Chapter 5

The Self-Adjoint Second-Order Differential Equation

5.1 Basic Definitions

In this chapter we are concerned with the second-order (formally) self-adjoint linear differential equation

\[(p(t)x')' + q(t)x = h(t).\]

We assume throughout that \(p, q, \) and \(h\) are continuous on some given interval \(I\) and \(p(t) > 0\) on \(I\). Let

\[\mathbb{D} := \{x: x \text{ and } px' \text{ are continuously differentiable on } I\}.\]

We then define the linear operator \(L\) on \(\mathbb{D}\) by

\[Lx(t) = (p(t)x'(t))' + q(t)x(t),\]

for \(t \in I\). Then the self-adjoint equation can be written briefly as \(Lx = h(t)\).

If \(h(t) \equiv 0\), we get the homogeneous self-adjoint differential equation \(Lx = 0\); otherwise we say the differential equation \(Lx = h(t)\) is nonhomogeneous.

Definition 5.1 If \(x \in \mathbb{D}\) and \(Lx(t) = h(t)\) for \(t \in I\), we say that \(x\) is a solution of \(Lx = h(t)\) on the interval \(I\).

Example 5.2 In this example we show that any second-order linear differential equation of the form

\[p_2(t)x'' + p_1(t)x' + p_0(t)x = g(t),\]

where we assume that \(p_2(t) \neq 0\) on \(I\) and \(p_i, i = 0, 1, 2,\) and \(g\) are continuous on an interval \(I\) can be written in self-adjoint form.

Assume \(x\) is a solution of (5.1); then

\[x''(t) + \frac{p_1(t)}{p_2(t)} x'(t) + \frac{p_0(t)}{p_2(t)} x(t) = \frac{g(t)}{p_2(t)},\]

for \(t \in I\). Multiplying by the integrating factor \(e^{\int \frac{p_1(t)}{p_2(t)} dt}\) for the first two terms, we obtain

\[e^{\int \frac{p_1(t)}{p_2(t)} dt} x''(t) + \frac{p_1(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} dt} x'(t) + \frac{p_0(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} dt} x(t) = \frac{g(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} dt}.\]
\[ \{e^{\int \frac{p_1(t)}{p_2(t)} \, dt} x'(t)\}' + \frac{p_0(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} \, dt} x(t) = \frac{g(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} \, dt}. \]

Hence we obtain that \( x \) is a solution of the self-adjoint equation
\[(p(t)x')' + q(t)x = h(t),\]
where
\[
\begin{align*}
p(t) &= e^{\int \frac{p_1(t)}{p_2(t)} \, dt} > 0, \\
q(t) &= \frac{p_0(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} \, dt}, \\
h(t) &= \frac{g(t)}{p_2(t)} e^{\int \frac{p_1(t)}{p_2(t)} \, dt},
\end{align*}
\]
for \( t \in I \). Note that \( p, q, \) and \( h \) are continuous on \( I \). Actually \( p \) is continuously differentiable on \( I \). In many studies of the self-adjoint differential equation it is assumed that \( p \) is continuously differentiable on \( I \), but we will only assume \( p \) is continuous on \( I \). In this case the self-adjoint equation \( Lx = h(t) \) is more general than the linear differential equation (5.1). \( \triangle \)

**Example 5.3** Write the differential equation
\[ t^2 x'' + 3tx' + 6x = t^4, \]
for \( t \in I := (0, \infty) \) in self-adjoint form.

Dividing by \( t^2 \), we obtain
\[ x'' + \frac{3}{t} x' + \frac{6}{t^2} x = t^2. \]

Hence
\[ e^{\int \frac{3}{t} \, dt} = e^{3 \log t} = t^3 \]
is an integrating factor for the first two terms. Multiplying by the integrating factor \( t^3 \) and simplifying, we obtain the self-adjoint differential equation
\[(t^3 x')' + 6tx = t^5. \]

\( \triangle \)

We now state and prove the following existence-uniqueness theorem for the self-adjoint nonhomogeneous differential equation \( Lx = h(t) \).

**Theorem 5.4** (Existence-Uniqueness Theorem) Assume that \( p, q, \) and \( h \) are continuous on \( I \) and \( p(t) > 0 \) on \( I \). If \( a \in I \), then the initial value problem (IVP)
\[
Lx = h(t), \quad x(a) = x_0, \quad x'(a) = x_1,
\]
where \( x_0 \) and \( x_1 \) are given constants, has a unique solution and this solution exists on the whole interval \( I \).
Proof We first write $Lx = h(t)$ as an equivalent vector equation. Let $x$ be a solution of $Lx = h(t)$ and let

$$y(t) := p(t)x'(t).$$

Then

$$x'(t) = \frac{1}{p(t)}y(t).$$

Also, since $x$ is a solution of $Lx = h(t)$,

$$y'(t) = (p(t)x'(t))' = -q(t)x(t) + h(t).$$

Hence if we let

$$z(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

then $z$ is a solution of the vector equation

$$z' = A(t)z + b(t),$$

where

$$A(t) := \begin{bmatrix} 0 & \frac{1}{p(t)} \\ -q(t) & 0 \end{bmatrix}, \quad b(t) := \begin{bmatrix} 0 \\ h(t) \end{bmatrix}. $$

Note that the matrix function $A$ and the vector function $b$ are continuous on $I$. Conversely, it is easy to see that if

$$z(t) := \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

defines a solution of the vector equation

$$z' = A(t)z + b(t),$$

then $x$ is a solution of the scalar equation $Lx = h(t)$ and $y(t) = p(t)x'(t)$. By Theorem 2.3 there is a unique solution $z$ of the IVP

$$z' = A(t)z + b(t), \quad z(a) = \begin{bmatrix} x_0 \\ y_1 \end{bmatrix},$$

and this solution is a solution on the whole interval $I$. But this implies that there is a unique solution of $Lx = h(t)$ satisfying

$$x(a) = x_0, \quad p(a)x'(a) = y_1,$$

and this solution exists on the whole interval $I$. It follows that the initial value problem

$$Lx = (p(t)x')' + q(t)x = h(t), \quad x(a) = x_0, \quad x'(a) = x_1$$

has a unique solution on $I$. Some authors (for good reasons) prefer to call the conditions $x(a) = x_0, p(a)x'(a) = y_1$ initial conditions, but we will call the conditions $x(a) = x_0, x'(a) = x_1$ initial conditions. □
Definition 5.5 Assume $x, y$ are differentiable functions on an interval $I$; then we define the Wronskian of $x$ and $y$ by
\[
w[x(t), y(t)] = \begin{vmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{vmatrix} := x(t)y'(t) - x'(t)y(t),
\]
for $t \in I$.

Theorem 5.6 (Lagrange Identity) Assume $x, y \in \mathbb{D}$; then
\[y(t)Lx(t) - x(t)Ly(t) = \{y(t); x(t)\}',
\]
for $t \in I$, where $\{y(t); x(t)\}$ is the Lagrange bracket of $y$ and $x$, which is defined by
\[\{y(t); x(t)\} := p(t)w[y(t), x(t)], \quad t \in I,
\]
where $w[y(t), x(t)]$ is the Wronskian of $y$ and $x$.

**Proof** Let $x, y \in \mathbb{D}$ and consider
\[
\{y(t); x(t)\}' = \{y(t)p(t)x'(t) - x(t)p(t)y'(t)\}',
\]
\[= y(t)\left[(p(t)x'(t))' + y'(t)p(t)x'(t) - x(t)(p(t)y'(t))' - x'(t)p(t)y'(t)\right] - x(t)\left[(p(t)y'(t))' + q(t)x(t)\right],
\]
\[= y(t)Lx(t) - x(t)Ly(t),
\]
for $t \in I$, which is what we wanted to prove.

Corollary 5.7 (Abel’s Formula) If $x, y$ are solutions of $Lx = 0$ on $I$, then
\[w[x(t), y(t)] = \frac{C}{p(t)},
\]
for all $t \in I$, where $C$ is a constant.

**Proof** Assume $x, y$ are solutions of $Lx = 0$ on $I$. By the Lagrange identity,
\[x(t)Ly(t) + y(t)Lx(t) = \{x(t); y(t)\}',
\]
for all $t \in I$. Since $x$ and $y$ are solutions on $I$,
\[\{x(t); y(t)\}' = 0,
\]
for all $t \in I$. Hence
\[\{x(t); y(t)\} = C,
\]
where $C$ is a constant. It follows that
\[w[x(t), y(t)] = \frac{C}{p(t)},
\]
for all $t \in I$, which is Abel’s formula.

Definition 5.8 If $x$ and $y$ are continuous on $[a, b]$, we define the inner product of $x$ and $y$ by
\[<x, y> = \int_{a}^{b} x(t)y(t) \, dt.
\]
Corollary 5.9 (Green’s Formula) If \([a, b] \subset I \) and \(x, y \in \mathbb{D} \), then
\[
<y, Lx> - <Ly, x> = \{y(t); x(t)\}_a^b,
\]
where \(\{F(t)\}_a^b := F(b) - F(a)\).

**Proof** Let \(x, y \in \mathbb{D} \); then by the Lagrange identity,
\[
y(t)Lx(t) - x(t)Ly(t) = \{y(t); x(t)\}',
\]
for all \(t \in I\). Integrating from \(a\) to \(b\), we get the desired result
\[
<y, Lx> - <Ly, x> = \{y(t); x(t)\}_a^b.
\]

Corollary 5.10 If \(u, v\) are solutions of \(Lx = 0\), then either
\((a)\) \(w[u(t), v(t)] \neq 0\) for all \(t \in I\)
or
\((b)\) \(w[u(t), v(t)] = 0\) for all \(t \in I\).

Case \((a)\) occurs iff \(u, v\) are linearly independent on \(I\) and case \((b)\) occurs iff \(u, v\) are linearly dependent on \(I\).

**Proof** Assume \(u, v\) are solutions of \(Lx = 0\). Then by Abel’s formula (Corollary 5.7),
\[
w[u(t), v(t)] = \frac{C}{p(t)},
\]
for all \(t \in I\), where \(C\) is a constant. If \(C \neq 0\), then part \((a)\) of this theorem holds, while if \(C = 0\), part \((b)\) of this theorem holds. The remainder of the proof of this corollary is left to the reader (see Exercise 5.5).

We now show that the differential equation \(Lx = 0\) has two linearly independent solutions on \(I\). To see this let \(a \in I\) and let \(u\) be the solution of the IVP
\[
Lu = 0, \quad u(a) = 1, \quad u'(a) = 0,
\]
and let \(v\) be the solution of the IVP
\[
Lv = 0, \quad v(a) = 0, \quad v'(a) = 1.
\]
Since the Wronskian of these two solutions at \(a\) is different from zero, these two solutions are linearly independent on \(I\) by Corollary 5.10.

**Theorem 5.11** If \(x_1, x_2\) are linearly independent solutions of \(Lx = 0\) on \(I\), then
\[
x = c_1x_1 + c_2x_2
\]
is a general solution of \(Lx = 0\). By this we mean every function in this form is a solution of \(Lx = 0\) and all solutions of \(Lx = 0\) are in this form.
Proof Assume $x_1, x_2$ are linearly independent solutions on $I$ of $Lx = 0$. Let $x$ be of the form (5.2). Then
\[
Lx = L[c_1x_1 + c_2x_2] = c_1Lx_1 + c_2Lx_2 = 0,
\]
so $x$ is a solution of $Lx = 0$.

Conversely, assume that $x$ is a solution of $Lx = 0$ on $I$. Let $t_0 \in I$ and let
\[
x_0 := x(t_0), \quad x_1 := x'(t_0).
\]
Let
\[
y(t) = c_1x_1(t) + c_2x_2(t).
\]
We now show that we can pick constants $c_1, c_2$ such that
\[
y(t_0) = x_0, \quad y'(t_0) = x_1.
\]
These last two equations are equivalent to the equations
\[
c_1x_1(t_0) + c_2x_2(t_0) = x_0, \quad c_1x'_1(t_0) + c_2x'_2(t_0) = x_1.
\]
The determinant of the coefficients for this system is
\[
w[x_1, x_2](t_0) \neq 0
\]
by Corollary 5.10. Let $c_1, c_2$ be the unique solution of this system. Then
\[
y = c_1x_1 + c_2x_2
\]
is a solution of $Lx = 0$ satisfying the same initial conditions as $x$ at $t_0$. By the uniqueness of solutions of IVPs (Theorem 5.4), $x$ and $y$ are the same solution. Hence we get the desired result
\[
x = c_1x_1 + c_2x_2.
\]

\section*{5.2 An Interesting Example}

In this section we indicate how Theorem 5.4 can be used to define functions and how we can use this to derive properties of these functions.

\textbf{Definition 5.12} We define $s$ and $c$ to be the solutions of the IVPs
\[
\begin{align*}
 s''' + s &= 0, & s(0) &= 0, & s'(0) &= 1, \\
 c''' + c &= 0, & c(0) &= 1, & c'(0) &= 0,
\end{align*}
\]
respectively.

\textbf{Theorem 5.13} (Properties of $s$ and $c$)
\begin{enumerate}
  \item $s'(t) = c(t), \quad c'(t) = -s(t)$
  \item $s^2(t) + c^2(t) = 1$
\end{enumerate}
(iii) \( s(-t) = -s(t), \quad c(-t) = c(t) \)
(iv) \( s(t + \alpha) = s(t)c(\alpha) + s(\alpha)c(t), \quad c(t + \alpha) = c(t)c(\alpha) - s(t)s(\alpha) \)
(v) \( s(t - \alpha) = s(t)c(\alpha) - s(\alpha)c(t), \quad c(t - \alpha) = c(t)c(\alpha) + s(t)s(\alpha) \)

for \( t, \alpha \in \mathbb{R} \).

**Proof** Since \( c \) and \( s' \) solve the same IVP
\[
x'' + x = 0, \quad x(0) = 1, \quad x'(0) = 0,
\]
we get by the uniqueness theorem that
\[
s'(t) = c(t), \quad \text{for} \quad t \in \mathbb{R}.
\]
Similarly, \(-s\) and \( c'\) solve the same IVP
\[
x'' + x = 0, \quad x(0) = 0, \quad x'(0) = -1,
\]
and so by the uniqueness theorem
\[
c'(t) = -s(t), \quad \text{for} \quad t \in \mathbb{R}.
\]
By Abel’s theorem the Wronskian of \( c \) and \( s \) is a constant, so
\[
\left| \begin{array}{cc} c(t) & s(t) \\ c'(t) & s'(t) \end{array} \right| = \left| \begin{array}{cc} c(0) & s(0) \\ c'(0) & s'(0) \end{array} \right| = 1.
\]
It follows that
\[
\left| \begin{array}{cc} c(t) & s(t) \\ -s(t) & c(t) \end{array} \right| = 1,
\]
and hence we obtain
\[
s^2(t) + c^2(t) = 1.
\]
Next note that since \( s(-t) \) and \(-s(t)\) both solve the IVP
\[
x'' + x = 0, \quad x(0) = 0, \quad x'(0) = -1,
\]
we have by the uniqueness theorem that
\[
s(-t) = -s(t), \quad \text{for} \quad t \in \mathbb{R}.
\]
By a similar argument we get that \( c \) is an even function. Since \( s(t + \alpha) \) and \( s(t)c(\alpha) + s(\alpha)c(t) \) both solve the IVP
\[
x'' + x = 0, \quad x(0) = s(\alpha), \quad x'(0) = c(\alpha),
\]
we have by the uniqueness theorem that
\[
s(t + \alpha) = s(t)c(\alpha) + s(\alpha)c(t), \quad \text{for} \quad t \in \mathbb{R}.
\]
By a similar argument
\[
c(t + \alpha) = c(t)c(\alpha) - s(t)s(\alpha), \quad \text{for} \quad t \in \mathbb{R}.
\]
Finally, using parts (iii) and (iv), we can easily prove part (v). \(\Box\)
5.3 Cauchy Function and Variation of Constants Formula

In this section we will derive a variation of constants formula for the nonhomogeneous second-order self-adjoint differential equation

\[ Lx = (p(t)x')' + q(t)x = h(t), \quad (5.3) \]

where we assume \( h \) is a continuous function on \( I \).

**Theorem 5.14** If \( u, v \) are linearly independent solutions on \( I \) of the homogeneous differential equation \( Lx = 0 \) and \( z \) is a solution on \( I \) of the nonhomogeneous differential equation \( Lx = h(t) \), then

\[ x = c_1 u + c_2 v + z, \]

where \( c_1 \) and \( c_2 \) are constants, is a general solution of the nonhomogeneous differential equation \( Lx = h(t) \).

**Proof** Let \( u, v \) be linearly independent solutions of the homogeneous differential equation \( Lx = 0 \), let \( z \) be a solution of the nonhomogeneous differential equation \( Lx = h(t) \), and let

\[ x := c_1 u + c_2 v + z, \]

where \( c_1 \) and \( c_2 \) are constants. Then

\[ Lx = c_1 Lu + c_2 Lv + Lz = h. \]

Hence for any constants \( c_1 \) and \( c_2 \), \( x := c_1 u + c_2 v + z \) is a solution of the nonhomogeneous differential equation \( Lx = h(t) \).

Conversely, assume \( x_0 \) is a solution of the nonhomogeneous differential equation \( Lx = h(t) \) and let

\[ x := x_0 - z. \]

Then

\[ Lx = Lx_0 - Lz = h - h = 0. \]

Hence \( x \) is a solution of the homogeneous differential equation \( Lx = 0 \) and so by Theorem 5.11 there are constants \( c_1 \) and \( c_2 \) such that

\[ x = c_1 u + c_2 v \]

and this implies that

\[ x_0 = c_1 u + c_2 v + z. \]

\[ \square \]

**Definition 5.15** Define the Cauchy function \( x(\cdot, \cdot) \) for \( Lx = 0 \) to be the function \( x : I \times I \to \mathbb{R} \) such that for each fixed \( s \in I \), \( x(\cdot, s) \) is the solution of the initial value problem

\[ Lx = 0, \quad x(s) = 0, \quad x'(s) = \frac{1}{p(s)}. \]
Example 5.16 Find the Cauchy function for \((p(t)x')' = 0\).

For each fixed \(s\),
\[(p(t)x'(t, s))' = 0, \quad t \in I.\]

Hence
\[p(t)x'(t, s) = \alpha(s).\]

The condition \(x'(s, s) = \frac{1}{p(s)}\) implies that \(\alpha(s) = 1\). It follows that
\[x'(t, s) = \frac{1}{p(t)}.\]

Integrating from \(s\) to \(t\) and using the condition \(x(s, s) = 0\), we get that
\[x(t, s) = \int_{s}^{t} \frac{1}{p(\tau)} d\tau.\]

\(\triangle\)

Example 5.17 Find the Cauchy function for
\[
\left( e^{-5t}x' \right)' + 6e^{-5t}x = 0.
\]

Expanding this equation out and simplifying, we get the equivalent equation
\[x'' - 5x' + 6x = 0.\]

It follows that the Cauchy function is of the form
\[x(t, s) = \alpha(s)e^{2t} + \beta(s)e^{3t}.\]

The initial conditions
\[x(s, s) = 0, \quad x'(s, s) = \frac{1}{p(s)} = e^{5s}\]
lead to the equations
\[\alpha(s)e^{2s} + \beta(s)e^{3s} = 0,\]
\[2\alpha(s)e^{2s} + 3\beta(s)e^{3s} = e^{5s}.\]

Solving these simultaneous equations, we get
\[\alpha(s) = -e^{3s}, \quad \beta(s) = e^{2s},\]
and so
\[x(t, s) = e^{3t}e^{2s} - e^{2t}e^{3s}.\]

\(\triangle\)

Theorem 5.18 If \(u\) and \(v\) are linearly independent solutions of \(Lx = 0\), then the Cauchy function for \(Lx = 0\) is given by
\[x(t, s) = \frac{\begin{vmatrix} u(s) & v(s) \\ u(t) & v(t) \end{vmatrix}}{p(s) \begin{vmatrix} u(s) & v(s) \\ u'(s) & v'(s) \end{vmatrix}},\]
for \(t, s \in I.\)
Proof Since $u$ and $v$ are linearly independent solutions of $Lx = 0$, the Wronskian of these two solutions is never zero, by Corollary 5.10, so we can define

$$y(t, s) := \frac{\begin{vmatrix} u(s) & v(s) \\ u(t) & v(t) \end{vmatrix}}{p(s) \begin{vmatrix} u(s) & v(s) \\ u'(s) & v'(s) \end{vmatrix}},$$

for $t, s \in I$. Since for each fixed $s$ in $I$, $y(\cdot, s)$ is a linear combination of $u$ and $v$, $y(\cdot, s)$ is a solution of $Lx = 0$. Also, $y(s, s) = 0$ and $y'(s, s) = \frac{1}{p(s)}$ so we have by the uniqueness of solutions of initial value problems that $y(t, s) = x(t, s)$ for $t \in I$ for each fixed $s \in I$, which gives the desired result.

