets.

Quite generally, with or without an assumption of local convexity, every neighborhood of 0 contains a balanced neighborhood of 0. Thus, in a locally convex topological vectorspace there is a local basis $U$ at 0 consisting of balanced convex open sets. That is, these are open sets $U$ which are convex, and so that for scalar $\alpha$ with $|\alpha| \leq 1$ and $v \in U$ we have $\alpha v \in U$.

For

**3. Completeness: a reminder**

We need to recall the general notion of completeness for topological vectorspaces, which does not presume that there is a metric. And then there is the generalization to this setting of the notion of completion of a topological vectorspace.
Let $S$ be a poset, that is, a set with a partial ordering $\geq$. We assume further that, given two elements $s, t \in S$, there is $z \in S$ so that $z \geq s$ and $z \geq t$. Then $S$ is said to be a directed set.

A net in $V$ is a subset $\{x_s : s \in S\}$ of $V$ indexed by a directed set $S$. A net $\{x_s : s \in S\}$ in a topological vectorspace $V$ is a Cauchy net if, for every neighborhood $U$ of 0 in $V$, there is an index $s_0$ so that for $s, t \geq s_0$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is convergent if there is $x \in V$ so that, for every neighborhood $U$ of 0 in $V$ there is an index $s_0$ so that for $s \geq s_0$ we have $x - x_s \in U$. Since points are closed, there can be at most one point to which a net converges. Thus, a convergent net is Cauchy. A topological vectorspace is complete if (also) every Cauchy net is convergent.

Given a topological vectorspace $V$, not necessarily metrizable, we construct its completion in a manner analogous to the more common construction of completion when a metric is available. The main issue is accommodation of the fact that sequences must be replaced by larger nets, but that some restriction must be put on the size of the nets in order to avoid set-theoretic troubles.

Let $A$ be the directed set of all neighborhoods of 0, where we write $U \geq U'$ if $U \subset U'$. Consider all Cauchy nets $\{v_\alpha : \alpha \in A\}$ in $V$ indexed by $A$. We make this collection into a complex vectorspace $V^#$ by

$$\{v_\alpha\} + \{w_\alpha\} = \{v_\alpha + w_\alpha\}$$
$$c \cdot \{v_\alpha\} = \{c \cdot v_\alpha\}$$

consider the subspace $V^o$ of $V^#$ consisting of all nets in $V$ indexed by $A$ which are Cauchy with limit $0 \in V$. It is easy to check that this really is a complex subspace. We claim that the quotient

$$\tilde{V} = V^#/V^o$$

is ‘the completion’ of $V$, in the following sense:

We need to have a better notation: nets indexed by $A$ are, of course, functions from $A$ to $V$. In this perspective, the natural imbedding of the original space $V$ imbeds into $V^#$ is readily described by

$$v \mapsto f_v$$

where $f_v : A \to V$ is the function (net)

$$f_v(\alpha) = v \quad \text{for all } \alpha \in A$$

That is, these are constant nets. It is also clear that in the composite

$$V \to V^# \to V^#/V^o = \tilde{V}$$

is still an injection. We identify $V$ with its image.

Then the claim is that we can give $\tilde{V}$ a topological vectorspace structure so that the original topology on $V$ is the subspace topology from $\tilde{V}$, and so that $V$ is dense in $\tilde{V}$.

### 4. Completions of spaces with seminorms

There are two little things here. First we must accommodate the notion of completion in