The cardinality of the set of real numbers

Jailton C. Ferreira

ABSTRACT: Each real number of the interval [0, 1] can be represented by an infinite path in a given binary tree. The binary tree is projected on a grid N x N and it is shown that the set of the infinite paths corresponds one-to-one to the set N. A proof that there are non-listable sets with cardinality of N is given. Cantor’s proof of 1891 is examined. It is shown that being any denumerable list incomplete, this does not affirm or deny that the cardinalities of the set of the real numbers of the interval [0, 1] and of N are equal.

1 Introduction

Each real number of the interval [0, 1] can be represented by an infinite path in a given binary tree. In section 2 the binary tree is projected on a grid N x N and it is shown that the set of the infinite paths corresponds one-to-one to the set N. The Theorems 1 and 2 give the first proof. The second proof (Theorem 3) is certainly less intuitive than the first proof. In section 2.4 is given the proof that there are non-listable sets with cardinality |N|. In this section a third proof on the cardinality of the real numbers with base in the Theorem 4 is given.

Section 3 examines the Cantor’s proof of 1891. The section shows that (i) if the diagonal method is correct, then any denumerable list L to which the diagonal method was applied is incomplete (Theorem 6), (ii) if some complete list exists and if the diagonal method is correct, then a complete list cannot be represented in the form used in Cantor’s proof (Theorem 7) and (iii) being L incomplete nothing affirms or denies that |N| is the cardinality of the set of real numbers of the interval [0, 1] (Theorem 8).

2 The proof of |F| = |N|

2.1 Theorem 1

The cardinality of the set of real numbers of the interval [0, 1] is equal to |N|.

Proof.

Let us denote by F the set of real numbers of the interval [0, 1]. Any element of F can be represented in the binary system by

\[ f_i \times 2^{-1} + f_2 \times 2^{-2} + f_3 \times 2^{-3} + f_4 \times 2^{-4} + f_5 \times 2^{-5} + \ldots \]  \hspace{1cm} (2-1)

where

\[ f_i \in \{0, 1\} \quad \text{and} \quad i \in N \quad \text{and} \quad i \neq 0 \]  \hspace{1cm} (2-2)

The representation (2-1) can be simplified to

\[ .f_1f_2f_3f_4f_5 \ldots \ldots \]  \hspace{1cm} (2-3)

where the first character of the sequence is the point. Each f_i of (2-3) is substituted by 0 or 1 to represent a given number. The i-th 0 or 1 on the right corresponds to f_i. The set F can also be represented by a binary tree where each node has two children. Each infinite path on the binary tree, \( .f_1f_2f_3f_4f_5 \ldots \ldots \), represents an element of F. The Figure 1 shows the binary tree up to the level 4.
Let us now consider the Figure 2. The horizontal sequence of finite natural numbers, presented in increasing order from left to right, contains all numbers of \( \mathbb{N} \); to each natural number of the sequence corresponds a vertical line. The vertical sequence of numbers, presented in increasing order from top to bottom, contains all numbers of \( \mathbb{N} \); to each natural number of the sequence corresponds a horizontal line. To each node of the grid formed by the horizontal and vertical lines corresponds a pair \((m, n)\), where \(m\) belongs to the horizontal sequence of numbers and \(n\) belongs to the vertical sequence of numbers.

Let us project the binary tree of the Figure 1 on the grid of the Figure 2 in the following way: to the root of the tree corresponds the pair \((0, 0)\); to the 2 nodes of the level 1 correspond the pairs \((0, 1)\) and \((1, 0)\); to the 4 nodes of the level 2 correspond the pairs \((0, 3)\), \((1, 2)\), \((2, 1)\) and \((3, 0)\); to the 8 nodes of the level 3 correspond the pairs \((0, 7)\), \((1, 6)\), \((2, 5)\), \((3, 4)\), \((4, 3)\), \((5, 2)\), \((6, 1)\) and \((7, 0)\) and to the \(2^k\) nodes of the level \(k\) correspond the pairs

\[
(0, 2^{k-1}), (1, 2^{k-2}), (2, 2^{k-3}), \ldots, (2^{k-3}, 2), (2^{k-2}, 1) \text{ and } (2^{k-1}, 0) \tag{2.4}
\]

The Figure 3 shows the projection of the binary tree up to the level 3 on the grid of the Figure 2.
It is known that the bijection \( f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \), where \( \mathbb{N} \) is the set of finite natural numbers, can be defined applying the diagonal method.