Example 5.19 Use Theorem 5.18 to find the Cauchy function for

$$(p(t)x')' = 0.$$

Let $a \in I$; then $u(t) := 1$, $v(t) := \int_a^t \frac{1}{p(\tau)} d\tau$ define linearly independent solutions of $(p(t)x')' = 0$. Hence by Theorem 5.18, the Cauchy function is given by

$$x(t, s) = \frac{1}{p(s)} \begin{vmatrix} 1 & \int_a^s \frac{1}{p(\tau)} d\tau \\ 1 & \int_a^t \frac{1}{p(\tau)} d\tau \end{vmatrix} = \int_s^t \frac{1}{p(\tau)} d\tau.$$

Example 5.20 Use Theorem 5.18 to find the Cauchy function for

$$(e^{-5t}x')' + 6e^{-5t}x = 0.$$

From Example 5.17,

$$u(t) = e^{2t}, \quad v(t) = e^{3t}, \quad t \in \mathbb{R}$$

define (linearly independent) solutions of $(e^{-5t}x')' + 6e^{-5t}x = 0$ on $\mathbb{R}$. Hence by Theorem 5.18, the Cauchy function is given by

$$x(t, s) = \frac{\begin{vmatrix} e^{2s} & e^{3s} \\ e^{2t} & e^{3t} \end{vmatrix}}{e^{-5s} \begin{vmatrix} e^{2s} & 3s \\ 2e^{2s} & 3e^{3s} \end{vmatrix}} = e^{3t}e^{2s} - e^{2t}e^{3s}.$$
Theorem 5.21 Assume \( f(\cdot, \cdot) \) and the first-order partial derivative \( f_t(\cdot, \cdot) \) are continuous real-valued functions on \( I \times I \) and \( a \in I \). Then
\[
\frac{d}{dt} \int_a^t f(t, s) \, ds = \int_a^t f_t(t, s) \, ds + f(t, t),
\]
for \( t \in I \).

**Proof** Letting \( x = x(t) \) and \( y = y(t) \) in the appropriate places and using the chain rule of differentiation,
\[
\frac{d}{dt} \int_a^t f(t, s) \, ds = \frac{d}{dt} \int_a^x f(y, s) \, ds
= \frac{\partial}{\partial x} \left( \int_a^x f(y, s) \, ds \right) \frac{dx}{dt} + \frac{\partial}{\partial y} \left( \int_a^x f(y, s) \, ds \right) \frac{dy}{dt},
\]
then with \( x(t) = t \) and \( y(t) = t \) we get
\[
f(y, x) \frac{dx}{dt} + \int_a^x f_y(y, s) \, ds \frac{dy}{dt} = \int_a^t f_t(t, s) \, ds + f(t, t),
\]
for \( t \in I \). \( \square \)

In the next theorem we derive the important variation of constants formula.

**Theorem 5.22** (Variation of Constants Formula) Assume \( h \) is continuous on \( I \) and assume \( a \in I \). Then the solution of the initial value problem
\[
Lx = h(t), \quad x(a) = 0, \quad x'(a) = 0
\]
is given by
\[
x(t) = \int_a^t x(t, s)h(s) \, ds, \quad t \in I,
\]
where \( x(\cdot, \cdot) \) is the Cauchy function for \( Lx = 0 \).

**Proof** Let
\[
x(t) := \int_a^t x(t, s)h(s) \, ds,
\]
for \( t \in I \) and note that \( x(a) = 0 \). We will let
\[
x'(t, s) := x_t(t, s).
\]
Then
\[
x'(t) = \int_a^t x'(t, s)h(s) \, ds + x(t, t)h(t)
= \int_a^t x'(t, s)h(s) \, ds.
\]
Hence \( x'(a) = 0 \) and
\[
p(t)x'(t) = \int_a^t p(t)x'(t, s)h(s) \, ds.
\]
It follows that

\[(p(t)x'(t))' = \int_a^t (p(t)x'(t, s))' h(s) \, ds + p(t)x'(t, t)h(t)\]

\[= \int_a^t (p(t)x'(t, s))' h(s) \, ds + h(t).\]

Hence

\[Lx(t) = \int_a^t \left\{(p(t)x'(t, s))' + q(t)x(t, s)\right\} h(s) \, ds + h(t)\]

\[= \int_a^t Lx(t, s)h(s) \, ds + h(t)\]

\[= h(t),\]

for \(t \in I\). The uniqueness follows from Theorem 5.4. \(\Box\)

**Corollary 5.23** Assume \(h\) is continuous on \(I\) and assume \(a \in I\). The solution of the initial value problem

\[Lx = h(t),\]

\[x(a) = A, \quad x'(a) = B,\]

where \(A\) and \(B\) are constants, is given by

\[x(t) = u(t) + \int_a^t x(t, s)h(s) \, ds,\]

for \(t \in I\), where \(u\) is the solution of the IVP \(Lu = 0, \quad u(a) = A, \quad u'(a) = B,\)

and \(x(\cdot, \cdot)\) is the Cauchy function for \(Lx = 0\).

**Proof** Let

\[x(t) := u(t) + \int_a^t x(t, s)h(s) \, ds,\]

where \(u, x(\cdot, \cdot)\), and \(h\) are as in the statement of this theorem. Then

\[x(t) = u(t) + v(t),\]

if we let

\[v(t) := \int_a^t x(t, s)h(s) \, ds.\]

Then by Theorem 5.22,

\[Lv(t) = h(t), \quad t \in I,\]

\[v(a) = 0, \quad v'(a) = 0.\]

It follows that

\[x(a) = u(a) + v(a) = A,\]

and

\[x'(a) = u'(a) + v'(a) = B.\]

Finally, note that

\[Lx(t) = Lu(t) + Lv(t) = h(t),\]
for $t \in I$. □

We now give a simple example to illustrate the variation of constants formula.

**Example 5.24** Use the variation of constants formula to solve the IVP
\[
(e^{-5t}x')' + 6e^{-5t}x = e^t,
\]
\[
x(0) = 0, \quad x'(0) = 0.
\]

By Example 5.17 (or Example 5.20) we get that the Cauchy function for $(e^{-5t}x')' + 6e^{-5t}x = 0$ is given by
\[
x(t, s) = e^{3t}e^{2s} - e^{2t}e^{3s},
\]
for $t, s \in I$. Hence the desired solution is given by
\[
x(t) = \int_0^t x(t, s)h(s) \, ds
\]
\[
= \int_0^t (e^{3t}e^{2s} - e^{2t}e^{3s}) e^s \, ds
\]
\[
= e^{3t} \int_0^t e^{3s} \, ds - e^{2t} \int_0^t e^{4s} \, ds
\]
\[
= \frac{1}{4}e^{2t} - \frac{1}{3}e^{3t} + \frac{1}{12}e^{6t}.
\]

△

### 5.4 Sturm-Liouville Problems

In this section we will be concerned with the *Sturm-Liouville differential equation*
\[
(p(t)x')' + (\lambda r(t) + q(t))x = 0.
\]  
(5.4)

In addition to the standard assumptions on the coefficient functions $p$ and $q$ we assume throughout that the coefficient function $r$ is a real-valued continuous function on $I$ and $r(t) \geq 0$, but is not identically zero, on $I$. Note that equation (5.4) can be written in the form
\[
Lx = -\lambda r(t)x.
\]

In this section we will be concerned with the Sturm-Liouville problem (SLP)
\[
Lx = -\lambda r(t)x,
\]  
(5.5)
\[
\alpha x(a) - \beta x'(a) = 0,
\]  
(5.6)
\[
\gamma x(b) + \delta x'(b) = 0,
\]  
(5.7)

where $\alpha, \beta, \gamma, \delta$ are constants satisfying
\[
\alpha^2 + \beta^2 > 0, \quad \gamma^2 + \delta^2 > 0.
\]
Definition 5.25 We say $\lambda_0$ is an eigenvalue for the SLP (5.5)–(5.7) provided the SLP (5.5)–(5.7) with $\lambda = \lambda_0$ has a nontrivial solution $x_0$ (by a nontrivial solution we mean a solution that is not identically zero). We say that $x_0$ is an eigenfunction corresponding to $\lambda_0$ and we say that $\lambda_0, x_0$ is an eigenpair for the SLP (5.5)–(5.7).

Note that if $\lambda_0, x_0$ is an eigenpair for the SLP (5.5)–(5.7), then if $k$ is any nonzero constant, $\lambda_0, kx_0$ is also an eigenpair of the SLP (5.5)–(5.7). We say $\lambda_0$ is a simple eigenvalue of the SLP (5.5)–(5.7) provided there is only one linearly independent eigenfunction corresponding to $\lambda_0$.

Example 5.26 Find eigenpairs for the Sturm-Liouville problem

$$x'' = -\lambda x,$$  \hspace{1cm} (5.8)

$$x(0) = 0, \quad x(\pi) = 0.$$ \hspace{1cm} (5.9)

The form for a general solution of (5.8) is different for the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ so when we try to find eigenpairs for the SLP (5.8), (5.9) we will consider these three cases separately. First assume $\lambda < 0$. In this case let $\lambda = -\mu^2$, where $\mu > 0$. Then a general solution of (5.8) is defined by

$$x(t) = c_1 \cosh(\mu t) + c_2 \sinh(\mu t), \quad t \in [0, \pi].$$

To satisfy the first boundary condition $x(0) = 0$ we are forced to take $c_1 = 0$. The second boundary condition gives

$$x(\pi) = c_2 \sinh(\mu \pi) = 0,$$

which implies $c_2 = 0$ and hence when $\lambda < 0$ the SLP (5.8), (5.9) has only the trivial solution. Hence the SLP (5.8), (5.9) has no negative eigenvalues. Next we check to see if $\lambda = 0$ is an eigenvalue of the SLP (5.8), (5.9). In this case a general solution of (5.8) is

$$x(t) = c_1 t + c_2.$$

Since

$$x(0) = c_2 = 0,$$

we get $c_2 = 0$. Then the second boundary condition gives

$$x(\pi) = c_1 \pi = 0,$$

which implies $c_1 = 0$. Hence when $\lambda = 0$ the only solution of the boundary value problem (BVP) is the trivial solution and so $\lambda = 0$ is not an eigenvalue. Finally, assume $\lambda > 0$; then $\lambda = \mu^2 > 0$, where $\mu > 0$ and so a general solution of (5.8) in this case is given by

$$x(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t), \quad t \in [0, \pi].$$

The boundary condition $x(0) = 0$ implies that $c_1 = 0$. The boundary condition $x(\pi) = 0$ leads to the equation

$$c_2 \sin(\mu \pi) = 0.$$
This last equation is true for \( \mu = \mu_n := n, \ n = 1, 2, 3, \cdots \). It follows that the eigenvalues of the SLP (5.8), (5.9) are

\[
\lambda_n = n^2, \quad n = 1, 2, 3, \cdots,
\]

and corresponding eigenfunctions are defined by

\[
x_n(t) = \sin(nt), \quad n = 1, 2, 3, \cdots,
\]

for \( t \in [0, \pi] \). Hence

\[
\lambda_n = n^2, \quad x_n(t) = \sin(nt), \quad n = 1, 2, 3, \cdots
\]

are eigenpairs for (5.8), (5.9).

**Definition 5.27** Assume that \( r : [a, b] \to \mathbb{R} \) is continuous and \( r(t) \geq 0 \), but not identically zero, on \([a, b]\). We define the inner product with respect to the weight function \( r \) of the continuous functions \( x \) and \( y \) on \([a, b]\) by

\[
<x, y>_r = \int_a^b r(t)x(t)y(t) \, dt.
\]

We say that \( x \) and \( y \) are orthogonal with respect to the weight function \( r \) on the interval \([a, b]\) provided

\[
<x, y>_r = \int_a^b r(t)x(t)y(t) \, dt = 0.
\]

**Example 5.28** The functions defined by \( x(t) = t^2, \ y(t) = 4 - 5t \) for \( t \in [0, 1] \) are orthogonal with respect to the weight function defined by \( r(t) = t \) on \([0, 1]\).

This is true because

\[
<x, y>_r = \int_a^b r(t)x(t)y(t) \, dt = \int_0^1 t \cdot t^2 \cdot (4 - 5t) \, dt = \int_0^1 (4t^3 - 5t^4) \, dt = 0.
\]

**Theorem 5.29** All eigenvalues of the SLP (5.5)–(5.7) are real and simple. Corresponding to each eigenvalue there is a real-valued eigenfunction. Eigenfunctions corresponding to distinct eigenvalues of the SLP (5.5)–(5.7) are orthogonal with respect to the weight function \( r \) on \([a, b]\).

**Proof** Assume \( \lambda_1, x_1 \) and \( \lambda_2, x_2 \) are eigenpairs for the SLP (5.5)–(5.7). By the Lagrange identity (Theorem 5.6),

\[
x_1(t)Lx_2(t) - x_2(t)Lx_1(t) = \{p(t)w[x_1(t), x_2(t)]\}',
\]
for \( t \in [a, b] \). Hence
\[
(\lambda_1 - \lambda_2) r(t) x_1(t) x_2(t) = \{ p(t) w[x_1(t), x_2(t)] \}'
\]
for \( t \in [a, b] \). Integrating both sides from \( a \) to \( b \), we get
\[
(\lambda_1 - \lambda_2) < x_1, x_2 >_r = \{ p(t) w[x_1(t), x_2(t)] \}_a^b.
\]
Using the fact that \( x_1 \) and \( x_2 \) satisfy the boundary conditions (5.6) and (5.7), it can be shown (see Exercise 5.19) that
\[
w[x_1(t), x_2(t)](a) = w[x_1(t), x_2(t)](b) = 0.
\]
Hence we get that
\[
(\lambda_1 - \lambda_2) < x_1, x_2 >_r = 0. \tag{5.10}
\]
If \( \lambda_1 \neq \lambda_2 \), then we get that \( x_1, x_2 \) are orthogonal with respect to the weight function \( r \) on \( [a, b] \). Assume \( \lambda_0, x_0 \) is an eigenpair for the SLP (5.5)–(5.7). From Exercise 5.18 we get that \( \lambda_0, x_0 \) is an eigenpair for the SLP (5.5)–(5.7). Hence from (5.10) for the eigenpairs \( \lambda_0, x_0 \) and \( \lambda_0, x_0 \) we get
\[
(\lambda_0 - \lambda_0) < x_0, x_0 >_r = 0.
\]
It follows that \( \lambda_0 = \lambda_0 \), that is, \( \lambda_0 \) is real.

Next assume that \( \lambda_0 \) is an eigenvalue and \( x_1, x_2 \) are corresponding eigenfunctions. Since \( x_1, x_2 \) satisfy the boundary condition (5.6), we have by Exercise 5.19 that
\[
w[x_1(t), x_2(t)](a) = 0.
\]
Since we also know that \( x_1, x_2 \) satisfy the same differential equation \( Lx = \lambda_0 r(t)x \), we get \( x_1, x_2 \) are linearly dependent on \( [a, b] \). Hence all eigenvalues of the SLP (5.5)–(5.7) are simple.

Finally, let \( \lambda_0, x_0 = u + iv \), where \( u, v \) are real-valued functions on \( [a, b] \), be an eigenpair for the SLP (5.5)–(5.7). Earlier we proved that all eigenvalues are real, so \( \lambda_0 \) is real. It is easy to see that \( \lambda_0, x_0 = u + iv \) is an eigenpair implies that (since at least one of \( u, v \) is not identically zero) either \( u \) or \( v \) is a real-valued eigenfunction corresponding to \( \lambda_0 \).

In the next example we show how finding nontrivial solutions of a partial differential equation leads to solving a SLP.

**Example 5.30** (Separation of Variables) In this example we use separation of variables to show how we can get solutions of the two-dimensional Laplace’s equation
\[
u_{xx} + u_{yy} = 0.
\]
We look for solutions of the form
\[
u(x, y) = X(x)Y(y).
\]
From Laplace’s equation we get
\[
X''Y + XY'' = 0.
\]
Separating variables and assuming \( X(x) \neq 0, Y(y) \neq 0 \), we get
\[
\frac{X''}{-X} = \frac{Y''}{Y}.
\]
Since the left-hand side of this equation depends only on \( x \) and the right-hand side depends only on \( y \), we get that
\[
\frac{X''(x)}{-X(x)} = \frac{Y''(y)}{Y(y)} = \lambda,
\]
where \( \lambda \) is a constant. This leads to the Sturm-Liouville differential equations
\[
X'' = -\lambda X, \quad Y'' = \lambda Y. \tag{5.11}
\]
It follows that if \( X \) is a solution of the first differential equation in (5.11) and \( Y \) is a solution of the second equation in (5.11), then
\[
u(x, y) = X(x)Y(y)
\]
is a solution of Laplace’s partial differential equation. Also note that if we want \( u(x, y) = X(x)Y(y) \) to be nontrivial and satisfy the boundary conditions
\[
\alpha u(a, y) - \beta u_x(a, y) = 0, \quad \gamma u(b, y) + \delta u_x(b, y) = 0,
\]
then we would want \( X \) to satisfy the boundary conditions
\[
\alpha X(a) - \beta X'(a) = 0, \quad \gamma X(b) + \delta X'(b) = 0.
\]
\[\triangle\]

**Theorem 5.31** If \( q(t) \leq 0 \) on \([a, b]\), \( \alpha \beta \geq 0 \), and \( \gamma \delta \geq 0 \), then all eigenvalues of the SLP (5.5)–(5.7) are nonnegative.

**Proof** Assume \( \lambda_0 \) is an eigenvalue of the SLP (5.5)–(5.7). By Theorem 5.29 there is a real-valued eigenfunction \( x_0 \) corresponding to \( \lambda_0 \). Then
\[
(p(t)x_0'(t))' + (\lambda_0 r(t) + q(t)) x_0(t) = 0,
\]
for \( t \in [a, b] \). Multiplying both sides by \( x_0(t) \) and integrating from \( a \) to \( b \), we have
\[
\int_a^b x_0(t) (p(t)x_0'(t))' dt + \lambda_0 \int_a^b r(t)x_0^2(t) dt + \int_a^b q(t)x_0^2(t) dt = 0.
\]
Since \( q(t) \leq 0 \) on \([a, b]\),
\[
\lambda_0 \int_a^b r(t)x_0^2(t) dt \geq - \int_a^b x_0(t)[p(t)x_0'(t)]' dt.
\]
Integrating by parts, we get
\[
\lambda_0 \int_a^b r(t)x_0^2(t) dt \geq - \{p(t)x_0(t)x_0'(t)\}_a^b + \int_a^b [p(t) [x_0'(t)]^2 dt
\geq - \{p(t)x_0(t)x_0'(t)\}_a^b. \tag{5.12}
\]
Now we will use the fact that \( x_0 \) satisfies the boundary condition (5.6) to show that
\[
p(a)x_0(a)x_0'(a) \geq 0.
\]
If \( \beta = 0 \), then \( x_0(a) = 0 \) and consequently
\[
p(a)x_0(a)x_0'(a) = 0.
\]
On the other hand, if \( \beta \neq 0 \), then
\[
p(a)x_0(a)x_0'(a) = \frac{\alpha}{\beta} |x_0(a)|^2 \geq 0.
\]
Similarly, using \( x_0 \) satisfies the boundary condition (5.7) and the fact that \( \gamma \delta \geq 0 \), it follows that
\[
p(b)x_0(b)x_0'(b) \leq 0.
\]
It now follows from (5.12) that
\[
\lambda_0 \geq 0.
\]
\[\square\]

The following example (see page 179, [9]) is important in the study of the temperatures in an infinite slab, \( 0 \leq x \leq 1, -\infty < y < \infty \), where the left edge at \( x = 0 \) is insulated and surface heat transfer takes place at the the right edge \( x = 1 \) into a medium with temperature zero.

**Example 5.32** Find eigenpairs for the SLP
\[
X'' = -\lambda X, \quad (5.13)
\]
\[
X'(0) = 0, \quad hX(1) + X'(1) = 0, \quad (5.14)
\]
where \( h \) is a positive constant.

Since \( q(t) = 0 \leq 0 \) on \( [0, 1] \), \( \alpha \beta = 0 \geq 0 \), and \( \gamma \delta = h \geq 0 \), we have from Theorem 5.31 that all eigenvalues of the SLP (5.13), (5.14) are nonnegative. If \( \lambda = 0 \), then \( X(x) = c_1 x + c_2 \). Then \( X'(0) = 0 \) implies that \( c_1 = 0 \). Also,
\[
hX(1) + X'(1) = c_2 h = 0
\]
implies that \( c_2 = 0 \) and hence \( X \) is the trivial solution. Therefore, zero is not an eigenvalue of the SLP (5.13), (5.14). Next assume that \( \lambda = \mu^2 > 0 \), where \( \mu > 0 \); then a general solution of (5.13) is given by
\[
X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x), \quad x \in [0, 1].
\]
Then
\[
X'(0) = c_2 \mu = 0
\]
implies that \( c_2 = 0 \) and so
\[
X(x) = c_1 \cos(\mu x), \quad x \in [0, 1].
\]
It follows that
\[
hX(1) + X'(1) = c_1 h \cos(\mu) - c_1 \mu \sin(\mu) = 0.
\]
Hence we want to choose $\mu > 0$ so that
\[ \tan(\mu) = \frac{h}{\mu}. \]
Let $0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the positive numbers satisfying
\[ \tan(\mu_n) = \frac{h}{\mu_n}, \]
n = 1, 2, 3, \cdots; then
\[ \lambda_n = \mu_n^2 \]
are the eigenvalues and
\[ X_n(x) = \cos(\mu_n x) \]
are corresponding eigenfunctions.

We can also prove the following theorem. For this result and other results of this type, see Sagan [45].

**Theorem 5.33** The eigenvalues for the SLP (5.5)–(5.7) satisfy
\[ \lambda_1 < \lambda_2 < \lambda_3 < \cdots \]
and
\[ \lim_{n \to \infty} \lambda_n = \infty. \]
Furthermore, if $x_n$ is an eigenfunction corresponding to the eigenvalue $\lambda_n$, $n = 1, 2, 3, \cdots$, then $x_{n+1}$ has exactly $n$ zeros in $(a, b)$.

We end this section by briefly considering the periodic Sturm-Liouville problem (PSLP)
\[ Lx = -\lambda r(t)x, \quad (5.15) \]
\[ x(a) = x(b), \quad x'(a) = x'(b). \quad (5.16) \]
The following example is important (see Churchill and Brown [9]) in the study of the heat flow in a circular plate that is insulated on its two faces and where the initial temperature distribution is such that along each radial ray emanating from the center of the disk and having polar angle $\theta$, $-\pi \leq \theta \leq \pi$ the temperature is constant.

**Example 5.34** Find eigenpairs for the PSLP
\[ u'' = -\lambda u, \quad (5.17) \]
\[ u(-\pi) = u(\pi), \quad u'(-\pi) = u'(-\pi). \quad (5.18) \]
If $\lambda < 0$, let $\lambda = -\mu^2$, where $\mu > 0$. In this case a general solution of (5.17) is given by
\[ u(\theta) = c_1 \cosh(\mu \theta) + c_2 \sinh(\mu \theta), \quad \theta \in [-\pi, \pi]. \]
The first boundary condition implies
\[ c_1 \cosh(\mu \pi) - c_2 \sinh(\mu \pi) = c_1 \cosh(\mu \pi) + c_2 \sinh(\mu \pi), \]
which is equivalent to the equation
\[ 2c_2 \sinh(\mu \pi) = 0. \]
This implies that \( c_2 = 0. \) Hence \( u(\theta) = c_1 \cosh(\mu \theta) \). The second boundary condition gives
\[ -c_1 \mu \sinh(\mu \pi) = c_1 \mu \sinh(\mu \pi), \]
which implies that \( c_1 = 0. \) Hence there are no negative eigenvalues. Next assume \( \lambda = 0; \) then a general solution of (5.17) is
\[ u(\theta) = c_1 \theta + c_2, \quad \theta \in [-\pi, \pi]. \]
The first boundary condition gives us
\[ -c_1 \pi + c_2 = c_1 \pi + c_2, \]
which implies that \( c_1 = 0. \) Hence \( u(\theta) = c_2 \), which satisfies the second boundary condition. Hence \( \lambda_0 = 0 \) is an eigenvalue and \( u_0(\theta) = 1 \) defines a corresponding eigenfunction. Finally, consider the case when \( \lambda > 0 \). In this case we let \( \lambda = \mu^2 \), where \( \mu > 0. \) A general solution of (5.17) in this case is
\[ u(\theta) = c_1 \cos(\mu \theta) + c_2 \sin(\mu \theta), \quad \theta \in [-\pi, \pi]. \]
The first boundary condition gives us
\[ c_1 \cos(\mu \pi) - c_2 \sin(\mu \pi) = c_1 \cos(\mu \pi) + c_2 \sin(\mu \pi), \]
which is equivalent to the equation
\[ c_2 \sin(\mu \pi) = 0. \]
Hence the first boundary condition is satisfied if we take
\[ \mu = \mu_n := n, \]
\[ n = 1, 2, 3, \ldots \] It is then easy to check that the second boundary condition is also satisfied. Hence
\[ \lambda_n = n^2, \]
\[ n = 1, 2, 3, \ldots \] are eigenvalues and corresponding to each of these eigenvalues are two linearly independent eigenfunctions given by \( u_n(\theta) = \cos(n \theta) \), \( v_n(\theta) = \sin(n \theta) \), \( \theta \in [-\pi, \pi] \). \( \triangle \)

**Theorem 5.35** Assume \( p(a) = p(b) \); then eigenfunctions corresponding to distinct eigenvalues of the PSLP (5.15), (5.16) are orthogonal with respect to the weight function \( r \) on \( [a, b] \).

**Proof** Assume \( \lambda_1, x_1 \) and \( \lambda_2, x_2 \) are eigenpairs for the PSLP (5.15), (5.16), with \( \lambda_1 \neq \lambda_2 \). By the Lagrange identity (Theorem 5.6),
\[ x_1(t)Lx_2(t) - x_2(t)Lx_1(t) = \{p(t)w[x_1(t), x_2(t)]\}', \]
for \( t \in [a, b] \). It follows that
\[ (\lambda_1 - \lambda_2)r(t)x_1(t)x_2(t) = \{p(t)w[x_1(t), x_2(t)]\}'. \]
for \( t \in [a, b] \). After integrating from \( a \) to \( b \), we get
\[
(\lambda_1 - \lambda_2) < x_1, x_2>_r = \{p(t)w[x_1(t), x_2(t)]\}_a^b.
\]
Since
\[
\{p(t)w[x_1(t), x_2(t)]\}(b) = p(b)[x_1(b)x_2'(b) - x_2(b)x_1'(b)]
\]
\[
= p(a)[x_1(a)x_2'(a) - x_2(a)x_1'(a)]
\]
\[
= \{p(t)w[x_1(t), x_2(t)]\}(a),
\]
we get that
\[
(\lambda_1 - \lambda_2) < x_1, x_2>_r = 0.
\]
Since \( \lambda_1 \neq \lambda_2 \), we get the desired result
\[
<x_1, x_2>_r = 0.
\]
\[\square\]

### 5.5 Zeros of Solutions and Disconjugacy

The study of the zeros of nontrivial solutions of \( Lx = 0 \) is very important, as we will see later in this chapter. We now give some elementary facts concerning zeros of nontrivial solutions of \( Lx = 0 \). First note for any nonempty subinterval \( J \) of \( I \) there is a nontrivial solution with a zero in \( J \). To see this, let \( t_0 \in J \), and let \( x \) be the solution of the initial value problem \( Lx = 0, x(t_0) = 0, x'(t_0) = 1 \). If \( x \) is a differentiable function satisfying \( x(t_0) = x'(t_0) = 0 \), then we say \( x \) has a double zero at \( t_0 \), whereas if \( x(t_0) = 0, x'(t_0) \neq 0 \), we say \( x \) has a simple zero at \( t_0 \). Note that by the uniqueness theorem there is no nontrivial solution with a double zero at some point in \( I \). Hence all nontrivial solutions of \( Lx = 0 \) have only simple zeros in \( I \). If we consider the equation \( x'' + x = 0 \) and \( J \subset I \) is an interval of length less than \( \pi \), then no nontrivial solution has two zeros in \( I \). This leads to the following definition.

**Definition 5.36** We say that \( Lx = 0 \) is disconjugate on \( J \subset I \) provided no nontrivial solution of \( Lx = 0 \) has two or more zeros in \( J \).

**Example 5.37** It is easy to see that the following equations are disconjugate on the corresponding intervals \( J \).

(i) \( x'' + x = 0, \quad J = [0, \pi) \)

(ii) \( x'' = 0, \quad J = \mathbb{R} \)

(iii) \( x'' - 5x' + 6x = 0, \quad J = \mathbb{R} \)

\(\triangle\)

At the other extreme there are self-adjoint differential equations \( Lx = 0 \), where there are nontrivial solutions with infinitely many zeros in \( I \), as shown in the following example.

**Example 5.38**
(i) \( x(t) := \sin t \) defines a nontrivial solution of \( x'' + x = 0 \) with infinitely many zeros in \( \mathbb{R} \);
(ii) \( x(t) := \sin \frac{t}{2} \) defines a nontrivial solution of \( (t^2x')' + \frac{1}{t^2}x = 0 \) with infinitely many zeros in the bounded interval \( J := (0,1] \). 
\( \triangle \)

This leads to the following definition.

**Definition 5.39** We say that a nontrivial solution \( x \) of \( Lx = 0 \) is oscillatory on \( J \subset I \) provided \( x \) has infinitely many zeros in \( J \). If \( Lx = 0 \) has a nontrivial oscillatory solution on \( J \), then we say the differential equation \( Lx = 0 \) is oscillatory on \( J \). If \( Lx = 0 \) has no nontrivial oscillatory solution on \( J \), then we say the differential equation \( Lx = 0 \) is nonoscillatory on \( J \). If \( x \) is a solution of \( Lx = 0 \) that does not have infinitely many zeros in \( J \), we say \( x \) is nonoscillatory on \( J \).

Hence \( x'' + x = 0 \) is oscillatory on \( \mathbb{R} \) and \( (t^2x')' + \frac{1}{t^2}x = 0 \) is oscillatory on \( (0,1] \).

The next example will motivate the next theorem.

**Example 5.40** The self-adjoint differential equation \( x'' + x = 0 \), \( t \in I := \mathbb{R} \) has the functions defined by \( x(t) = \cos t \), \( y(t) = \sin t \) as linearly independent solutions on \( \mathbb{R} \). Note that the zeros of these solutions separate each other in \( \mathbb{R} \). 
\( \triangle \)

**Theorem 5.41** (Sturm Separation Theorem) If \( x, y \) are linearly independent solutions on \( I \) of the self-adjoint differential equation \( Lx = 0 \), then their zeros separate each other in \( I \). By this we mean that \( x \) and \( y \) have no common zeros and between any two consecutive zeros of one of these solutions there is exactly one zero of the other solution.

**Proof** Assume that \( x, y \) are linearly independent solutions on \( I \) of the self-adjoint differential equation \( Lx = 0 \). Then by Corollary 5.10

\[
w(t) := w[x(t),y(t)] \neq 0,
\]

for \( t \in I \). Assume \( x \) and \( y \) have a common zero in \( I \); that is, there is a \( t_0 \in I \) such that

\[
x(t_0) = y(t_0) = 0.
\]

But then

\[
w(t_0) = \begin{vmatrix} 0 & 0 \\ x'(t_0) & y'(t_0) \end{vmatrix} = 0,
\]

which is a contradiction. Next assume \( x \) has consecutive zeros at \( t_1 < t_2 \) in \( I \). We claim that \( y \) has a zero in \( (t_1, t_2) \). Assume to the contrary that \( y(t) \neq 0 \) for \( t \in (t_1, t_2) \). Then without loss of generality we can assume that \( y(t) > 0 \) on the closed interval \( [t_1, t_2] \). Also without loss of generality we can assume that \( x(t) > 0 \) on \( (t_1, t_2) \). But then

\[
w(t_1) = \begin{vmatrix} 0 & y(t_1) \\ x'(t_1) & y'(t_1) \end{vmatrix} = -y(t_1)x'(t_1) < 0,
\]
and
\[
\begin{vmatrix}
0 & y(t_2) \\
x'(t_2) & y'(t_2)
\end{vmatrix} = -y(t_2)x'(t_2) > 0.
\]
Hence by the intermediate value theorem there is a point \( t_3 \in (t_1, t_2) \) such that
\[
w(t_3) = 0,
\]
which is a contradiction. By interchanging \( x \) and \( y \) in the preceding argument we get that between any two consecutive zeros of \( y \) there has to be a zero of \( x \). It follows that the zeros of \( x \) and \( y \) have to separate each other in \( I \). □

Note that it follows from the Sturm separation theorem that either all nontrivial solutions of \( Lx = 0 \) are oscillatory on \( I \) or all nontrivial solutions are nonoscillatory on \( I \).

**Definition 5.42** An interval \( I \) is said to be a *compact interval* provided it is a closed and bounded interval.

**Theorem 5.43** If \( J \) is a compact subinterval of \( I \), then \( Lx = 0 \) is nonoscillatory on \( J \).

**Proof** To the contrary, assume that \( Lx = 0 \) is oscillatory on a compact subinterval \( J \) of \( I \). Then there is a nontrivial solution \( x \) of \( Lx = 0 \) with infinitely many zeros in \( J \). It follows that there is an infinite sequence \( \{t_n\} \) of distinct points contained in \( J \) such that
\[
x(t_n) = 0,
\]
for \( n = 1, 2, 3, \ldots \) and
\[
\lim_{n \to \infty} t_n = t_0,
\]
where \( t_0 \in I \). Without loss of generality we can assume that the sequence \( \{t_n\} \) is either strictly increasing or strictly decreasing with
\[
t_0 := \lim_{n \to \infty} t_n \in J.
\]
We will only consider the case where the sequence \( \{t_n\} \) is strictly increasing since the other case is similar. Since
\[
x(t_n) = x(t_{n+1}) = 0,
\]
we have by Rolle’s theorem that there is a \( t'_n \in (t_n, t_{n+1}) \) such that
\[
x'(t'_n) = 0.
\]
It follows that
\[
x(t_0) = \lim_{n \to \infty} x(t_n) = 0
\]
and
\[
x'(t_0) = \lim_{n \to \infty} x'(t'_n) = 0.
\]
But by the uniqueness theorem (Theorem 5.4), \( x(t_0) = 0, \ x'(t_0) = 0, \) implies that \( x \) is the trivial solution which is a contradiction. □
The following example shows that boundary value problems are not as nice as initial value problems.

**Example 5.44** Consider the conjugate BVP

\[ x'' + x = 0, \]
\[ x(0) = 0, \quad x(\pi) = B. \]

If \( B = 0 \), this BVP has infinitely solutions \( x(t) = c_1 \sin t \), where \( c_1 \) is a constant. On the other hand, if \( B \neq 0 \), then this BVP has no solutions. △

**Theorem 5.45** If \( Lx = 0 \) is disconjugate on \([a, b] \subset I\), then the conjugate BVP

\[ Lx = h(t), \]
\[ x(a) = A, \quad x(b) = B, \]

where \( A \) and \( B \) are given constants and \( h \) is a given continuous function on \([a, b]\), has a unique solution.

**Proof** Let \( u, v \) be linearly independent solutions of the homogeneous equation \( Lx = 0 \) and let \( z \) be a solution of the nonhomogeneous equation \( Lx = h(t) \). Then, by Theorem 5.14,

\[ x = c_1 u + c_2 v + z \]

is a general solution of \( Lx = h(t) \). Hence there is a solution of the given BVP iff there are constants \( c_1, c_2 \) such that

\[ c_1 u(a) + c_2 v(a) = A - z(a), \]
\[ c_1 u(b) + c_2 v(b) = B - z(b). \]

Hence our BVP has a unique solution iff

\[ \begin{vmatrix} u(a) & v(a) \\ u(b) & v(b) \end{vmatrix} \neq 0. \]

Assume

\[ \begin{vmatrix} u(a) & v(a) \\ u(b) & v(b) \end{vmatrix} = 0. \]

Then there are constants \( a_1, a_2 \), not both zero, such that

\[ a_1 u(a) + a_2 v(a) = 0, \]
\[ a_1 u(b) + a_2 v(b) = 0. \]

Let \( x := a_1 u + a_2 v; \) then \( x \) is a nontrivial solution of \( Lx = 0 \) satisfying

\[ x(a) = a_1 u(a) + a_2 v(a) = 0, \]
\[ x(b) = a_1 u(b) + a_2 v(b) = 0, \]

which contradicts that \( Lx = 0 \) is disconjugate on \([a, b]\). □

**Theorem 5.46** If \( Lx = 0 \) has a positive solution on \( J \subset I \), then \( Lx = 0 \) is disconjugate on \( J \). Conversely, if \( J \) is a compact subinterval of \( I \) and \( Lx = 0 \) is disconjugate on \( J \), then \( Lx = 0 \) has a positive solution on \( J \).
Proof Assume that $Lx = 0$ has a positive solution on $J \subset I$. It follows from the Sturm separation theorem that no nontrivial solution can have two zeros in $J$. Hence $Lx = 0$ is disconjugate on $J$.

Conversely, assume $Lx = 0$ is disconjugate on a compact subinterval $J$ of $I$. Let $a < b$ be the endpoints of $J$ and let $u, v$ be the solutions of $Lx = 0$ satisfying the initial conditions

$$u(a) = 0, \quad u'(a) = 1,$$

and

$$v(b) = 0, \quad v'(b) = -1,$$

respectively. Since $Lx = 0$ is disconjugate on $J$, we get that

$$u(t) > 0 \quad \text{on} \quad (a, b],$$

and

$$v(t) > 0 \quad \text{on} \quad [a, b).$$

It follows that

$$x := u + v$$

is a positive solution of $Lx = 0$ on $J = [a, b]$. □

The following example shows that we cannot remove the word compact in the preceding theorem.

Example 5.47 The differential equation

$$x'' + x = 0$$

is disconjugate on the interval $J = [0, \pi)$ but has no positive solution on $[0, \pi)$. △

The following example is a simple application of Theorem 5.46.

Example 5.48 Since $x(t) = e^{2t}, \ t \in \mathbb{R}$, defines a positive solution of

$$x'' - 5x' + 6x = 0,$$

on $\mathbb{R}$, this differential equation is disconjugate on $\mathbb{R}$. △

We will now be concerned with the two self-adjoint equations

$$\begin{align*}
(p_1(t)x')' + q_1(t)x & = 0, \\
(p_2(t)x')' + q_2(t)x & = 0.
\end{align*}$$

We always assume that the coefficient functions in these two equations are continuous on an interval $I$ and $p_i(t) > 0$ on $I$ for $i = 1, 2.$
Theorem 5.49 (Picone Identity) Assume $u$ and $v$ are solutions of (5.19) and (5.20), respectively, with $v(t) \neq 0$ on $[a, b] \subset I$. Then

$$\left\{ \left( \frac{u(t)}{v(t)} \right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)] \right\}^b_a$$

$$= \int_a^b [q_2(t) - q_1(t)]u^2(t) \, dt + \int_a^b [p_1(t) - p_2(t)][u'(t)]^2 \, dt$$

$$+ \int_a^b p_2(t)v^2(t) \left[ \left( \frac{u(t)}{v(t)} \right)' \right]^2 \, dt.$$

**Proof** Consider

$$\left\{ \left( \frac{u(t)}{v(t)} \right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)] \right\}'$$

$$= u(t)[p_1(t)u'(t)]' + p_1(t)[u'(t)]^2$$

$$- \frac{u^2(t)}{v(t)}[p_2(t)v'(t)]' - [p_2(t)v'(t)] \left( \frac{u^2(t)}{v(t)} \right)'$$

$$= -q_1(t)u^2(t) + p_1(t)[u'(t)]^2 + q_2(t)u^2(t)$$

$$- [p_2(t)v'(t)] \left( \frac{2v(t)u(t)u'(t) - u^2(t)v'(t)}{v^2(t)} \right)$$

$$= [q_2(t) - q_1(t)]u^2(t) + [p_1(t) - p_2(t)][u'(t)]^2$$

$$+ p_2(t) \left( \frac{v^2(t)[u'(t)]^2 - 2u(t)u'(t)v(t)v'(t) + u^2(t)[v'(t)]^2}{v^2(t)} \right)$$

$$= [q_2(t) - q_1(t)]u^2(t) + [p_1(t) - p_2(t)][u'(t)]^2$$

$$+ p_2(t)v^2(t) \left( \frac{v(t)u'(t) - u(t)v'(t)}{v^2(t)} \right)^2$$

$$= [q_2(t) - q_1(t)]u^2(t) + [p_1(t) - p_2(t)][u'(t)]^2 + p_2(t)v^2(t) \left[ \left( \frac{u(t)}{v(t)} \right)' \right]^2,$$

for $t \in [a, b]$. Integrating both sides from $a$ to $b$, we get the Picone identity (5.21).