The Figure 4 shows how starting from the pair \((0, 0)\) and following the line in red we can establish the one-to-one correspondence between the set of the pairs \((m, n)\) and \(\mathbb{N}\), this is,

\[
\begin{array}{cccccc}
(0, 0) & (1, 0) & (0, 1) & (0, 2) & (1, 1) & (2, 0) & (3, 0) \\
& & & & & & \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

(2-5)

Let \( A \) be the set of nodes of the binary tree of the Figure 1 and \( B \) the set of nodes of the grid of the Figure 2. Considering that to any node of the binary tree corresponds a node of the grid and that there are nodes in the grid without corresponding nodes in the binary tree, we have that

\[
A \subseteq B
\]

(2-6)

and

\[
|A| \leq |B|
\]

(2-7)
Considering that the number of infinite paths on the binary tree is less than or equal to the number of nodes of the binary tree, we have

\[ |F| \leq |A| \]  

(2-8)

From (2-7) and (2-8), we have

\[ |F| \leq |B| \]  

(2-9)

Considering that there are proper subsets of \( F \) that can be put into one-to-one correspondence with \( N \), we have

\[ |F| \geq |N| \]  

(2-10)

From (2-5) we have

\[ |B| = |N| \]  

(2-11)

Substituting (2-11) in (2-9), we have

\[ |F| \leq |N| \]  

(2-12)

From (2-10) and (2-12) we conclude that

\[ |F| = |N| \]  

(2-13)

2.3 Theorem 2

Let \( B \) be a binary tree such that every node has two children. Let \( A \) be the set of the nodes of the binary tree \( B \) and \( F \) the set of the infinite paths of \( B \). The cardinality of \( F \) is less than or equal to the cardinality of \( A \).

Proof.

Any element of \( F \) can be represented by

\[ .f_1f_2f_3f_4f_5 \ldots \ldots \]  

(2-14)

where

\[ f_i \in \{0, 1\} \quad \text{and} \quad i \in N \quad \text{and} \quad i \neq 0 \]  

(2-15)

and the symbol “.” on the left of \( f_i \) corresponds to the root of the binary tree. The Figure 1 shows the binary tree up to the level 4.

To each subscript \( i \) of \( f_i \), we can associate a binary tree, denoted by \( B_i \), where each node has two children and the depth of the binary tree is \( i \). Let \( P_i \) be the set of paths from the root of \( B_i \) to its leaves.

Let \( P \) be the well-ordered set of all paths starting from the root

\[ P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6 \cup \ldots \]  

(2-16)

Let us notice that to each \( f_i \) in (2-14) corresponds one-to-one \( P_i \) in (2-16). In \( P \) there are not just sets of finite paths, there is the emergent property given by
For any infinite path \( t \), and any level \( n \) of \( t \), \( \mathbf{P} \) contains all the part of \( t \), besides the level \( n \). From (2-17) we have

\[
|\mathbf{F}| \leq |\mathbf{P}|
\]  

(2-18)

Since the number of nodes in the level \( i \) of the binary tree \( \mathbf{B} \) is equal to \( |\mathbf{P}_i| \), we have that

\[
|\mathbf{P}_i| = |\mathbf{A}|
\]  

(2-19)

Substituting (2-19) in (2-18), we conclude

\[
|\mathbf{F}| \leq |\mathbf{A}|
\]  

(2-20)

2.3 Theorem 3

The cardinal number of the set of infinite paths of the binary tree \( \mathbf{B} \) is equal to \( |\mathbf{N}| \).

Proof.

Be the infinite path of \( \mathbf{B} \)

\[
0, f_1 f_2 f_3 f_4 f_5, ...
\]  

(2-21)

where \( f_i = 1 \) for all \( i \in \mathbb{N} \), projected in the grid as shows the Figure 3. In the coordinates of the grid, the path (2-21) corresponds to the pairs

\[
(0, 0), (0, 1), (0, 3), (0, 7), (0, 15), ..., (0, 2^k-1), ...
\]  

(2-22)

Let us notice that in \( (0, 0) \) all infinite paths begin. By the pair \( (0, 1) \) a part of all infinite paths passes, by \( (0, 3) \) a part of the infinite paths that passed by \( (0, 1) \) passes, by \( (0, 7) \) a part of the infinite paths that passed by \( (0, 3) \) passes and so forth. When an entire path is accomplished, the path exists. In the examined case, (2-21) is the accomplished path.