We next use the Picone identity to prove the Sturm comparison theorem, which is a generalization of the Sturm separation theorem.

**Theorem 5.50** (Sturm Comparison Theorem) Assume $u$ is a solution of (5.19) with consecutive zeros at $a < b$ in $I$ and assume that

$$q_2(t) \geq q_1(t), \quad 0 < p_2(t) \leq p_1(t),$$

(5.22)

for $t \in [a, b]$. If $v$ is a solution of (5.20) and if for some $t \in [a, b]$ one of the inequalities in (5.22) is strict or if $u$ and $v$ are linearly independent on $[a, b]$, then $v$ has a zero in $(a, b)$. 
Proof Assume $u$ is a solution of (5.19) with consecutive zeros at $a < b$ in $I$. Assume the conclusion of this theorem is not true. That is, assume $v$ is a solution of (5.20) with $v(t) \neq 0, \quad t \in (a, b)$.

Assume $a < c < d < b$, then by the Picone identity (5.21) with $a$ replaced by $c$ and $b$ replaced by $d$ we have

$$
\int_c^d [q_2(t) - q_1(t)]u^2(t)\,dt + \int_c^d [p_1(t) - p_2(t)][u'(t)]^2\,dt \quad (5.23)
$$

$$
+ \int_c^d p_2(t)v^2(t) \left(\frac{u(t)}{v(t)}\right)^2 \,dt
$$

$$
= \left\{\left(\frac{u(t)}{v(t)}\right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)]\right\}_c^d.
$$

Using the inequalities in (5.22) and the fact that either one of the inequalities in (5.22) is strict at some point $t \in [a, b]$, or $u$ and $v$ are linearly independent on $[a, b]$, we get, letting $c \to a+$ and letting $d \to b-$,

$$
\lim_{c \to a+,\,d \to b-} \left\{\left(\frac{u(t)}{v(t)}\right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)]\right\}_c^d > 0.
$$

We would get a contradiction if the two limits in the preceding inequality are zero. We will only show this for the right-hand limit at $a$ (see Exercise 5.28 for the other case). Note that if $v(a) \neq 0$, then clearly

$$
\lim_{t \to a+} \left\{\left(\frac{u(t)}{v(t)}\right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)]\right\} = 0.
$$

Now assume that $v(a) = 0$; then since $v$ only has simple zeros, $v'(a) \neq 0$. Consider

$$
\lim_{t \to a+} \left\{\left(\frac{u(t)}{v(t)}\right) [p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t)]\right\}
$$

$$
= - \lim_{t \to a+} \frac{u^2(t)p_2(t)v'(t)}{v(t)}
$$

$$
= - \lim_{t \to a+} \frac{u^2(t)[p_2(t)v'(t)]' + 2u(t)u'(t)p_2(t)v'(t)}{v'(t)}
$$

$$
= 0,
$$

where we have used l’Hôpital’s rule \hfill \square

Corollary 5.51 Assume that

$$
q_2(t) \geq q_1(t), \quad 0 < p_2(t) \leq p_1(t),
$$

(5.24)

for $t \in J \subset I$. If (5.20) is disconjugate on $J$, then (5.19) is disconjugate on $J$. If (5.19) is oscillatory on $J$, then (5.20) is oscillatory on $J$. 


Proof Assume equation (5.19) is oscillatory on \( J \). Then there is a nontrivial solution \( u \) of equation (5.19) with infinitely many zeros in \( J \). Let \( v \) be a nontrivial solution of equation (5.20). If \( u \) and \( v \) are linearly dependent on \( J \), then \( v \) has the same zeros as \( u \) and it would follow that the differential equation (5.20) is oscillatory on \( J \). On the other hand, if \( u \) and \( v \) are linearly independent on \( J \), then by Theorem 5.50, \( v \) has at least one zero between each pair of consecutive zeros of \( u \) and it follows that the differential equation (5.20) is oscillatory on \( J \). The other part of this proof is similar (see Exercise 5.30).

Example 5.52 Show that if

\[
0 < p(t) \leq t^2, \quad \text{and} \quad q(t) \geq b > \frac{1}{4},
\]

for \( t \in [1, \infty) \), then the self-adjoint equation \( Lx = 0 \) is oscillatory on \( [1, \infty) \).

To see this we use Corollary 5.51. We compare \( Lx = 0 \) with the differential equation

\[
(t^2 x')' + bx = 0.
\]

Expanding this equation out, we get the Euler–Cauchy differential equation

\[
t^2 x'' + 2tx' + bx = 0.
\]

By Exercise 5.29 this equation is oscillatory since \( b > \frac{1}{4} \). Hence by Corollary 5.51 we get that \( Lx = 0 \) is oscillatory on \( [1, \infty) \). △

5.6 Factorizations and Recessive and Dominant Solutions

Theorem 5.53 (Polya Factorization) Assume \( Lx = 0 \) has a positive solution \( u \) on \( J \subset I \). Then, for \( x \in \mathbb{D} \),

\[
Lx(t) = \rho_1(t) \{ \rho_2(t)[\rho_1(t)x(t)]' \}',
\]

for \( t \in J \), where

\[
\rho_1(t) := \frac{1}{u(t)} > 0, \quad \text{and} \quad \rho_2(t) := p(t)u^2(t) > 0,
\]

for \( t \in J \).
**Proof** Assume \( u \) is a positive solution of \( Lx = 0 \) and assume that \( x \in \mathbb{D} \); then by the Lagrange identity,

\[
\begin{align*}
    u(t)Lx(t) &= \{p(t)w[u(t), x(t)]\}' \\
    &= \{p(t)[u(t)x'(t) - u'(t)x(t)]\}' \\
    &= \left\{ p(t)u^2(t) \frac{[u(t)x'(t) - u'(t)x(t)]'}{u^2(t)} \right\}' \\
    &= \left\{ p(t)u^2(t) \left[ x(t) \left( \frac{x(t)'}{u(t)} \right) \right]' \right\}' \\
    &= \left\{ \rho_2(t)[\rho_1(t)x(t)]' \right\}',
\end{align*}
\]

for \( t \in J \), which leads to the desired result. \( \square \)

We say that the differential equation

\[
\rho_1(t) \left\{ \rho_2(t)[\rho_1(t)x(t)]' \right\}' = 0
\]

is the Polya factorization of the differential equation \( Lx = 0 \). Note that by the proof of Theorem 5.53 we only need to assume that \( Lx = 0 \) has a solution \( u \) without zeros on \( J \) to get the factorization, but if \( u(t) < 0 \) on \( J \), then \( \rho_1(t) < 0 \) on \( J \). When \( \rho_i(t) > 0 \) on \( J \) for \( i = 1, 2 \) we call our factorization a Polya factorization; otherwise we just call it a factorization.

**Example 5.54** Find a Polya factorization of the differential equation \( Lx = x'' + x = 0 \) on \( J = (0, \pi) \). Here \( u(t) = \sin t, t \in J \), defines a positive solution on \( J = (0, \pi) \). By Theorem 5.53 a Polya factorization is

\[
Lx(t) = \frac{1}{\sin t} \left\{ \sin^2 t \left[ \frac{x(t)'}{\sin t} \right]' \right\}' = 0,
\]

for \( x \in \mathbb{D}, t \in (0, \pi) \). \( \triangle \)

The following example shows that Polya factorizations are not unique, and we will also indicate how this example motivates Theorem 5.58.

**Example 5.55** Find two Polya factorizations for the differential equation

\[
Lx = (e^{-6t}x')' + 8e^{-6t}x = 0. \tag{5.25}
\]

Expanding this differential equation out and then dividing by \( e^{-6t} \), we get the equivalent equation

\[
x'' - 6x' + 8x = 0.
\]

Note that \( u_1(t) := e^{4t} \) defines a positive solution of the differential equation (5.25) on \( \mathbb{R} \) [which implies that the differential equation (5.25) is disconjugate on \( \mathbb{R} \)] and hence from Theorem 5.53 we get the Polya factorization

\[
Lx(t) = \rho_1(t) \left\{ \rho_2(t)[\rho_1(t)x(t)]' \right\}' = 0,
\]
for \( x \in \mathbb{D}, \ t \in \mathbb{R} \), where
\[
\rho_1(t) = e^{-4t} \quad \text{and} \quad \rho_2(t) = e^{2t},
\]
for \( t \in \mathbb{R} \). Note that \( u_2(t) := e^{2t} \) also defines a positive solution of the differential equation (5.25) and hence from Theorem 5.53 we get the Polya factorization
\[
Lx(t) = \gamma_1(t) \{ \gamma_2(t)[\gamma_1(t)x(t)]' \}' = 0,
\]
for \( t \in \mathbb{R} \), where
\[
\gamma_1(t) = e^{-2t} \quad \text{and} \quad \gamma_2(t) = e^{-2t},
\]
for \( t \in \mathbb{R} \). Hence we have found two distinct factorizations of equation (5.25). Note in the first factorization we get
\[
\int_0^\infty \frac{1}{\rho_2(t)} \, dt = \int_0^\infty e^{-2t} \, dt < \infty,
\]
whereas in the second factorization we get
\[
\int_0^\infty \frac{1}{\gamma_2(t)} \, dt = \int_0^\infty e^{2t} \, dt = \infty.
\]
In the next theorem we show that under the hypothesis of Theorem 5.53 we can always get a factorization of \( Lx = 0 \) (called a Trench factorization), where the \( \gamma_2 \) in the Polya factorization satisfies
\[
\int_a^b \frac{1}{\gamma_2(t)} \, dt = \infty
\]
(usually we will want \( b = \infty \) as in the preceding example). We will see in Theorem 5.59 why we want this integral to be infinite.

\( \triangle \)

We get the following result from Theorem 5.53.

**Theorem 5.56 (Reduction of Order)** If \( u \) is a solution of \( Lx = 0 \) without zeros in \( J \subset I \) and \( t_0 \in J \), then
\[
v(t) := u(t) \int_{t_0}^t \frac{1}{p(\tau)u^2(\tau)} \, d\tau, \quad t \in J
\]
defines a second linearly independent solution on \( J \).

**Proof** Assume \( u \) is a solution of \( Lx = 0 \) without zeros in \( J \subset I \) and \( t_0 \in J \). Then by Theorem 5.53 we have the factorization
\[
Lx(t) = \frac{1}{u(t)} \left( p(t)u^2(t) \left( \frac{x(t)}{u(t)} \right)' \right)' = 0,
\]
for \( t \in J \). It follows from this that the solution \( v \) of the IVP
\[
p(t)u^2(t) \left( \frac{v}{u(t)} \right)' = 1, \quad v(t_0) = 0
\]
is a solution of $Lx = 0$ on $J$. It follows that

$$\left( \frac{v(t)}{u(t)} \right)' = \frac{1}{p(t)u^2(t)},$$

for $t \in J$. Integrating both sides from $t_0$ to $t$ and solving for $v(t)$, we get

$$v(t) = u(t) \int_{t_0}^{t} \frac{1}{p(\tau)u^2(\tau)} d\tau, \quad t \in J.$$

To see that $u$, $v$ are linearly independent on $J$, note that

$$w[u(t), v(t)] = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = \begin{vmatrix} u(t) & u(t) \int_{t_0}^{t} \frac{1}{p(\tau)u^2(\tau)} d\tau \\ u'(t) & u'(t) \int_{t_0}^{t} \frac{1}{p(\tau)u^2(\tau)} d\tau + \frac{1}{p(t)u(t)} \end{vmatrix} = \begin{vmatrix} u(t) & 0 \\ u'(t) & \frac{1}{p(t)u(t)} \end{vmatrix} = \frac{1}{p(t)} \neq 0,$$

for $t \in J$. Hence $u$, $v$ are linearly independent solutions of $Lx = 0$ on $J$. □

**Example 5.57** Given that $u(t) = e^t$ defines a solution, solve the differential equation

$$\left( \frac{1}{te^{2t}}x' \right)' + \left( \frac{1+t}{t^2e^{2t}} \right)x = 0,$$

$t \in I := (0, \infty)$.

By the reduction of order theorem (Theorem 5.56), a second linearly independent solution on $I = (0, \infty)$ is given by

$$v(t) = u(t) \int_{1}^{t} \frac{1}{p(s)u^2(s)} ds = e^t \int_{1}^{t} \frac{se^{2s}}{e^{2s}} ds = e^t \int_{1}^{t} s ds = \frac{1}{2} t^2 e^t - \frac{1}{2} e^t.$$

Hence a general solution is given by

$$x(t) = c_1 e^t + c_2 \left( \frac{1}{2} t^2 e^t - \frac{1}{2} e^t \right) = \alpha e^t + \beta t^2 e^t.$$

△
Theorem 5.58 (Trench Factorization) Assume $Lx = 0$ has a positive solution on $[a,b) \subset I$, where $-\infty < a < b \leq \infty$. Then there are positive functions $\gamma_i$, $i = 1,2$ on $[a,b)$ such that for $x \in D$

$$Lx(t) = \gamma_1(t) \{ \gamma_2(t)[\gamma_1(t)x(t)]' \}' ,$$

for $t \in [a,b)$, and

$$\int_a^b \frac{1}{\gamma_2(t)} \, dt = \infty .$$

Proof Assume $Lx = 0$ has a positive solution on $[a,b) \subset I$, where $-\infty < a < b \leq \infty$. Then by Theorem 5.53 the operator $L$ has a Polya factorization on $[a,b)$. That is, if $x \in D$, then

$$Lx(t) = \rho_1(t) \{ \rho_2(t)[\rho_1(t)x(t)]' \}' ,$$

for $t \in [a,b)$. Let

$$\alpha_i(t) = \frac{1}{\rho_i(t)} ,$$

$i = 1,2$ for $t \in [a,b)$. Then

$$Lx(t) = \frac{1}{\alpha_1(t)} \left\{ \frac{1}{\alpha_2(t)} \left[ \frac{x(t)}{\alpha_1(t)} \right]' \right\}' ,$$

for $x \in D$, $t \in [a,b)$. If

$$\int_a^b \alpha_2(t) \, dt = \infty$$

we have a Trench factorization and the proof is complete in this case. Now assume that

$$\int_a^b \alpha_2(t) \, dt < \infty .$$

In this case we let

$$\beta_1(t) := \alpha_1(t) \int_t^b \alpha_2(s) \, ds , \quad \beta_2(t) := \frac{\alpha_2(t)}{\left[ \int_t^b \alpha_2(s) \, ds \right]^2} ,$$

for $t \in [a,b)$. Then

$$\int_a^b \beta_2(t) \, dt = \lim_{c \to b^-} \int_a^c \frac{\alpha_2(t)}{\left[ \int_t^b \alpha_2(s) \, ds \right]^2} \, dt$$

$$= \lim_{c \to b^-} \left\{ \left[ \int_t^b \alpha_2(s) \, ds \right]^{-1} \right\}_a^c$$

$$= \infty .$$
Let \( x \in \mathbb{D} \) and consider

\[
\begin{align*}
\left\{ \frac{x(t)}{\beta_1(t)} \right\}' &= \left\{ \frac{x(t)}{\alpha_1(t)} \right\}' \left( \int_t^b \alpha_2(s) \, ds \right) \\
&= \int_t^b \alpha_2(s) \, ds \left[ \frac{x(t)}{\alpha_1(t)} \right]' - \frac{x(t)}{\alpha_1(t)} \left[ -\alpha_2(t) \right] \\
&= \int_t^b \alpha_2(s) \, ds \left[ \frac{x(t)}{\alpha_1(t)} \right]' + \frac{x(t)}{\alpha_1(t)} \left[ \alpha_2(t) \right] \\
&= \left[ \int_t^b \alpha_2(s) \, ds \right]^2.
\end{align*}
\]

Hence

\[
\frac{1}{\beta_2(t)} \left[ \frac{x(t)}{\beta_1(t)} \right]' = \left\{ \int_t^b \alpha_2(s) \, ds \right\} \left\{ \frac{1}{\alpha_2(t)} \left[ \frac{x(t)}{\alpha_1(t)} \right]' \right\} + \frac{x(t)}{\alpha_1(t)}
\]

and so

\[
\left\{ \frac{1}{\beta_2(t)} \left[ \frac{x(t)}{\beta_1(t)} \right]' \right\}' = \int_t^b \alpha_2(s) \, ds \left\{ \frac{1}{\alpha_2(t)} \left[ \frac{x(t)}{\alpha_1(t)} \right]' \right\}'.
\]

Finally, we get that

\[
\frac{1}{\beta_1(t)} \left\{ \frac{1}{\beta_2(t)} \left[ \frac{x(t)}{\beta_1(t)} \right]' \right\}' = \frac{1}{\alpha_1(t)} \left\{ \frac{1}{\alpha_2(t)} \left[ \frac{x(t)}{\alpha_1(t)} \right]' \right\} = Lx(t),
\]

for \( t \in [a, b) \). \( \square \)

We now can use the Trench factorization to prove the existence of recessive and dominant solutions at \( b \).

**Theorem 5.59** Assume \( Lx = 0 \) is nonoscillatory on \( [a, b) \subset I \), where \(-\infty < a < b \leq \infty\); then there is a solution \( u \), called a recessive solution at \( b \), such that if \( v \) is any second linearly independent solution, called a dominant solution at \( b \), then

\[
\lim_{t \to b^-} \frac{u(t)}{v(t)} = 0,
\]

\[
\int_{t_0}^b \frac{1}{p(t)u^2(t)} \, dt = \infty,
\]

and

\[
\int_{t_0}^b \frac{1}{p(t)v^2(t)} \, dt < \infty,
\]
for some $t_0 < b$ sufficiently close. Furthermore,

$$\frac{p(t)v'(t)}{v(t)} > \frac{p(t)u'(t)}{u(t)},$$

for $t < b$ sufficiently close. Also, the recessive solution is unique up to multiplication by a nonzero constant.

**Proof** Since $Lx = 0$ is nonoscillatory on $[a, b)$, there is a $c \in [a, b)$ such that $Lx = 0$ has a positive solution on $[c, b)$. It follows that the operator $L$ has a Trench factorization on $[c, b)$. That is, for any $x \in \mathbb{D}$,

$$Lx(t) = \gamma_1(t) \{ \gamma_2(t)[\gamma_1(t)x(t)]' \}',$$

for $t \in [c, b)$, where $\gamma_i(t) > 0$ on $[c, b)$, $i = 1, 2$ and

$$\int_c^b \frac{1}{\gamma_2(t)} dt = \infty.$$

Then from the factorization we get that

$$u(t) := \frac{1}{\gamma_1(t)},$$

defines a solution of $Lx = 0$ that is positive on $[c, b)$. Let $z$ be the solution of the IVP

$$\gamma_2(t) (\gamma_1(t)z)' = 1,$$

$$z(c) = 0.$$

Then from the factorization we get that $z$ is a solution of $Lx = 0$ that is given by

$$z(t) = \frac{1}{\gamma_1(t)} \int_c^t \frac{1}{\gamma_2(s)} ds,$$

for $t \in [c, d)$. Note that

$$\lim_{t \to b^-} \frac{u(t)}{z(t)} = \lim_{t \to b^-} \frac{1}{\gamma_1(t)} \int_c^t \frac{1}{\gamma_2(s)} ds$$

$$= \lim_{t \to b^-} \frac{1}{\int_c^t 1/\gamma_2(s) ds}$$

$$= 0.$$

Now let $v$ be any solution such that $u$ and $v$ are linearly independent; then

$$v(t) = c_1 u(t) + c_2 z(t),$$
where $c_2 \neq 0$. Then

$$
\lim_{t \to b^-} \frac{u(t)}{v(t)} = \lim_{t \to b^-} \frac{u(t)}{c_1 u(t) + c_2 z(t)} = \lim_{t \to b^-} \frac{u(t)}{c_1 \frac{u(t)}{z(t)} + c_2} = 0.
$$

Next consider, for $t \in [c, d)$,

$$
\left[ \frac{z(t)}{u(t)} \right]' = \frac{u(t)z'(t) - z(t)u'(t)}{u^2(t)} = \frac{w[u(t), z(t)]}{u^2(t)} = \frac{C}{p(t)u^2(t)},
$$

where $C$ is a nonzero constant. Integrating from $c$ to $t$, we get that

$$
\frac{z(t)}{u(t)} - \frac{z(c)}{u(c)} = C \int_c^t \frac{1}{p(t)u^2(t)} dt.
$$

Letting $t \to b^-$, we get that

$$
\int_c^b \frac{1}{p(t)u^2(t)} dt = \infty.
$$

Next let $v$ be a solution of $Lx = 0$ such that $u$ and $v$ are linearly independent. Pick $d \in [c, b)$ so that $v(t) \neq 0$ on $[d, b)$. Then consider

$$
\left[ \frac{u(t)}{v(t)} \right]' = \frac{v(t)u'(t) - u(t)v'(t)}{v^2(t)} = \frac{w[v(t), u(t)]}{v^2(t)} = \frac{D}{p(t)v^2(t)},
$$

where $D$ is a nonzero constant. Integrating from $d$ to $t$, we get that

$$
\frac{u(t)}{v(t)} - \frac{u(d)}{v(d)} = D \int_d^t \frac{1}{p(s)v^2(s)} ds.
$$

Letting $t \to b^-$, we get that

$$
\int_d^b \frac{1}{p(s)v^2(s)} ds < \infty.
$$

Finally, let $u$ be as before and let $v$ be a second linearly independent solution. Pick $t_0 \in [a, b)$ so that $v(t) \neq 0$ on $[t_0, b)$. The expression

$$
\frac{p(t)v'(t)}{v(t)}
$$
is the same if we replace \( v(t) \) by \(-v(t)\) so, without loss of generality, we can assume \( v(t) > 0 \) on \([t_0, b]\). For \( t \in [t_0, b)\), consider

\[
\frac{p(t)v'(t)}{v(t)} - \frac{p(t)u'(t)}{u(t)} = \frac{p(t)w[u(t), v(t)]}{u(t)v(t)}
\]

\[
= \frac{H}{u(t)v(t)},
\]

where \( H \) is a constant. It remains to show that \( H > 0 \). To see this note that

\[
\left[ \frac{v(t)}{u(t)} \right]' = \frac{u(t)v'(t) - v(t)u'(t)}{u^2(t)}
\]

\[
= \frac{w[u(t), v(t)]}{u^2(t)}
\]

\[
= \frac{H}{p(t)u^2(t)},
\]

for \( t \in [t_0, b)\). Since

\[
\lim_{t \to b^-} \frac{v(t)}{u(t)} = \infty,
\]

it follows that \( H > 0 \). The proof of the last statement in the statement of this Theorem is Exercise 5.36.

**Example 5.60** Find a recessive and dominant solution of

\[
x'' - 3x' + 2x = 0
\]

at \( \infty \) and show directly that the conclusions of Theorem 5.59 concerning these two solutions are true.

The self-adjoint form of this equation is

\[
(e^{-3t}x')' + 2e^{-3t}x = 0,
\]

so \( p(t) = e^{-3t} \) and \( q(t) = 2e^{-3t} \). Two solutions of this differential equation are \( e^t \) and \( e^{2t} \). It follows that our given differential equation is disconjugate on \( \mathbb{R} \), but we want to show directly that if we take \( u(t) = e^t \) and \( v(t) = e^{2t} \) then the conclusions of Theorem 5.59 concerning these two solutions are true. First note that

\[
\lim_{t \to \infty} \frac{u(t)}{v(t)} = \lim_{t \to \infty} \frac{e^t}{e^{2t}} = \lim_{t \to \infty} \frac{1}{e^t} = 0.
\]

Also,

\[
\int_0^\infty \frac{1}{p(t)u^2(t)}\,dt = \int_0^\infty e^t\,dt = \infty.
\]

In addition,

\[
\int_0^\infty \frac{1}{p(t)v^2(t)}\,dt = \int_0^\infty e^{-t}\,dt < \infty.
\]
Finally, consider
\[ \frac{p(t)v'(t)}{v(t)} = 2e^{-3t} > e^{-3t} = \frac{p(t)u'(t)}{u(t)}, \]
for \( t \in \mathbb{R}. \)

**Example 5.61** Find a recessive and dominant solution of
\[ x'' + x = 0 \]
at \( \pi \) and show directly that the conclusions of Theorem 5.59 concerning these two solutions are true.

Let \( u(t) := \sin t \) and \( v(t) := \cos t; \) then
\[ \lim_{t \to \pi^-} \frac{u(t)}{v(t)} = \lim_{t \to \pi^-} \tan t = 0. \]

Also,
\[ \int_{\frac{\pi}{2}}^{\pi} \frac{1}{p(t)u^2(t)} \, dt = \int_{\frac{\pi}{2}}^{\pi} \csc^2 t \, dt = \infty. \]

In addition,
\[ \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{p(t)v^2(t)} \, dt = \int_{\frac{2\pi}{3}}^{\pi} \sec^2 t \, dt < \infty. \]

Finally, consider
\[ \frac{p(t)v'(t)}{v(t)} = -\tan t > \cot t = \frac{p(t)u'(t)}{u(t)}, \]
for \( t \in (\frac{\pi}{2}, \pi). \)

**Theorem 5.62** *(Mammana Factorization)* Assume \( Lx = 0 \) has a solution \( u \) with \( u(t) \neq 0 \) on \( J \subset I. \) Then if \( x \in \mathbb{D}(J), \)
\[ Lx(t) = \left[ \frac{d}{dt} + \rho(t) \right] p(t) \left[ \frac{d}{dt} - \rho(t) \right] x(t), \]
for \( t \in J, \) where
\[ \rho(t) := \frac{u'(t)}{u(t)}, \]
for \( t \in J. \)

**Proof** Let \( x \in \mathbb{D}(J) \) and consider
\[ \left[ \frac{d}{dt} - \rho(t) \right] x(t) = \left[ \frac{d}{dt} - \frac{u'(t)}{u(t)} \right] x(t) = x'(t) - \frac{u'(t)x(t)}{u(t)}, \]
for \( t \in J \). Hence

\[
\left[ \frac{d}{dt} + \rho(t) \right] p(t) \left[ \frac{d}{dt} - \rho(t) \right] x(t) = \left[ \frac{d}{dt} + \rho(t) \right] \left[ p(t)x'(t) - p(t)u'(t) \frac{x(t)}{u(t)} \right]
\]

\[
= [p(t)x'(t)]' - [p(t)u'(t)]' \frac{x(t)}{u(t)} - p(t)u'(t) \frac{u(t)x'(t) - x(t)u'(t)}{u^2(t)}
\]

\[
+ p(t)x'(t) \frac{u'(t)}{u(t)} - p(t)u'(t) \frac{x(t)u'(t)}{u^2(t)}
\]

\[
= Lx(t),
\]

for \( t \in J \).

\begin{example}

Find a Mammana factorization of the equation \( Lx = x'' + x = 0 \) on \( I = (0, \pi) \).

Here \( u(t) = \sin t \) is a positive solution on \( I = (0, \pi) \). By Theorem 5.62 the Mammana factorization is

\[
Lx(t) = \left[ \frac{d}{dt} + \cot t \right] \left[ \frac{d}{dt} - \cot t \right] x(t) = 0,
\]

for \( t \in (0, \pi) \).
\end{example}

### 5.7 The Riccati Equation

Assume throughout this section that \( p, q \) are continuous on an interval \( I \) and that \( p(t) > 0 \) on \( I \). We define the Riccati operator \( R : C^1(I) \to C(I) \) by

\[
Rz(t) = z'(t) + q(t) + \frac{1}{p(t)} z^2(t),
\]

for \( t \in I \). The nonlinear (see Exercise 5.38) first-order differential equation

\[
Rz = z' + q(t) + \frac{1}{p(t)} z^2 = 0
\]

is called the \textit{Riccati differential equation}. This Riccati equation can be written in the form \( z' = f(t, z) \), where \( f(t, z) := -q(t) - \frac{z^2}{p(t)} \). Since \( f(t, z) \) and \( f_z(t, z) = -\frac{2z}{p(t)} \) are continuous on \( I \times \mathbb{R} \), we have by Theorem 1.3 that solutions of IVPs in \( I \times \mathbb{R} \) for the Riccati equation \( Rz = 0 \) exist, are unique, and have maximal intervals of existence. The following example shows that solutions of the Riccati equation do not always exist on the whole interval \( I \). Recall that by Theorem 5.4 solutions of the self-adjoint differential equation \( Lx = 0 \) always exist on the whole interval \( I \).

\begin{example}

The solution of the IVP

\[
Rz = z' + 1 + z^2 = 0, \quad z(0) = 0
\]

\end{example}
is given by
\[ z(t) = -\tan t, \]
for \( t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \). In this example \( p(t) = q(t) = 1 \) are continuous on \( \mathbb{R} \) and \( p(t) > 0 \) on \( \mathbb{R} \) so \( I = \mathbb{R} \), but the solution of the preceding IVP only exists on \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). \( \triangle \)

The following theorem gives a relationship between the self-adjoint operator \( L \) and the Riccati operator \( R \).

**Theorem 5.65 (Factorization Theorem)** Assume \( x \in \mathbb{D}(J) \) and \( x(t) \neq 0 \) on \( J \subset I \); then if
\[ z(t) := \frac{p(t)x'(t)}{x(t)}, \quad t \in J, \]
then
\[ Lx(t) = x(t)Rz(t), \quad t \in J. \]

**Proof** Assume \( x \in \mathbb{D}(J) \), \( x(t) \neq 0 \) on \( J \), and make the Riccati substitution
\[ z(t) := \frac{p(t)x'(t)}{x(t)}, \quad t \in J. \]
Then
\[
x(t)Rz(t) = x(t)\left[ z'(t) + q(t) + \frac{z^2(t)}{p(t)} \right] \\
= x(t)z'(t) + q(t)x(t) + \frac{z^2(t)x(t)}{p(t)} \\
= x(t)\left( \frac{p(t)x'(t)}{x(t)} \right)' + q(t)x(t) + \frac{z^2(t)x(t)}{p(t)} \\
= x(t)\frac{x(t)\left[ p(t)x'(t) \right]' - p(t)xx'(t)x'(t)}{x^2(t)} + q(t)x(t) + \frac{x(t)z^2(t)}{p(t)} \\
= (p(t)x'(t))' - \frac{x(t)z^2(t)}{p(t)} + q(t)x(t) + \frac{x(t)z^2(t)}{p(t)} \\
= (p(t)x'(t))' + q(t)x(t) \\
= Lx(t),
\]
for \( t \in J. \)

**Theorem 5.66** Assume \( J \subset I \). The self-adjoint differential equation \( Lx = 0 \) has a solution \( x \) without zeros on \( J \) iff the Riccati equation \( Rz = 0 \) has a solution \( z \) that exists on \( J \). These two solutions satisfy
\[ z(t) = \frac{p(t)x'(t)}{x(t)}, \quad t \in J. \]
**Proof** Assume that $x$ is a solution of the self-adjoint differential equation $Lx = 0$ such that $x(t) \neq 0$ on $J$. Make the Riccati substitution $z(t) := \frac{p(t)x'(t)}{x(t)}$, for $t \in J$; then by Theorem 5.65

$$x(t)Rz(t) = Lx(t) = 0, \quad t \in J.$$ 

It follows that $z$ is a solution of the Riccati equation $Rz = 0$ on $J$.

Conversely, assume that the Riccati equation $Rz = 0$ has a solution $z$ on the interval $J$. Let $x$ be the solution of the IVP

$$x' = \frac{z(t)}{p(t)}x, \quad x(t_0) = 1,$$

where $t_0 \in J$. Then $x(t) > 0$ on $J$. Furthermore, since

$$z(t) = \frac{p(t)x'(t)}{x(t)},$$

we get from Theorem 5.65 that

$$Lx(t) = x(t)Rz(t) = 0, \quad t \in J,$$

and so $x$ is a solution of $Lx = 0$ without zeros in $J$. \hfill \square

**Example 5.67** Solve the Riccati differential equation (DE)

$$z' + \frac{2}{t^4} + t^2z^2 = 0, \quad t > 0.$$ 

Here

$$p(t) = \frac{1}{t^2}, \quad q(t) = \frac{2}{t^4}.$$ 

Hence the corresponding self-adjoint DE is

$$\left(\frac{1}{t^2}x'\right)' + \frac{2}{t^4}x = 0.$$ 

Expanding this equation out we obtain the Euler–Cauchy DE

$$t^2x'' - 2tx' + 2x = 0.$$ 

A general solution to this equation is

$$x(t) = At + Bt^2.$$ 

Therefore,

$$z(t) = \frac{p(t)x'(t)}{x(t)}$$

$$= \frac{A}{At + Bt^2} + \frac{2B}{t^3}.$$ 

When $B = 0$ we get the solution

$$z(t) = \frac{1}{t^3}.$$
When $B \neq 0$ we can divide the numerator and denominator by $\frac{B}{t^2}$ to get that
\[
z(t) = \frac{C + 2t}{Ct^3 + t^4},
\]
where $C$ is a constant is a solution. \hfill \triangle

Next we state and prove the most famous oscillation result for the self-adjoint differential equation $Lx = 0$.

**Theorem 5.68** (Fite-Wintner Theorem) Assume $I = [a, \infty)$. If
\[
\int_a^\infty \frac{1}{p(t)} dt = \int_a^\infty q(t) dt = \infty,
\]
then the self-adjoint differential equation $Lx = 0$ is oscillatory on $[a, \infty)$.

**Proof** Assume that the differential equation $Lx = 0$ is nonoscillatory on $[a, \infty)$. Then by Theorem 5.59 there is a dominant solution $v$ of $Lx = 0$ and a number $T \in [a, \infty)$ such that $v(t) > 0$ on $[T, \infty)$ and
\[
\int_T^\infty \frac{1}{p(t)v^2(t)} dt < \infty. \tag{5.26}
\]
If we make the Riccati substitution
\[
z(t) := \frac{p(t)v'(t)}{v(t)}, \quad t \in [T, \infty),
\]
then by Theorem 5.66, $z$ is a solution of the Riccati equation $Rz = 0$ on $[T, \infty)$. Hence
\[
z'(t) = -q(t) - \frac{z^2(t)}{p(t)} \leq -q(t), \quad t \in [T, \infty).
\]
Integrating from $T$ to $t$, we get
\[
z(t) - z(T) \leq - \int_T^t q(s) \, ds.
\]
It follows that
\[
\lim_{t \to \infty} z(t) = -\infty.
\]
Pick $T_1 \geq T$ so that
\[
z(t) = \frac{p(t)v'(t)}{v(t)} < 0, \quad t \in [T_1, \infty).
\]
It follows that
\[
v'(t) < 0, \quad t \in [T_1, \infty),
\]
and therefore $v$ is decreasing on $[T_1, \infty)$. Then we get that for $t \geq T_1$,
\[
\int_{T_1}^t \frac{1}{p(s)v^2(s)} \, ds \geq \frac{1}{v^2(T_1)} \int_{T_1}^t \frac{1}{p(s)} \, ds,
\]
which implies that
\[
\int_{T_1}^\infty \frac{1}{p(t)v^2(t)} dt = \infty,
\]
which contradicts (5.26). □

**Example 5.69** Since $\int_1^\infty \frac{1}{t} \, dt = \infty$, we get from the Fite-Wintner theorem (Theorem 5.68) that for any positive constant $\alpha$ the differential equation

\[
(tx')' + \frac{\alpha}{t}x = 0
\]

is oscillatory on $[1, \infty)$. △

The following lemma is important in the calculus of variations, which we study briefly in Section 5.8.

**Lemma 5.70** (Completing the Square Lemma) Assume $[a, b] \subset I$ and that $\eta : [a, b] \to \mathbb{R}$ is continuous and assume $\eta'$ is piecewise continuous on $[a, b]$. If $z$ is a solution of the Riccati equation $Rz = 0$ on $[a, b]$, then

\[
[z(t)\eta^2(t)]' = \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\}
\]

\[
- \left[ \sqrt{p(t)}\eta'(t) - \frac{1}{\sqrt{p(t)}}\eta(t)z(t) \right]^2,
\]

for those $t \in [a, b]$ where $\eta'(t)$ exists.

**Proof** Assume that $\eta$ is as in the statement of this theorem and $z$ is a solution of the Riccati equation $Rz = 0$ on $[a, b]$, and consider

\[
[z(t)\eta^2(t)]' = z'(t)\eta^2(t) + 2z(t)\eta(t)\eta'(t)
\]

\[
= [-q(t) - \frac{1}{p(t)}z^2(t)]\eta^2(t) + 2z(t)\eta(t)\eta'(t)
\]

\[
= \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\}
\]

\[
- \left\{ p(t)[\eta'(t)]^2 - 2z(t)\eta(t)\eta'(t) + \frac{1}{p(t)}\eta^2(t)z^2(t) \right\}
\]

\[
= \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\}
\]

\[
- \left[ \sqrt{p(t)}\eta'(t) - \frac{1}{\sqrt{p(t)}}\eta(t)z(t) \right]^2,
\]

for those $t \in [a, b]$ where $\eta'(t)$ exists. □

We next define a quadratic functional $Q$ that is very important in the calculus of variations.

**Definition 5.71** First we let $A$ be the set of all functions $\eta : [a, b] \to \mathbb{R}$ such that $\eta$ is continuous, $\eta'$ is piecewise continuous on $[a, b]$, and $\eta(a) = \eta(b) = 0$. Then we define the quadratic functional $Q : A \to \mathbb{R}$ by

\[
Q[\eta] = \int_a^b \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\} \, dt.
\]

We call $A$ the set of admissible functions.
**Definition 5.72** We say that $Q$ is positive definite on $A$ provided $Q \eta \geq 0$ for all $\eta \in A$ and $Q \eta = 0$ iff $\eta = 0$.

**Theorem 5.73** Let $[a, b] \subset I$. Then $Lx = 0$ is disconjugate on $[a, b]$ iff the quadratic functional $Q$ is positive definite on $A$.

**Proof** Assume $Lx = 0$ is disconjugate on $[a, b]$. Then, by Theorem 5.46, $Lx = 0$ has a positive solution on $[a, b]$. It then follows from Theorem 5.66 that the Riccati equation $Rz = 0$ has a solution $z$ that exists on $[a, b]$. Let $\eta \in A$; then by the completing the square lemma (Lemma 5.70)

$$[z(t)\eta^2(t)]' = \{p(t)[\eta'(t)]^2 - q(t)\eta^2(t)\} - \left(\sqrt{p(t)}\eta'(t) - \frac{1}{\sqrt{p(t)}}\eta(t)z(t)\right)^2,$$

for those $t \in [a, b]$ where $\eta'(t)$ exists. Integrating from $a$ to $b$ and simplifying, we get

$$Q\eta = [z(t)\eta^2(t)]_a^b + \int_a^b \left(\sqrt{p(t)}\eta'(t) - \frac{1}{\sqrt{p(t)}}\eta(t)z(t)\right)^2 dt$$

$$= \int_a^b \left(\sqrt{p(t)}\eta'(t) - \frac{1}{\sqrt{p(t)}}\eta(t)z(t)\right)^2 dt \geq 0.$$  

Also note that $Q\eta = 0$ only if

$$\eta'(t) = \frac{z(t)}{p(t)}\eta(t),$$

for $t \in [a, b]$. Since we also know that $\eta(a) = 0$, we get $Q\eta = 0$ only if $\eta = 0$. Hence we have that $Q$ is positive definite on $A$.

Conversely, assume $Q$ is positive definite on $A$. We will show that $Lx = 0$ is disconjugate on $[a, b]$. Assume not; then there is a nontrivial solution $x$ of $Lx = 0$ with

$$x(c) = 0 = x(d),$$

where $a \leq c < d \leq b$. Define the function $\eta$ by

$$\eta(t) = \begin{cases} 0, & \text{if } t \in [a, c] \\ x(t), & \text{if } t \in [c, d] \\ 0, & \text{if } t \in [d, b]. \end{cases}$$

Since $\eta \in A$ and $\eta \neq 0$ (the zero function), we get that

$$Q[\eta] > 0.$$
But, using integration by parts,

\[
Q[\eta] = \int_a^b \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\} \, dt
\]

\[
= \int_c^d \left\{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \right\} \, dt
\]

\[
= [p(t)\eta(t)\eta'(t)]_c^d - \int_c^d [p(t)\eta'(t)]'\eta(t) \, dt - \int_c^d q(t)\eta^2(t) \, dt
\]

\[
= -\int_c^d Lx(t)x(t) \, dt
\]

\[
= 0,
\]

which is a contradiction. \(\square\)

**Corollary 5.74** Assume (5.19) is disconjugate on \(J \subset I\) and

\[ q_1(t) \geq q_2(t) \quad \text{and} \quad 0 < p_1(t) \leq p_2(t), \]

for \(t \in J\); then (5.20) is disconjugate on \(J\).