Each pair \( (0, 2^k-1) \) of the sequence (2-22) belongs to the set of pairs given by (2-4), this is,

\[
(0, 2^{k-1}), (1, 2^{k-2}), (2, 2^{k-3}), ..., (2^{k-3}, 2), (2^{k-2}, 1) \text{ and } (2^{k-1}, 0)
\]  

(2-23)

Let us denote by \( \mathbf{G}_k \) the set of pairs given by (2-23) for \( k \). For any \( k \) the cardinal number of the set of infinite paths that passes by \( (0, 2^{k-1}) \) is equal to the cardinal number of the set of infinite paths that pass by any other pair of (2-23). For any \( k \) the distance from \( (0, 0) \) to \( (0, 2^{k-1}) \) in the grid is equal to \( |\mathbf{G}_k| \). When we examine the pairs of (2-22), from left to right, and we accomplished the path (2-21), \( \mathbf{G}_k \) becomes the set of the infinite path whose cardinality is equal to the cardinality of the set of the nodes of the path (2-21).

The cardinality of the set of the infinite paths is \( |\mathbf{F}| \) and the cardinality of the set of nodes of any infinite path is \( |\mathbf{N}| \). Therefore

\[
|\mathbf{F}| = |\mathbf{N}|
\]  

(2-24)
2.4  Non-listable sets of cardinality N

Definition 1
A ordered set X is listable if the elements of X can be put into one-to-one correspondence with N in a list such that (i) the elements of N in the list are in increasing order, (ii) to the element x, that belongs to X it corresponds the element i of N in the list and (iii) if x_i and x_j belongs to X and i<j, then x_i precedes x_j in X. A set is called non-listable if it is not listable.

Theorem 4
There are non-listable sets of cardinality |N|.

Proof.
Let us consider the set

\[ A = \{ a \mid a = \{ a_0, a_1, a_2, ... \} \text{ and } a_i \in \{0, 1\} \} \tag{2-25} \]

where each element of A (i) is equivalent to N and (ii) for i>1 there is some a_i equal to 0 and some a_i equal to 1. Let us reordain all of the elements of A in according with the procedure

\[ a_0, a_1 \]
\[ a_0, a_2, a_1 \]
\[ a_0, a_3, a_2, a_4, a_1 \]
\[ a_0, a_5, a_3, a_6, a_2, a_7, a_4, a_8, a_1 \]
\[ ............. \]

Be \( A_R \) the set A reordained in accordance with (2-26). By definition in set theory, a set X is equal to a set Y if they both have the same elements. However \( A_R \) is not A because they are ordered differently. Let us do

\[ A_{(R)} = \{ a \mid a = b - \{a_0, a_1\} \text{ and } b \in A_R, \text{ where } a_0 \in b \text{ and } a_1 \in b \} \tag{2-27} \]

The elements of \( A_{(R)} \) do not possess immediate successor or immediate predecessor. Any element of \( A_{(R)} \) is a non-listable set of cardinality \(|N|\).

The Theorem 4 can be used to prove \(|F| = |N|\), since we use (2-29) as an axiom.

Theorem 5
The cardinality of F is equal to |N|.

Proof.
Let us consider the set of real numbers of the interval (0, 1)

\[ F_o = F - \{0, 1\} \tag{2-28} \]

To each element of \( F_o \) we associate at random 0 or 1, such that there is some element equal to 0 and some element equal to 1 in the set of symbols 0 and 1. We denote by \( \Delta \) (beth) the obtained set of symbols 0 and 1.

The sets which are elements of \( A_{(R)} \) have cardinality \(|N|\) and the cardinality of \( F_o \) is equal to the cardinality of \( \Delta \). Since

\[ There \ is \ an \ element \ of \ A_{(R)} \ which \ is \ the \ set \ \Delta \tag{2-29} \]
and

\[ |F| = |F_a| \]  \hspace{2cm} (2-30) 

we can conclude \(|F| = |\mathbb{N}|\).

3 The Cantor’s Proof

3.1 The 1891 proof

**Theorem**  The set \(F\) is not countable.

**Proof.** We assume that the set \(F\) is countable. This means, by the definition of countable sets, that \(F\) is finite or denumerable. Let us notice that \(|F|\) is not less than \(|\mathbb{N}|\). Be

\[ F = \{a_0, a_1, a_2, \ldots\} \]

where the cardinality of \(\{a_0, a_1, a_2, \ldots\}\) is \(|\mathbb{N}|\). We can write their decimal expansions as follows:

\[
\begin{align*}
a_0 &= 0.d_{1,1} d_{1,2} d_{1,3} \ldots \\
a_1 &= 0.d_{2,1} d_{2,2} d_{2,3} \ldots \\
a_2 &= 0.d_{3,1} d_{3,2} d_{3,3} \ldots \\
&\vdots
\end{align*}
\]  \hspace{2cm} (3-0)

where the \(d’s\) are digits 0 - 9. Now we define the number

\[ x = 0.d_1 d_2 d_3 \ldots \]

by selecting \(d_1 \neq d_{1,1}, d_2 \neq d_{2,2}, d_3 \neq d_{3,3}, \ldots\). This gives a number not in the set \(\{a_0, a_1, a_2, \ldots\}\), but \(x \in F\). Therefore, \(F\) is not countable.