**Proof** It suffices to show that (5.20) is disconjugate on any closed subinterval \([a,b] \subset J\). Define \(Q_i : A \to \mathbb{R}\) by

\[
Q_i[\eta] := \int_a^b \left\{ p_i(t)[\eta'(t)]^2 - q_i(t)\eta^2(t) \right\} \, dt,
\]

for \(i = 1, 2\). Assume (5.19) is disconjugate on \(J \subset I\). Then, by Theorem 5.73, \(Q_1\) is positive definite on \(A\). Note that

\[
Q_2[\eta] = \int_a^b \left\{ p_2(t)[\eta'(t)]^2 - q_2(t)\eta^2(t) \right\} \, dt
\]

\[
\geq \int_a^b \left\{ p_1(t)[\eta'(t)]^2 - q_1(t)\eta^2(t) \right\} \, dt
\]

\[
= Q_1[\eta],
\]

for all \(\eta \in A\). It follows that \(Q_2\) is positive definite on \(A\) and so by Theorem 5.73 the self-adjoint equation (5.20) is disconjugate on \([a,b]\). \(\square\)

Similar to the proof of Corollary 5.74, we can prove the following Corollary (see Exercise 5.40).

**Corollary 5.75** Assume (5.19) and (5.20) are disconjugate on \(J \subset I\) and

\[ p(t) = \lambda_1 p_1(t) + \lambda_2 p_2(t) \quad \text{and} \quad q(t) = \lambda_1 q_1(t) + \lambda_2 q_2(t), \]

for \(t \in J\), where \(\lambda_1, \lambda_2 \geq 0\), not both zero; then \(Lx = 0\) is disconjugate on \(J\).
Definition 5.76 Assume $Lx = 0$ is nonoscillatory on $[a, b) \subset I$, where $-\infty < a < b \leq \infty$. We say $z_m$ is the minimum solution of the corresponding Riccati differential equation $Rz = 0$ for $t < b$, sufficiently close, provided $z_m$ is a solution of the Riccati equation $Rz = 0$ for $t < b$, sufficiently close, and if $z$ is any solution of the Riccati equation $Rz = 0$ for $t < b$, sufficiently close, then

$$z_m(t) \leq z(t),$$

for all $t < b$, sufficiently close.

Theorem 5.77 Assume $Lx = 0$ is nonoscillatory on $[a, b) \subset I$, where $-\infty < a < b \leq \infty$; then the minimum solution $z_m$ of the corresponding Riccati differential equation $Rz = 0$ for $t < b$, sufficiently close, is given by

$$z_m(t) := \frac{p(t)u'(t)}{u(t)},$$

where $u$ is a recessive solution of $Lx = 0$ at $b$. Assume that the solution $z_m$ exists on $[t_0, b)$; then for any $t_1 \in [t_0, b)$ the solution $z$ of the IVP

$$Rz = 0, \quad z(t_1) = z_1,$$

where $z_1 < z_m(t_1)$ has right maximal interval of existence $[t_1, \omega)$, where $\omega < b$ and

$$\lim_{t \to \omega^-} z(t) = -\infty.$$

Proof Recall that by Theorem 5.66 if $J \subset I$, then the self-adjoint differential equation $Lx = 0$ has a solution $x$ without zeros on $J$ iff the Riccati equation $Rz = 0$ has a solution $z$ that exists on $J$ and these two solutions are related by

$$z(t) = \frac{p(t)x'(t)}{x(t)}, \quad t \in J.$$

By Theorem 5.59 the differential equation $Lx = 0$ has a recessive solution $u$ at $b$ and for any second linearly independent solution $v$ it follows that

$$z_m(t) := \frac{p(t)u'(t)}{u(t)} < \frac{p(t)v'(t)}{v(t)},$$

for $t < b$ sufficiently close. Note that if $x$ is a solution of $Lx = 0$ that is linearly dependent of $u$, then $x = ku$, where $k \neq 0$ and

$$\frac{p(t)x'(t)}{x(t)} = \frac{p(t)ku'(t)}{ku(t)} = \frac{p(t)u'(t)}{u(t)} = z_m(t),$$

for $t < b$ sufficiently close (this also shows that no matter what recessive solution you pick at $b$ the $z_m$ is the same and hence $z_m$ is well defined).

Finally, assume $z_m$ is a solution on $[t_0, b)$ and assume $z$ is the solution of the IVP

$$Rz = 0, \quad z(t_1) = z_1,$$
where \( z_1 < z_m(t_1) \). From the uniqueness of solutions of IVPs, \( z(t) < z_m(t) \) on the right maximal interval of existence \([t_1, \omega)\) of the solution \( z \). It follows that \( \omega < b \) and, using Theorem 1.3, we get that

\[
\lim_{t \to \omega} z(t) = -\infty. \quad \Box
\]

**Example 5.78** Find the minimum solution \( z_m \) of the Riccati differential equation

\[
Rz = z' + \frac{2}{t^4} + t^2 z^2 = 0
\]

that exists for all sufficiently large \( t \).

From Example 5.67, \( u(t) = t \) is a recessive solution of the corresponding self-adjoint differential equation \( Lx = 0 \) at \( \infty \). It follows from Theorem 5.77 that the minimum solution of the Riccati equation for all sufficiently large \( t \) is

\[
z_m(t) = \frac{p(t)u'(t)}{u(t)} = \frac{1}{t^3}.
\]

\( \triangle \)

**Example 5.79** Find the minimum solution \( z_m \) of the Riccati differential equation

\[
Rz = z' + 1 + z^2 = 0
\]

that exists for all \( t < \pi \) sufficiently close.

The corresponding self-adjoint differential equation is \( x'' + x = 0 \). A recessive solution of this equation at \( \pi \) is \( u(t) = \sin t \). It follows from Theorem 5.77 that the minimum solution at \( \pi \) is

\[
z_m(t) = \frac{p(t)u'(t)}{u(t)} = \cot t,
\]

for \( t \in (0, \pi) \). It can be shown that a general solution of this Riccati equation is given by

\[
z(t) = \cot(t - \alpha),
\]

where \( \alpha \) is an arbitrary constant. Let \( t_1 \in (0, \pi) \) and let \( z \) be the solution of the IVP

\[
Rz = 0, \quad z(t_1) = z_1.
\]

Note that if the constant

\[
z_1 < z_m(t_1) = \cot(t_1),
\]

then there is an \( \alpha_1 \in \left(-\frac{\pi}{2}, 0\right) \) such that

\[
z(t) = \cot(t - \alpha_1),
\]

which has right maximal interval of existence \([t_1, \alpha_1 + \pi)\). Note that \( \alpha_1 + \pi < \pi \) and

\[
\lim_{t \to (\alpha_1 + \pi)-} z(t) = -\infty.
\]

On the other hand, if \( z_1 \geq z_m(t_1) = \cot(t_1) \), then there is an \( \alpha_2 \in [0, \frac{\pi}{2}) \) such that

\[
z(t) = \cot(t - \alpha_2) \geq z_m(t),
\]
for $t < \pi$ sufficiently close. \hfill \triangle$

In the next theorem we summarize some of the important results (with one improvement) in this chapter. Ahlbrandt and Hooker [1] called this theorem the Reid roundabout theorem to honor W. T. Reid’s work [43], [42] in this area. There have been numerous research papers written concerning various versions of this theorem. See, for example, Ahlbrandt and Peterson [2] for a general discrete Reid roundabout theorem.

**Theorem 5.80** (Reid Roundabout Theorem) The following are equivalent:

(i) $Lx = 0$ is disconjugate on $[a, b]$.

(ii) $Lx = 0$ has a positive solution on $[a, b]$.

(iii) $Q$ is positive definite on $A$.

(iv) The Riccati differential inequality $Rw \leq 0$ has a solution that exists on the whole interval $[a, b]$.

**Proof** By Theorem 5.46, (i) and (ii) are equivalent. By Theorem 5.73, (i) and (iii) are equivalent. By Theorem 5.66, we get that (ii) implies (iv). It remains to prove that (iv) implies (i). To this end, assume that the Riccati inequality $Rw \leq 0$ has a solution $w$ that exists on the whole interval $[a, b]$. Let

$$h(t) := Rw(t), \quad t \in [a, b];$$

then $h(t) \leq 0$, for $t \in [a, b]$ and

$$w'(t) + (q(t) - h(t)) + \frac{w^2(t)}{p(t)} = 0,$$

for $t \in [a, b]$, but then from Theorem 5.66 we have that

$$(p(t)x')' + (q(t) - h(t))x = 0$$

has a solution without zeros in $[a, b]$ and hence is disconjugate on $[a, b]$. But since

$$q(t) - h(t) \geq q(t),$$

for $t \in [a, b]$, we get from the Sturm comparison theorem (Theorem 5.50) that $Lx = 0$ is disconjugate on $[a, b]$. \hfill \square

We end this section with another oscillation theorem for $Lx = 0$.

**Theorem 5.81** Assume $I = [a, \infty)$. If $\int_a^\infty \frac{1}{p(t)} dt = \infty$ and there is a $t_0 \geq a$ and a $u \in C^1[t_0, \infty)$ such that $u(t) > 0$ on $[t_0, \infty)$ and

$$\int_{t_0}^\infty \left[ q(t)u^2(t) - p(t)(u'(t))^2 \right] dt = \infty;$$

then $Lx = 0$ is oscillatory on $[a, \infty)$. 

We prove this theorem by contradiction. So assume $Lx = 0$ is nonoscillatory on $[a, \infty)$. By Theorem 5.59, $Lx = 0$ has a dominant solution $v$ at $\infty$ such that for $t_1 \geq a$, sufficiently large,

$$\int_{t_1}^{\infty} \frac{1}{p(t)v^2(t)} dt < \infty,$$

and we can assume that $v(t) > 0$ on $[t_1, \infty)$. Let $t_0$ and $u$ be as in the statement of this theorem. Let $T = \max\{t_0, t_1\}$; then let

$$z(t) := \frac{p(t)v'(t)}{v(t)}, \quad t \geq T.$$

Then by Lemma 5.70, we have for $t \geq T$

$$\{z(t)u^2(t)\}' = p(t)(u'(t))^2 - q(t)u^2(t)$$

$$- \left\{ \sqrt{p(t)u'(t)} - \frac{u(t)z(t)}{\sqrt{p(t)}} \right\}^2$$

$$\leq p(t)(u'(t))^2 - q(t)u^2(t),$$

for $t \geq T$. Integrating from $T$ to $t$, we get

$$z(t)u^2(t) \leq z(T)u^2(T) - \int_T^t \left[ q(t)u^2(t) - p(t)(u'(t))^2 \right] dt,$$

which implies that

$$\lim_{t \to \infty} z(t)u^2(t) = -\infty.$$

But then there is a $T_1 \geq T$ such that

$$z(t) = \frac{p(t)v'(t)}{v(t)} < 0,$$

for $t \geq T_1$. This implies that $v'(t) < 0$ for $t \geq T_1$ and hence $v$ is decreasing for $t \geq T_1$. But then

$$\int_{T_1}^{\infty} \frac{1}{p(s)} ds = v^2(T_1) \int_{T_1}^{\infty} \frac{1}{p(s)v^2(T_1)} ds$$

$$\leq v^2(T_1) \int_{T_1}^{\infty} \frac{1}{p(s)v^2(s)} ds < \infty,$$

which is a contradiction. \qed

**Example 5.82** Show that if $(a > 0)$

$$\int_a^{\infty} t^\alpha q(t) dt = \infty,$$

where $\alpha < 1$, then $x'' + q(t)x = 0$ is oscillatory on $[a, \infty)$.

We show that this follows from Theorem 5.81. First note that

$$\int_{a}^{\infty} \frac{1}{p(t)} dt = \int_{a}^{\infty} 1 dt = \infty.$$
Now let
\[ u(t) := t^{\alpha}, \]
and consider
\[ \int_{a}^{\infty} \{q(t)u^2(t) - p(t)[u'(t)]^2\} \, dt = \int_{a}^{\infty} \{t^{\alpha}q(t) - \frac{\alpha^2}{4}t^{\alpha-2}\} \, dt = \infty, \]
since \( \alpha < 1 \) implies
\[ \int_{a}^{\infty} t^{\alpha-2} \, dt < \infty. \]
Hence \( x'' + q(t)x = 0 \) is oscillatory on \([a, \infty)\) from Theorem 5.81. \( \triangle \)

5.8 Calculus of Variations

In this section we will introduce the calculus of variations and show how the previous material in this chapter is important in the calculus of variations.

Let’s look at the following example for motivation.

**Example 5.83** The problem is to find the curve \( x = x(t) \) joining the given points \((a, x_a)\) and \((b, x_b)\) such that \( x \) is continuous and \( x' \) and \( x'' \) are piecewise continuous on \([a, b]\) and the length of this curve is a minimum.

Using the formula for the length of a curve, we see that we want to minimize
\[ I[x] := \int_{a}^{b} \sqrt{1 + [x'(t)]^2} \, dt \]
subject to
\[ x(a) = x_a, \quad x(b) = x_b, \]
where \( x_a, \ x_b \) are given constants and \( x \) is continuous and \( x' \) and \( x'' \) are piecewise continuous on \([a, b]\). \( \triangle \)

We assume throughout this section that \( f = f(t, u, v) \) is a given continuous real-valued function on \([a, b] \times \mathbb{R}^2\) such that \( f \) has continuous partial derivatives up through the third order with respect to each of its variables on \([a, b] \times \mathbb{R}^2\). The simplest problem of the calculus of variations is to extremize
\[ I[x] := \int_{a}^{b} f(t, x(t), x'(t)) \, dt \]
subject to
\[ x(a) = x_a, \quad x(b) = x_b, \]
where \( x_a \) and \( x_b \) are given constants and \( x \) is continuous and \( x' \) and \( x'' \) are piecewise continuous on \([a, b]\). In this section we will let \( D \) be the set of all continuous \( x : [a, b] \to \mathbb{R} \) such that \( x' \) and \( x'' \) are piecewise continuous on \([a, b]\), with \( x(a) = x_a, \ x(b) = x_b \). We define a norm on \( D \) by
\[ \|x\| = \sup \{\max\{|x(t)|, |x'(t)|, |x''(t)|\}\}, \]
where the sup is over those \( t \in [a, b]\), where \( x'(t) \) and \( x''(t) \) exist. We then define the set of admissible variations \( A \) to be the set of all continuous
\( \eta : [a, b] \rightarrow \mathbb{R} \) such that \( \eta' \) and \( \eta'' \) are piecewise continuous on \([a, b]\) and \( \eta(a) = \eta(b) = 0 \). Note that if \( x \in D \) and \( \eta \in A \), then \( x + \epsilon \eta \in D \), for any number \( \epsilon \). Also, \( x, y \in D \) implies \( \eta := x - y \in A \).

**Definition 5.84** We say that

\[
I[x] := \int_{a}^{b} f(t, x(t), x'(t)) \, dt,
\]
subject to \( x \in D \) has a local minimum at \( x_0 \) provided \( x_0 \in D \) and there is a \( \delta > 0 \) such that if \( x \in D \) with \( \|x - x_0\| < \delta \), then

\[
I[x] \geq I[x_0],
\]
for all \( x \in D \) and equality holds only if \( x = x_0 \).

**Definition 5.85** Fix \( x_0 \in D \) and let the function \( h : \mathbb{R} \rightarrow \mathbb{R} \) be defined by

\[
h(\epsilon) := I[x_0 + \epsilon \eta],
\]
where \( \eta \in A \). Then we define the first variation \( J_1 \) of \( I \) along \( x_0 \) by

\[
J_1[\eta] = J_1[\eta; x_0] := h'(0), \quad \eta \in A.
\]

The second variation \( J_2 \) of \( I \) along \( x_0 \) is defined by

\[
J_2[\eta] = J_2[\eta; x_0] := h''(0), \quad \eta \in A.
\]

**Theorem 5.86** Let \( x_0 \in D \); then the first and second variations along \( x_0 \) are given by

\[
J_1[\eta] = \int_{a}^{b} \left\{ f_u(t, x_0(t), x'_0(t)) - \frac{d}{dt} [f_v(t, x_0(t), x'_0(t))] \right\} \eta(t) \, dt,
\]
and

\[
J_2[\eta] = \int_{a}^{b} \{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \} \, dt,
\]
respectively, for \( \eta \in A \), where

\[
p(t) = R(t), \quad q(t) = Q'(t) - P(t),
\]
where

\[
P(t) = f_{uu}(t, x_0(t), x'_0(t)), \quad Q(t) = f_{uv}(t, x_0(t), x'_0(t))
\]
and

\[
R(t) = f_{vv}(t, x_0(t), x'_0(t)),
\]
for those \( t \in [a, b] \) such that \( x'_0 \) and \( x''_0 \) are continuous.
Proof Fix $x_0 \in D$, let $\eta \in \mathcal{A}$, and consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(\epsilon) := I[x_0 + \epsilon \eta] = \int_a^b f(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \, dt.$$ 

Then

$$h'(\epsilon) = \int_a^b \left[ f_u(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta(t) \right] \, dt$$

$$+ \int_a^b \left[ f_v(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta'(t) \right] \, dt$$

$$= \int_a^b \left[ f_u(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta(t) \right] \, dt$$

$$+ \int_a^b \left[ f_v(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta(t) \right] \eta'(t) \, dt$$

$$- \int_a^b \frac{d}{dt} \left[ f_v(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \right] \eta(t) \, dt$$

$$= \int_a^b f_u(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \, dt$$

$$- \int_a^b \frac{d}{dt} \left[ f_v(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \right] \eta(t) \, dt,$$

where we have integrated by parts. It follows that

$$J_1[\eta] = h'(0) = \int_a^b \left\{ f_u(t, x_0(t), x_0'(t)) - \frac{d}{dt} [f_v(t, x_0(t), x_0'(t))] \right\} \eta(t) \, dt,$$

for $\eta \in \mathcal{A}$. Differentiating $h'(\epsilon)$, we get

$$h''(\epsilon) = \int_a^b f_{uu}(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta^2(t) \, dt$$

$$+ 2 \int_a^b f_{uv}(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t)) \eta(t) \eta'(t) \, dt$$

$$+ \int_a^b [f_{vv}(t, x_0(t) + \epsilon \eta(t), x_0'(t) + \epsilon \eta'(t))] \eta'(t)^2 \, dt.$$ 

Hence

$$J_2[\eta] = h''(0) = \int_a^b P(t) \eta^2(t) \, dt + 2 \int_a^b Q(t) \eta(t) \eta'(t) \, dt + \int_a^b R(t) \eta'(t)^2 \, dt,$$

where

$$P(t) = f_{uu}(t, x_0(t), x_0'(t)), \quad Q(t) = f_{uv}(t, x_0(t), x_0'(t)),$$

and

$$R(t) = f_{vv}(t, x_0(t), x_0'(t)).$$
But
\[ 2 \int_a^b Q(t)\eta(t)\eta'(t) \, dt = \int_a^b Q(t)[\eta^2(t)]' \, dt \]
\[ = [Q(t)\eta^2(t)]_a^b - \int_a^b Q'(t)\eta^2(t) \, dt \]
\[ = -\int_a^b Q'(t)\eta^2(t) \, dt, \]
where we have integrated by parts and used \( \eta(a) = \eta(b) = 0 \). It follows that
\[ J_2[\eta] = \int_a^b \{ p(t)(\eta'(t))^2 - q(t)\eta^2(t) \} \, dt, \]
where
\[ p(t) = R(t), \quad q(t) = Q'(t) - P(t), \]
for those \( t \in [a, b] \) where \( x'_0 \) and \( x''_0 \) are continuous.

Lemma 5.87 (Fundamental Lemma of the Calculus of Variations) Assume that \( g : [a, b] \to \mathbb{R} \) is piecewise continuous and
\[ \int_a^b g(t)\eta(t) \, dt = 0 \]
for all \( \eta \in A \); then \( g(t) = 0 \) for those \( t \in [a, b] \) where \( g \) is continuous.

Proof Assume that the conclusion of this theorem does not hold. Then there is a \( t_0 \in (a, b) \) such that \( g \) is continuous at \( t_0 \) and \( g(t_0) \neq 0 \). We will only consider the case where \( g(t_0) > 0 \) as the other case (see Exercise 5.46) is similar. Since \( g \) is continuous at \( t_0 \), there is a \( 0 < \delta < \min\{t_0 - a, b - t_0\} \) such that \( g(t) > 0 \) on \( [t_0 - \delta, t_0 + \delta] \). Define \( \eta \) by
\[ \eta(t) = \begin{cases} 0, & \text{if } t \in [a, t_0 - \delta) \cup (t_0 + \delta, b] \\ (t - t_0 + \delta)^2(t - t_0 - \delta)^2, & \text{if } t \in [t_0 - \delta, t_0 + \delta]. \end{cases} \]
Then \( \eta \in A \) and
\[ \int_a^b g(t)\eta(t) \, dt = \int_{t_0-\delta}^{t_0+\delta} (t - t_0 + \delta)^2(t - t_0 - \delta)^2 g(t) \, dt > 0, \]
which is a contradiction.

Theorem 5.88 (Euler-Lagrange Equation) Assume
\[ I[x] := \int_a^b f(t, x(t), x'(t)) \, dt, \]
subject to \( x \in D \) has a local extremum at \( x_0 \). Then \( x_0 \) satisfies the Euler-Lagrange equation
\[ f_u(t, x(t), x'(t)) - \frac{d}{dt} [f_v(t, x(t), x'(t))] = 0, \quad (5.28) \]
for those \( t \in [a, b] \) where \( x'_0 \) and \( x''_0 \) are continuous.
Proof Assume

\[ I[x] := \int_a^b f(t, x(t), x'(t)) \, dt, \]

subject to \( x \in D \) has a local extremum at \( x_0 \). This implies that for each \( \eta \in \mathcal{A} \), \( h \) has a local extremum at \( \epsilon = 0 \) and hence

\[ h'(0) = 0, \]

for each \( \eta \in \mathcal{A} \). Hence

\[ J_1[\eta] = \int_a^b \left\{ f_u(t, x_0(t), x_0'(t)) - \frac{d}{dt} [f_v(t, x_0(t), x_0'(t))] \right\} \eta(t) \, dt = 0, \]

for all \( \eta \in \mathcal{A} \). It follows from Lemma 5.87 that

\[ f_u(t, x_0(t), x_0'(t)) - \frac{d}{dt} [f_v(t, x_0(t), x_0'(t))] = 0, \]

for those \( t \in [a, b] \) where \( x_0' \) and \( x_0'' \) are continuous. \( \square \)

Example 5.89 Assume that we are given that

\[ I[x] := \int_1^e \left\{ t [x'(t)]^2 - \frac{1}{t} x^2(t) \right\} \, dt, \]

subject to \( x \in D \), where the boundary conditions are

\[ x(1) = 0, \quad x(e) = 1, \]

has a global minimum at \( x_0 \) and \( x_0 \in C^2[1, e] \). Find \( x_0 \).

In this example

\[ f(t, u, v) = tv^2 - \frac{u^2}{t} \]

and so

\[ f_u(t, u, v) = -2 \frac{u}{t}, \quad f_v(t, u, v) = 2tv. \]

It follows that the Euler-Lagrange equation is the self-adjoint equation

\[ (tx')' + \frac{1}{t} x = 0. \]

It follows that \( x_0 \) is the solution of the BVP

\[ (tx')' + \frac{1}{t} x = 0, \]

\[ x(1) = 0, \quad x(e) = 1. \]

Solving this BVP, we get

\[ x_0(t) = \frac{\sin(\log t)}{\sin 1}. \]

Definition 5.90 The Euler-Lagrange equation for the second variation \( J_2 \) along \( x_0 \) is called the Jacobi equation for the simplest problem of the calculus of variations.
The following theorem shows that the self-adjoint equation is very important in the calculus of variations.

**Theorem 5.91** The Jacobi equation for the simplest problem of the calculus of variations is a self-adjoint second-order differential equation $Lx = 0$.

**Proof** The second variation is given by

$$J_2[\eta] = \int_a^b \{ p(t)(\eta'(t))^2 - q(t)\eta^2(t) \} \, dt.$$ 

Let $F$ be defined by

$$F(t, u, v) := p(t)v^2 - q(t)u^2.$$ 

Then

$$F_u(t, u, v) = -2q(t)u, \quad F_v(t, u, v) = 2p(t)v.$$ 

It follows that the Jacobi equation is the self-adjoint equation $Lx = 0$, where $p$ and $q$ are given by (5.27).

**Theorem 5.92** (Legendre’s Necessary Condition) Assume the simplest problem of the calculus of variations has a local extremum at $x_0$. In the local minimum case

$$f_{vv}(t, x_0(t), x_0''(t)) \geq 0,$$ 

for those $t \in [a, b]$ such that $x'_0$ and $x''_0$ are continuous. In the local maximum case

$$f_{vv}(t, x_0(t), x_0''(t)) \leq 0,$$ 

for those $t \in [a, b]$ such that $x'_0$ and $x''_0$ are continuous.

**Proof** Assume

$$I[x] := \int_a^b f(t, x(t), x'(t)) \, dt,$$ 

subject to $x \in D$ has a local minimum at $x_0$. Then if $\eta \in A$ and $h : \mathbb{R} \to \mathbb{R}$ is defined as usual by

$$h(\epsilon) := I[x_0 + \epsilon \eta],$$

it follows that $h$ has a local minimum at $\epsilon = 0$. This implies that

$$h'(0) = 0, \quad \text{and} \quad h''(0) \geq 0,$$

for each $\eta \in A$. Since $h''(0) = J_2[\eta]$, we have by Theorem 5.86 that

$$J_2[\eta] = \int_a^b \{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \} \, dt \geq 0,$$
for all \( \eta \in A \). Now fix an \( s \) in \((a, b)\), where \( x'_0 \) and \( x''_0 \) are continuous. Let \( \epsilon > 0 \) be a constant so that \( a \leq s - \epsilon < s + \epsilon \leq b \), and \( p \) is continuous on \([s - \epsilon, s + \epsilon]\). Next define \( \eta_\epsilon \) by

\[
\eta_\epsilon(t) = \begin{cases} 
0, & a \leq t \leq s - \epsilon \\
t - s + \epsilon, & s - \epsilon \leq t \leq s \\
-t + s + \epsilon, & s \leq t \leq s + \epsilon \\
0, & s + \epsilon \leq t \leq b.
\end{cases}
\]

Since \( \eta_\epsilon \in A \),

\[
J_2[\eta_\epsilon] = \int_a^b \{ p(t)[\eta'_\epsilon(t)]^2 - q(t)\eta^2_\epsilon(t) \} \, dt \geq 0.
\]

It follows that

\[
\int_{s-\epsilon}^{s+\epsilon} p(t) \, dt \geq \int_{s-\epsilon}^{s+\epsilon} q(t)\eta^2_\epsilon(t) \, dt.
\]

Let \( M > 0 \) be a constant such that \( q(t) \geq -M \) for all those \( t \in [a, b] \) such that \( x'_0 \) and \( x''_0 \) are continuous, then, using the mean value theorem from calculus, we get that

\[
p(\xi_\epsilon)(2\epsilon) \geq -M\epsilon^2(2\epsilon),
\]

where \( s - \epsilon < \xi_\epsilon < s + \epsilon \). It follows that

\[
p(\xi_\epsilon) \geq -M\epsilon^2.
\]

Letting \( \epsilon \to 0^+ \), we get that

\[
p(s) \geq 0.
\]

It then follows that

\[
p(t) = f_{vv}(t, x_0(t), x'_0(t)) \geq 0,
\]

at all \( t \in (a, b) \), where \( x'_0 \) and \( x''_0 \) are continuous. A similar argument can be given for \( t = a \) and \( t = b \). The proof of the local maximum case is left to the reader (see Exercise 5.54).

\[\square\]

**Example 5.93** Show that

\[
I[x] := \int_0^2 \left[(\cos t)x^2(t) - 3(\sin t)x(t)x'(t) - (t - 1)(x'(t))^2\right] \, dt,
\]

for \( x \in D \) with boundary conditions \( x(0) = 1, \ x(2) = 2 \) has no local maximums or local minimums.

Here

\[
f(t, u, v) = (\cos t)u^2 - 3(\sin t)uv - (t - 1)v^2,
\]

and hence

\[
f_{vv}(t, u, v) = -2(t - 1),
\]

which changes sign on \([0, 2]\) and hence by Theorem 5.92, \( I \) has no local extrema in \( D \).
Theorem 5.94 (Weierstrass Integral Formula) Assume that $x$ is a solution of $Lx = 0$ on $[a, b]$, $\eta \in \mathcal{A}$, and $z = x + \eta$; then

$$Q[z] = Q[x] + Q[\eta].$$

**Proof** Assume that $x$ is a solution of $Lx = 0$ on $[a, b]$, $\eta \in \mathcal{A}$, and $z = x + \eta$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(\epsilon) := Q[x + \epsilon \eta].$$

By Taylor’s formula,

$$h(1) = h(0) + \frac{1}{1!} h'(0) + \frac{1}{2!} h''(\xi),$$

where $\xi \in (0, 1)$. We next find $h'(0)$ and $h''(\xi)$. First note that

$$h'(\epsilon) = 2 \int_a^b \{ p(t)[x'(t) + \epsilon \eta'(t)]\eta'(t) - q(t)[x(t) + \epsilon \eta(t)]\eta(t) \} \, dt.$$ 

It follows that

$$h'(0) = 2 \int_a^b \{ p(t)x'(t)\eta'(t) - q(t)x(t)\eta(t) \} \, dt$$

$$= 2p(t)x'(t)\eta(t)|_a^b - \int_a^b Lx(t)\eta(t) \, dt$$

$$= 0,$$

where we have integrated by parts. Next note that

$$h''(\epsilon) = 2 \int_a^b \{ p(t)[\eta'(t)]^2 - q(t)\eta^2(t) \} \, dt$$

$$= 2Q[\eta].$$

Since $h(1) = Q[z]$, $h(0) = Q[x]$, $h'(0) = 0$, and $h''(\xi) = 2Q[\eta]$, we get from (5.29) the Weierstrass integral formula

$$Q[z] = Q[x] + Q[\eta].$$

□

Theorem 5.95 Assume $Lx = 0$ is disconjugate on $[a, b]$ and let $x_0$ be the solution of the BVP

$$Lx = 0,$$

$$x(a) = x_a, \quad x(b) = x_b.$$ 

Then

$$Q[x] = \int_a^b \{ p(t)[x'(t)]^2 - q(t)x^2(t) \} \, dt,$$

subject to $x \in D$ has a proper global minimum at $x_0$. 


Proof Since $Lx = 0$ is disconjugate on $[a, b]$, the BVP
\[ Lx = 0, \]
\[ x(a) = x_a, \quad x(b) = x_b, \]
has a unique solution $x_0$ by Theorem 5.45. Let $x \in D$ and let
\[ \eta := x - x_0, \]
then $\eta \in \mathcal{A}$ and $x = x_0 + \eta$. Hence, by the Weierstrass integral formula (Theorem 5.94),
\[ Q[x] = Q[x_0] + Q[\eta]. \]
But $Lx = 0$ is disconjugate on $[a, b]$ implies (see Theorem 5.73) that $Q$ is positive definite on $\mathcal{A}$ and hence is positive definite on $\mathcal{A}$. Therefore,
\[ Q[x] \geq Q[x_0], \]
for all $x \in D$ and equality holds only if $x = x_0$. Hence $Q$ has a proper global minimum at $x_0$. \qed

Example 5.96 Find the minimum value of
\[ Q[x] = \int_0^1 \left\{ e^{-4t}[x'(t)]^2 - 4e^{-4t}x^2(t) \right\} dt \]
subject to
\[ x(0) = 1, \quad x(1) = 0. \]
Here
\[ p(t) = e^{-4t}, \quad q(t) = 4e^{-4t} \]
and so the corresponding self-adjoint equation is
\[ (e^{-4t}x')' + 4e^{-4t}x = 0. \]
Expanding this equation out, we get the equivalent differential equation
\[ x'' - 4x' + 4x = 0. \]
Since $x(t) = e^{2t}$ defines a positive solution of this differential equation on $\mathbb{R}$, we have by Theorem 5.46 that this differential equation is disconjugate on $\mathbb{R}$ and hence on $[0, 1]$. Hence, by Theorem 5.95, $Q$ has a proper global minimum at $x_0$, where $x_0$ is the solution of the BVP
\[ x'' - 4x' + 4x = 0, \]
\[ x(0) = 1, \quad x(1) = 0. \]
Solving this BVP, we get
\[ x_0(t) = e^{2t} - te^{2t}. \]
Hence, after some routine calculations, we get that the global minimum for $Q$ on $D$ is
\[ Q[x_0] = -1. \]
Many calculus of variation problems come from Hamilton’s principle, which we now briefly discuss. Assume an object has some forces acting on it. Let \( T \) be the object’s kinetic energy and \( V \) its potential energy. The Lagrangian is then defined by
\[
L = T - V.
\]
In general,
\[
L = L(t, x, x').
\]
The integral
\[
I[x] = \int_a^b L(t, x(t), x'(t)) \, dt
\]
is called the action integral. We now state Hamilton’s principle in the simple case of rectilinear motion.

**Hamilton’s Principle:** Let \( x_0(t) \) be the position of an object under the influence of various forces on an \( x \)-axis at time \( t \). If \( x_0(a) = x_a \) and \( x_0(b) = x_b \), where \( x_a \) and \( x_b \) are given, then the action integral assumes a stationary value at \( x_0 \).

In the next example we apply Hamilton’s principle to a very simple example.

**Example 5.97** Consider the spring problem, where the kinetic energy function \( T \) and the potential energy function \( V \) are given by
\[
T = \frac{1}{2}m[x']^2 \quad \text{and} \quad V = \frac{1}{2}kx^2,
\]
respectively. Assume that we know that \( x(0) = 0, x(\frac{\pi}{2}\sqrt{\frac{m}{k}}) = 1 \). Then the action integral is given by
\[
I[x] = \int_0^{\frac{\pi}{2}\sqrt{\frac{m}{k}}} \left\{ \frac{1}{2}m[x'(t)]^2 - \frac{1}{2}kx^2(t) \right\} \, dt.
\]
Applying Theorem 5.95, we get that the displacement \( x_0 \) of the mass is the solution of the BVP
\[
mx'' + kx = 0,
\]
\[
x(0) = 0, \quad x\left(\frac{\pi}{2}\sqrt{\frac{m}{k}}\right) = 1.
\]
Solving this BVP, we get
\[
x_0(t) = \sin\left(\sqrt{\frac{m}{k}}t\right).
\]
\[\quad \triangle\]

We end this section by considering the special case of the simplest problem of the calculus of variations where \( f(t, u, v) = f(u, v) \) is independent of \( t \). In this case \( x(t) \) solves the Euler–Lagrange equation on \([a, b]\) provided
\[
f_u(x(t), x'(t)) - \frac{d}{dt} [f_v(x(t), x'(t))] = 0, \quad t \in [a, b]. \tag{5.30}
\]
We now show that if \( x(t) \) is a solution of (5.30) on \([a, b]\), then \( x(t) \) solves the integrated form of the Euler–Lagrange equation

\[
f_v(x, x')x' - f(x, x') = C
\]

where \( C \) is a constant. To see this, assume \( x(t) \) is a solution of (5.30) on \([a, b]\) and consider

\[
\frac{d}{dt} [f_v(x(t), x'(t))x'(t) - f(x(t), x'(t))] = \frac{d}{dt} [f_v(x(t), x'(t))x'(t) + f_v(x(t), x'(t))x''(t)]
\]

\[
- f_u(x(t), x'(t))x'(t) - f_v(x(t), x'(t))x''(t)
\]

\[
x'(t) \left[ \frac{d}{dt} f_u(x(t), x'(t)) - f_u(x(t), x'(t)) \right]
\]

\[
= 0
\]

for \( t \in [a, b] \).

We now use the integrated form of the Euler–Lagrange form (5.31) to solve the following example.

**Example 5.98** Find the \( C^1 \) curve \( x = x(t) \) joining the given points \((a, x_a) \) and \((b, x_b) \), \( a < b, x_a > 0, x_b > 0 \) such that the surface area of the surface of revolution obtained by revolving the curve \( x = x(t), a \leq t \leq b \) about the \( x \)-axis is a minimum.

Here we want to minimize

\[
I[x] = \int_a^b 2\pi \sqrt{1 + (x'(t))^2} \, dt
\]

subject to

\[
x(a) = x_a, \quad x(b) = x_b.
\]

We now solve the integrated form of the Euler–Lagrange equation (5.31). In this case \( f(u, v) = 2\pi u \sqrt{1 + v^2} \) so

\[
f_u(u, v) = 2\pi \sqrt{1 + v^2}, \quad f_v(u, v) = 2\pi \frac{uv}{\sqrt{1 + v^2}}.
\]

Hence from (5.31) we get the differential equation

\[
\frac{x(x')^2}{\sqrt{1 + (x')^2}} - x \sqrt{1 + (x')^2} = A.
\]

Simplifying this equation we get

\[-x = A \sqrt{1 + (x')^2}.
\]

This can be rewritten in the form

\[
\frac{Ax'}{\sqrt{x^2 - A^2}} = 1.
\]
Integrating we get
\[ t = A \ln \left[ \frac{x + \sqrt{x^2 - A^2}}{A} \right] + B. \]

Solving this equation we get the catenaries
\[ x(t) = A \cosh \frac{t - B}{A}. \]

We then want to find \( A \) and \( B \) so that the boundary conditions \( x(a) = x_a, \)
\( x(b) = x_b \) are satisfied. It can be shown (not routine) that if we fix \( a, \)
x\( a > 0, \) and \( x_b > 0, \) then as we vary \( b \) there is a value \( b_0 \) such that if
\( 0 < b < b_0, \) then there are two solutions one of which renders a minimum.
For \( b = b_0 \) there is a single extremum and it renders a minimum. Finally
if \( b > b_0, \) then there is no extremum.

5.9 Green’s Functions

At the outset of this section we will be concerned with the Green’s functions
for a general two-point boundary value problem for the self-adjoint
differential equation \( Lx = 0. \) At the end of this section we will consider the
Green’s function for the periodic BVP.

First we consider the homogeneous boundary value problem
\[ Lx = 0, \quad (5.32) \]
\[ \alpha x(a) - \beta x'(a) = 0, \quad (5.33) \]
\[ \gamma x(b) + \delta x'(b) = 0, \quad (5.34) \]

where we always assume that \([a, b] \subset I, \alpha^2 + \beta^2 > 0, \) and \( \gamma^2 + \delta^2 > 0. \)

**Theorem 5.99** Assume the homogeneous BVP (5.32)–(5.34) has only the
trivial solution. Then the nonhomogeneous BVP
\[ Lu = h(t), \quad (5.35) \]
\[ \alpha u(a) - \beta u'(a) = A, \quad (5.36) \]
\[ \gamma u(b) + \delta u'(b) = B, \quad (5.37) \]

where \( A \) and \( B \) are constants and \( h \) is continuous on \([a, b], \) has a unique
solution.

**Proof** Assume the BVP (5.32)–(5.34) has only the trivial solution. Let
\( x_1, x_2 \) be linearly independent solutions of \( Lx = 0; \) then
\[ x(t) = c_1 x_1(t) + c_2 x_2(t) \]
is a general solution of \( Lx = 0. \) Note that \( x \) satisfies boundary conditions
(5.33) and (5.34) iff \( c_1, c_2 \) are constants such that the two equations
\[ c_1[\alpha x_1(a) - \beta x_1'(a)] + c_2[\alpha x_2(a) - \beta x_2'(a)] = 0, \quad (5.38) \]
\[ c_1[\gamma x_1(b) + \delta x_1'(b)] + c_2[\gamma x_2(b) + \delta x_2'(b)] = 0 \quad (5.39) \]
hold. Since we are assuming that the BVP (5.32)–(5.34) has only the trivial solution, it follows that the only solution of the system (5.38), (5.39) is
\[ c_1 = c_2 = 0. \]
Therefore, the determinant of the coefficients in the system (5.38), (5.39) is different than zero; that is,
\[
\begin{vmatrix}
\alpha x_1(a) - \beta x'_1(a) & \alpha x_2(a) - \beta x'_2(a) \\
\gamma x_1(b) + \delta x'_1(b) & \gamma x_2(b) + \delta x'_2(b)
\end{vmatrix} \neq 0. \tag{5.40}
\]
Now we will show that the BVP (5.35)–(5.37) has a unique solution. Let \( u_0 \) be a fixed solution of \( Lu = h(t) \); then a general solution of \( Lu = h(t) \) is given by
\[ u(t) = a_1 x_1(t) + a_2 x_2(t) + u_0(t). \]
It follows that \( u \) satisfies the boundary conditions (BCs) (5.36), (5.37) iff \( a_1, a_2 \) are constants satisfying the system of equations
\[
\begin{align*}
\alpha x_1(a) - \beta x'_1(a) + a_2[\alpha x_2(a) - \beta x'_2(a)] &= A - \alpha u_0(a) + \beta u'_0(a), \\
\gamma x_1(b) + \delta x'_1(b) + a_2[\gamma x_2(b) + \delta x'_2(b)] &= B - \gamma u_0(a) - \delta u'_0(a) \tag{5.41}
\end{align*}
\]
Since (5.40) holds, the system (5.41), (5.42) has a unique solution \( a_1, a_2 \). This implies that the BVP (5.35)–(5.37) has a unique solution. \( \square \)

In the next example we give a BVP of the type (5.32)–(5.34) that does not have just the trivial solution.