When (3-0) is represented as (3-1) bellow, the proof can be called the written list form of the Cantor’s argument of 1891 [1].

3.2 Comments

Let us consider the set \(N\), the set \(F\) of the real numbers of the interval \([0, 1]\) and the hypothesis

\( (H_2) \) *There is a list, denoted by \(L_{n}\), that contains all members of \(N\) and all members of \(F\) and to each member of \(F\) of the list corresponds one-to-one a member of \(N\) of the list.*

For Cantor’s diagonal method to be applied, it is created a list \(L\)

\[
\begin{array}{c|ccc}
0 & 0.d_{1,1} d_{1,2} d_{1,3} \ldots \\
1 & 0.d_{2,1} d_{2,2} d_{2,3} \ldots \\
2 & 0.d_{3,1} d_{3,2} d_{3,3} \ldots \\
&\vdots
\end{array}
\]  \hspace{2cm} (3-1)

The diagonal method finds a member of \(F\) which is not in the list \(L\) and shows that the list cannot contain all members of \(F\). The Cantor’s proof supposes that the list \(L\) is the list of \((H_2)\) and concludes that the list of \((H_2)\) does not exist.
Theorem 6 below affirms that if the diagonal method is correct, then any denumerable list \( L \) to which the diagonal method has been applied is incomplete, this is, \( L \) does not contain all of real numbers of the interval \([0, 1]\). The Theorem 7 affirms that if some complete list exists and if the diagonal method is correct, then a complete list cannot be represented in the form \((3-1)\). The Theorem 8 affirms that being \( L \) incomplete nothing affirms or denies that \( |F| = |\mathbb{N}| \).

Considering the Theorem 4, we cannot affirm that \( L \) incomplete implies \( |F| \neq |\mathbb{N}| \).

### 3.3 Theorem 6

**If the diagonal method is correct, then any denumerable list \( L \) is incomplete.**

**Proof.**

Suppose that the diagonal method is correct. Let us consider the list \( L \) in which it is presumed that the real numbers of the interval \([0, 1]\) are put into one-to-one correspondence with the even natural numbers (0 included). We use the binary system for the real numbers. The list \( L \), where any \( d_{ij} \) is either 0 or 1, is the following:

\[
\begin{array}{cccc}
0 & 0.d_{1,1} d_{1,2} d_{1,3} d_{1,4} & \ldots \\
2 & 0.d_{2,1} d_{2,2} d_{2,3} d_{2,4} & \ldots \\
4 & 0.d_{3,1} d_{3,2} d_{3,3} d_{3,4} & \ldots \\
6 & 0.d_{4,1} d_{4,2} d_{4,3} d_{4,4} & \ldots \\
\end{array}
\]

Be the real number \( 0.d_{1,1} d_{2,2} d_{3,3} d_{4,4} \ldots \). Let us consider the number \( 0.k_1 k_2 k_3 k_4 \ldots \) , where \( k_i \neq d_{ij} \) for every \( i \in \mathbb{N} \). The number \( 0.k_1 k_2 k_3 k_4 \ldots \) belongs to the interval \([0, 1]\) and it is not in \((3-2)\). Therefore, the hypothesis that the list contains all of the real numbers of the interval \([0, 1]\) is false. Let us include \( 0.k_1 k_2 k_3 k_4 \ldots \) in the list:

\[
\begin{array}{cccc}
0 & 0.d_{1,1} d_{1,2} d_{1,3} d_{1,4} & \ldots \\
1 & 0.k_1 k_2 k_3 k_4 \ldots & \\
2 & 0.d_{2,1} d_{2,2} d_{2,3} d_{2,4} & \ldots \\
4 & 0.d_{3,1} d_{3,2} d_{3,3} d_{3,4} & \ldots \\
6 & 0.d_{4,1} d_{4,2} d_{4,3} d_{4,4} & \ldots \\
\end{array}
\]

Starting from of the hypothesis that the set of all real numbers of the interval \([0, 1]\) is in the list \((3-2)\) and it has cardinality \( |\mathbb{N}| \), we showed that there are real numbers of the interval \([0, 1]\) not included in the set of real numbers of \((3-2)\).

We can conclude that if the diagonal method is correct, then any denumerable list \( L \) is incomplete.

### 3.4 Theorem 7

**Be the list \((3-4)\) such that (i) the column on the left contains the elements of \( \mathbb{N} \) put from top to bottom in increasing order and (ii) the column on the right contains real numbers of the interval \([0, 1]\), where \( d_{ij} \in \{0, 1\} \). The sets of the numbers of the column on the right are equivalent to \( \mathbb{N} \).

\[
\begin{array}{cccc}
0 & 0.d_1 d_2 d_3 d_4 & \ldots \\
1 & 0.d_1 d_2 d_3 d_4 & \ldots \\
2 & 0.d_1 d_2 d_3 d_4 & \ldots \\
3 & 0.d_1 d_2 d_3 d_4 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
If some complete list exists and if the diagonal method is correct, then a complete list cannot be represented in the form (3-4).

Proof.

Let us separate the list (3-4) in two lists so that all of the odd lines belong to one of the lists and the remaining lines belong to the second list.

\[
\begin{array}{c}
0 & 0.d_1 d_2 d_3 d_4 \ldots \\
2 & 0.d_1 d_2 d_3 d_4 \ldots \\
4 & 0.d_1 d_2 d_3 d_4 \ldots \\
6 & 0.d_1 d_2 d_3 d_4 \ldots \\
\vdots & \vdots \\
\end{array}
\]

and

\[
\begin{array}{c}
1 & 0.d_1 d_2 d_3 d_4 \ldots \\
3 & 0.d_1 d_2 d_3 d_4 \ldots \\
5 & 0.d_1 d_2 d_3 d_4 \ldots \\
7 & 0.d_1 d_2 d_3 d_4 \ldots \\
\vdots & \vdots \\
1 & 0.d_1 d_2 d_3 d_4 \ldots \\
\end{array}
\]

(3-5)

(3-6)

Considering that the sets \{0, 2, 4, 6, \ldots\} and \{1, 3, 5, 7, \ldots\} are equivalent to \(\mathbb{N}\), we can rewrite the two lists, respectively, as

\[
\begin{array}{c}
0 & 0.d_1 d_2 d_3 d_4 \ldots \\
1 & 0.d_1 d_2 d_3 d_4 \ldots \\
2 & 0.d_1 d_2 d_3 d_4 \ldots \\
3 & 0.d_1 d_2 d_3 d_4 \ldots \\
\vdots & \vdots \\
\end{array}
\]

(3-7)

and

\[
\begin{array}{c}
0 & 0.d_1 d_2 d_3 d_4 \ldots \\
1 & 0.d_1 d_2 d_3 d_4 \ldots \\
2 & 0.d_1 d_2 d_3 d_4 \ldots \\
3 & 0.d_1 d_2 d_3 d_4 \ldots \\
\vdots & \vdots \\
\end{array}
\]

(3-8)

The set of real numbers of (3-7) and the set of real numbers of (3-8) are disjoint. Therefore, both (3-7) and (3-8) are incomplete, this is, any one of them does not contain all real numbers of the interval \([0, 1]\).

We can apply the diagonal method to (3-7), for instance, and we obtain a real number that is not in (3-7). That is consistent because (3-7) is incomplete.

There is uncertainty as to the completeness or not of (3-4). Considering that the representation (3-4) is the same of (3-7) and (3-8), the application of the diagonal method to (3-4) is to say that (3-4) is incomplete. Concluding, if some complete list exists and if the diagonal method is correct, then a complete list cannot be represented in the form (3-4).

3.5 Theorem 8

Being \(L\) incomplete nothing affirms or denies that \(|F| = |N|\).

Proof.

Suppose that the list \(L\) is the list \(L_{\text{max}}\) [section 3.2]. In this case a proof that \(L\) doesn't contain all members of \(F\) implies in \((H_2)\) false. This also means that the one-to-one correspondence between \(N\) and \(F\) in a list cannot be done. If \((H_2)\) is false, then \(L\) does not exist. However, \(L\) exists. Therefore, it is false.
the hypothesis that \( L \) is the list \( L_{\text{end}} \). Since “\( L \) is not \( L_{\text{end}} \)” nothing affirms the existence or not of \( L_{\text{end}} \). being \( L \) incomplete nothing affirms or denies that \(|F| = |N|\).

4 Reference