**Example 5.100** Find all solutions of the BVP
\[
(p(t)x')' = 0, \quad x'(a) = 0, \quad x'(b) = 0.
\]
This BVP is equivalent to a BVP of the form (5.32)–(5.34), where \( q(t) \equiv 0, \alpha = \gamma = 0, \beta \neq 0, \) and \( \delta \neq 0. \) A general solution of this differential equation is
\[ x(t) = c_1 + c_2 \int_a^t \frac{1}{p(s)} ds. \]
The boundary conditions lead to the equations
\[ x'(a) = c_2 \frac{1}{p(a)} = 0, \quad x'(b) = c_2 \frac{1}{p(b)} = 0. \]
Thus \( c_2 = 0 \) and there is no restriction on \( c_1. \) Hence for any constant \( c_1, x(t) = c_1 \) is a solution of our BVP. In particular, our given BVP has nontrivial solutions. \( \triangle \)

In the next theorem we give a necessary and sufficient condition for some boundary value problems of the form (5.32)–(5.34) to have only the trivial solution. The proof of this theorem is Exercise 5.57. Note that Theorem 5.101 gives the last statement in Example 5.100 as a special case.
Theorem 5.101 Let
\[
\rho := \alpha \gamma \int_a^b \frac{1}{p(s)} ds + \frac{\beta \gamma}{p(a)} + \frac{\alpha \delta}{p(b)}.
\]
Then the BVP
\[
(p(t)x')' = 0,
\]
\[
\alpha x(a) - \beta x'(a) = 0, \quad \gamma x(b) + \delta x'(b) = 0,
\]
has only the trivial solution iff \(\rho \neq 0\).

The function \(G(\cdot, \cdot)\) in the following theorem is called the Green’s function for the BVP (5.32)–(5.34).

Theorem 5.102 (Green’s Function for General Two-Point BVP) Assume the homogeneous BVP (5.32)–(5.34) has only the trivial solution. For each fixed \(s \in [a, b]\), let \(u(\cdot, s)\) be the solution of the BVP
\[
Lu = 0, \quad (5.43)
\]
\[
\alpha u(a, s) - \beta u'(a, s) = 0, \quad (5.44)
\]
\[
\gamma u(b, s) + \delta u'(b, s) = -\gamma x(b, s) - \delta x'(b, s), \quad (5.45)
\]
where \(x(\cdot, \cdot)\) is the Cauchy function for \(Lx = 0\). Define
\[
G(t, s) := \begin{cases} 
  u(t, s), & \text{if } a \leq t \leq s \leq b \\
  v(t, s), & \text{if } a \leq s \leq t \leq b,
\end{cases} \quad (5.46)
\]
where \(v(t, s) := u(t, s) + x(t, s)\), for \(t, s \in [a, b]\). Assume \(h\) is continuous on \([a, b]\); then
\[
x(t) := \int_a^b G(t, s) h(s) \, ds,
\]
for \(t \in [a, b]\), defines the unique solution of the nonhomogeneous BVP
\(Lx = h(t)\), (5.33), (5.34). Furthermore, for each fixed \(s \in [a, b]\), \(v(\cdot, s)\) is a solution of \(Lx = 0\) satisfying the boundary condition (5.34).

Proof The existence and uniqueness of \(u(t, s)\) is guaranteed by Theorem 5.99. Since \(v(t, s) := u(t, s) + x(t, s)\), we have for each fixed \(s\) that \(v(\cdot, s)\) is a solution of \(Lx = 0\). Using the boundary condition (5.45), it is easy to see that for each fixed \(s\), \(v(\cdot, s)\) satisfies the boundary condition (5.34). Let
\(G(t, s)\) be as in the statement of this theorem and consider
\[
x(t) = \int_a^b G(t, s)h(s) \, ds
\]
\[
= \int_a^t G(t, s)h(s) \, ds + \int_t^b G(t, s)h(s) \, ds
\]
\[
= \int_a^t v(t, s)h(s) \, ds + \int_t^b u(t, s)h(s) \, ds
\]
\[
= \int_a^t [u(t, s) + x(t, s)]h(s) \, ds + \int_t^b u(t, s)h(s) \, ds
\]
\[
= \int_a^b u(t, s)h(s) \, ds + \int_a^t x(t, s)h(s) \, ds
\]
\[
= \int_a^b u(t, s)h(s) \, ds + z(t),
\]
where, by the variation of constants formula (Theorem 5.22), \(z(t) := \int_a^t x(t, s)h(s) \, ds\) defines the solution of the IVP
\[
Lz = h(t), \quad z(a) = 0, \quad z'(a) = 0.
\]
Hence
\[
Lx(t) = \int_a^b Lu(t, s)h(s) \, ds + Lz(t)
\]
\[
= \int_a^b Lu(t, s)h(s) \, ds + h(t)
\]
\[
= h(t).
\]
Thus \(x\) is a solution of \(Lx = h(t)\). Note that
\[
\alpha x(a) - \beta x'(a) = \int_a^b [\alpha u(a, s) - \beta u'(a, s)] h(s) \, ds + \alpha z(a) - \beta z'(a)
\]
\[
= \int_a^b [\alpha u(a, s) - \beta u'(a, s)] h(s) \, ds
\]
\[
= 0,
\]
since for each fixed \(s \in [a, b]\), \(u(\cdot, s)\) satisfies the boundary condition (5.44). Therefore, \(x\) satisfies the boundary condition (5.33). It remains to show that \(x\) satisfies the boundary condition (5.34). Earlier in this proof we had that
\[
x(t) = \int_a^b u(t, s)h(s) \, ds + \int_a^t x(t, s)h(s) \, ds.
\]
It follows that
\[
x(t) = \int_a^b [v(t,s) - x(t,s)] h(s) \, ds + \int_t^b x(t,s) h(s) \, ds \\
= \int_a^b v(t,s) h(s) \, ds - \int_t^b x(t,s) h(s) \, ds \\
= \int_a^b v(t,s) h(s) \, ds + \int_t^b x(t,s) h(s) \, ds \\
= \int_a^b v(t,s) h(s) \, ds + w(t),
\]
where, by the variation of constants formula (Theorem 5.22), \( w(t) := \int_t^b x(t,s) h(s) \, ds \) defines the solution of the IVP
\[
Lw = h(t), \quad w(b) = 0, \quad w'(b) = 0.
\]
Hence
\[
\gamma x(b) + \delta x'(b) = \int_a^b [\gamma v(b,s) + \delta v'(b,s)] h(s) \, ds + \gamma w(b) + \delta w'(b) \\
= \int_a^b [\gamma v(b,s) + \delta v'(b,s)] h(s) \, ds \\
= 0,
\]
since for each fixed \( s \in [a,b] \), \( v(\cdot,s) \) satisfies the boundary condition (5.34). Hence \( x \) satisfies the boundary condition (5.34). \( \square \)

We then get the following corollary, whose proof is left as an exercise (Exercise 5.63).

**Corollary 5.103** Assume the BVP (5.32)–(5.34) has only the trivial solution. If \( h \) is continuous on \( [a,b] \), then the solution of the BVP
\[
Lx(t) = h(t), \\
\alpha x(a) - \beta x'(a) = A, \\
\gamma x(b) + \delta x'(b) = B,
\]
where \( A \) and \( B \) are constants, is given by
\[
x(t) = w(t) + \int_a^b G(t,s) h(s) \, ds,
\]
where \( G \) is the Green’s function for the BVP (5.32)–(5.34) and \( w \) is the solution of the BVP
\[
Lw = 0, \\
\alpha w(a) - \beta w'(a) = A, \quad \gamma w(b) + \delta w'(b) = B.
\]

In the next theorem we give another form of the Green’s function for the BVP (5.32)–(5.34) and note that the Green’s function is symmetric on the square \( [a,b]^2 \).
Theorem 5.104 Assume that the BVP (5.32)–(5.34) has only the trivial solution. Let \( \phi \) be the solution of the IVP

\[
L\phi = 0, \quad \phi(a) = \beta, \quad \phi'(a) = \alpha,
\]

and let \( \psi \) be the solution of the IVP

\[
L\psi = 0, \quad \psi(b) = \delta, \quad \psi'(b) = -\gamma.
\]

Then the Green’s function for the BVP (5.32)–(5.34) is given by

\[
G(t, s) := \begin{cases} 
\frac{1}{c} \phi(t) \psi(s), & \text{if } a \leq t \leq s \leq b \\
\frac{1}{c} \phi(s) \psi(t), & \text{if } a \leq s \leq t \leq b,
\end{cases}
\]

where \( c := p(t)w[\phi(t), \psi(t)] \) is a constant. Furthermore, the Green’s function is symmetric on the square \([a, b]^2\); that is,

\[
G(t, s) = G(s, t),
\]

for \( t, s \in [a, b] \).

Proof Let \( \phi, \psi, \) and \( c \) be as in the statement of this theorem. By Abel’s formula (Corollary 5.7), \( c = p(t)w[\phi(t), \psi(t)] \) is a constant. We will use Theorem 5.102 to prove that \( G \) defined by (5.47) is the Green’s function for the BVP (5.32)–(5.34). Note that

\[
\alpha \phi(a) - \beta \phi'(a) = \alpha \beta - \beta \alpha = 0,
\]

and

\[
\gamma \psi(b) + \delta \psi'(b) = \gamma \delta - \delta \gamma = 0.
\]

Hence \( \phi \) satisfies the boundary condition (5.33) and \( \psi \) satisfies the boundary condition (5.34).

Let

\[
u(t, s) := \frac{1}{c} \phi(t) \psi(s), \quad v(t, s) := \frac{1}{c} \phi(s) \psi(t),
\]

for \( t, s \in [a, b] \). Note that for each fixed \( s \in [a, b] \), \( u(\cdot, s) \) and \( v(\cdot, s) \) are solutions of \( Lx = 0 \) on \([a, b]\). Also, for each fixed \( s \in [a, b] \),

\[
\alpha u(a, s) - \beta u'(a, s) = \frac{1}{c} \psi(s) [\alpha \phi(a) - \beta \phi'(a)] = 0,
\]

and

\[
\gamma v(b, s) + \delta v'(b, s) = \frac{1}{c} \phi(s) [\gamma \psi(b) + \delta \psi'(b)] = 0.
\]

Hence for each fixed \( s \in [a, b] \), \( u(\cdot, s) \) satisfies the boundary condition (5.33) and \( v(\cdot, s) \) satisfies the boundary condition (5.34). Let

\[
k(t, s) := v(t, s) - u(t, s) = \frac{1}{c} [\phi(s) \psi(t) - \phi(t) \psi(s)].
\]
It follows that for each fixed \( s, k(\cdot, s) \) is a solution of \( Lx = 0 \). Also, \( k(s, s) = 0 \) and

\[
k'(s, s) = \frac{1}{c} \left[ \phi(s)\psi'(s) - \phi'(s)\psi(s) \right]
= \frac{\phi(s)\psi'(s) - \phi'(s)\psi(s)}{p(s)w[\phi(s), \psi(s)]}
= \frac{1}{p(s)}.
\]

Therefore, \( k(\cdot, \cdot) = x(\cdot, \cdot) \) is the Cauchy function for \( Lx = 0 \) and we have

\[
v(t, s) = u(t, s) + x(t, s).
\]

It remains to prove that for each fixed \( s \), \( u(\cdot, s) \) satisfies the boundary condition (5.45). To see this, consider

\[
\gamma u(b, s) + \delta u'(b, s) = [\gamma v(b, s) + \delta v'(b, s)] - [\gamma x(b, s) + \delta x'(b, s)]
= -[\gamma x(b, s) + \delta x'(b, s)].
\]

Hence by Theorem 5.102, \( G \) defined by equation (5.47) is the Green’s function for the BVP (5.32)–(5.34). It follows from (5.47) that the Green’s function \( G \) is symmetric on the square \([a, b]^2\).  

A special case of Theorem 5.102 is when the boundary conditions (5.33), (5.34) are the conjugate (Dirichlet) boundary conditions

\[
x(a) = 0, \quad x(b) = 0.
\]

In this case we get the following result.

**Corollary 5.105** Assume that the homogeneous conjugate BVP \( Lx = 0 \), (5.48), has only the trivial solution. Then the Green’s function for the conjugate BVP \( Lx = 0 \), (5.48), is given by (5.46), where for each \( s \in [a, b] \), \( u(\cdot, s) \) is the solution of the BVP \( Lx = 0 \),

\[
u(a, s) = 0, \quad u(b, s) = -x(b, s),
\]

and \( v(t, s) = u(t, s) + x(t, s) \), where \( x(\cdot, \cdot) \) is the Cauchy function for \( Lx = 0 \). For each fixed \( s \in [a, b] \), \( v(\cdot, s) \) is a solution of \( Lx = 0 \) satisfying the boundary condition \( v(b, s) = 0 \). Furthermore, the Green’s function is symmetric on the square \([a, b]^2\).

**Example 5.106** Using an appropriate Green’s function, we will solve the BVP

\[
x'' + x = t,
\]

\[
x(0) = 0, \quad x\left(\frac{\pi}{2}\right) = 0.
\]

It is easy to check that the homogeneous BVP \( x'' + x = 0 \), (5.51) has only the trivial solution, and hence we can use Corollary 5.105 to find the
solution of the nonhomogeneous BVP (5.50), (5.51). The Cauchy function for \( x'' + x = 0 \) is given by \[ x(t, s) = \sin(t - s). \]

Since for each fixed \( s \in [0, \frac{\pi}{2}] \), \( u(\cdot, s) \) is a solution of \( x'' + x = 0 \),

\[ u(t, s) = A(s) \cos t + B(s) \sin t. \]

Using the boundary conditions

\[ u(0, s) = 0, \quad u\left(\frac{\pi}{2}, s\right) = -x\left(\frac{\pi}{2}, s\right) = -\sin \left(\frac{\pi}{2} - s\right) = -\cos s, \]

we get

\[ A(s) = 0, \quad B(s) = -\cos(s). \]

Therefore,

\[ u(t, s) = -\sin t \cos s. \]

Next

\[ v(t, s) = u(t, s) + x(t, s) = -\sin t \cos s + \sin(t - s) = -\sin s \cos t. \]

Therefore, by Corollary 5.105, the Green’s function for the BVP \( x'' + x = 0 \), (5.51) is given by

\[ G(t, s) = \begin{cases} -\sin t \cos s, & \text{if } 0 \leq t \leq s \leq \frac{\pi}{2} \\ -\sin s \cos t, & \text{if } 0 \leq s \leq t \leq \frac{\pi}{2}. \end{cases} \]

Hence, by Theorem 5.102, the solution of the BVP (5.50), (5.51) is given by

\[ x(t) = \int_a^b G(t, s) h(s) \, ds \]

\[ = \int_0^{\frac{\pi}{2}} G(t, s) s \, ds \]

\[ = \int_0^t G(t, s) s \, ds + \int_t^{\frac{\pi}{2}} G(t, s) s \, ds \]

\[ = -\cos t \int_0^t s \sin s \, ds - \sin t \int_t^{\frac{\pi}{2}} s \cos s \, ds \]

\[ = -\cos t \left(-t \cos t + \sin t\right) - \sin t \left(\frac{\pi}{2} - t \sin t - \cos t\right) \]

\[ = t - \frac{\pi}{2} \sin t. \]

\( \triangle \)
Example 5.107 Find the Green’s function for the conjugate BVP

\[(p(t)x')' = 0,\]

\[x(a) = 0, \quad x(b) = 0.\]

By Example 5.16 (or Example 5.19), we get that the Cauchy function for \((p(t)x')' = 0\) is

\[x(t, s) = \int_s^t \frac{1}{p(\tau)} d\tau,\]

for \(t, s \in [a, b]\). By Corollary 5.105, for each fixed \(s\), \(u(\cdot, s)\) solves the BVP

\[Lu(t, s) = 0, \quad u(a, s) = 0, \quad u(b, s) = -x(b, s).\]

Since \(u(\cdot, s)\) is a solution for each fixed \(s\),

\[u(t, s) = c_1(s) \cdot 1 + c_2(s) \int_a^t \frac{1}{p(\tau)} d\tau.\]

But \(u(a, s) = 0\) implies that \(c_1(s) = 0\), so

\[u(t, s) = c_2(s) \int_a^t \frac{1}{p(\tau)} d\tau.\]

But \(u(b, s) = -x(b, s) = -\int_s^b \frac{1}{p(\tau)} d\tau\) implies that

\[c_2(s) = -\frac{\int_s^b \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau}.\]

Hence

\[G(t, s) = u(t, s) = -\frac{\int_a^t \frac{1}{p(\tau)} d\tau \int_s^b \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau},\]

for \(a \leq t \leq s \leq b\).

One could use the symmetry (by Theorem 5.104) of the Green’s function to get the form of the Green’s function for \(a \leq s \leq t \leq b\), but we will
find $G$ directly now. For $a \leq s \leq t \leq b,$

$$G(t, s) = v(t, s) = u(t, s) + x(t, s)$$

$$= -\frac{\int_a^t \frac{1}{p(\tau)} d\tau \int_s^b \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau} + \int_s^t \frac{1}{p(\tau)} d\tau$$

$$= -\frac{\int_a^t \frac{1}{p(\tau)} d\tau \int_s^b \frac{1}{p(\tau)} d\tau - \int_a^b \frac{1}{p(\tau)} d\tau \int_s^t \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau}$$

In summary, we get that the Green’s function for the BVP

$$(p(t)x')' = 0,$$

$$x(a) = 0, \quad x(b) = 0$$

is given by

$$G(t, s) := \begin{cases} 
-\frac{\int_a^t \frac{1}{p(\tau)} d\tau \int_s^b \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau}, & \text{if } a \leq t \leq s \leq b \\
-\frac{\int_a^b \frac{1}{p(\tau)} d\tau \int_s^t \frac{1}{p(\tau)} d\tau}{\int_a^b \frac{1}{p(\tau)} d\tau}, & \text{if } a \leq s \leq t \leq b.
\end{cases}$$

Letting $p(t) = 1$ in Example 5.107, we get the following example.

**Example 5.108** The Green’s function for the conjugate BVP

$$x'' = 0,$$

$$x(a) = 0, \quad x(b) = 0$$

is given by

$$G(t, s) := \begin{cases} 
-\frac{(t-a)(b-s)}{b-a}, & \text{if } a \leq t \leq s \leq b \\
-\frac{(s-a)(b-t)}{b-a}, & \text{if } a \leq s \leq t \leq b.
\end{cases}$$

We will use the following properties of the conjugate Green’s function later in this chapter and in Chapter 7.
Theorem 5.109 Let $G$ be the Green’s function for the BVP

\[
x'' = 0,
\]

\[
x(a) = 0, \quad x(b) = 0.
\]

Then

\[
-\frac{(b-a)}{4} \leq G(t,s) \leq 0,
\]

for $t, s \in [a, b]$,

\[
\int_a^b |G(t,s)| \, ds \leq \frac{(b-a)^2}{8},
\]

for $t \in [a, b]$, and

\[
\int_a^b |G'(t,s)| \, ds \leq \frac{(b-a)}{2},
\]

for $t \in [a, b]$.

Proof It is easy to see that $G(t,s) \leq 0$ for $t, s \in [a, b]$. For $a \leq t \leq s \leq b$,

\[
G(t,s) = u(t,s) = -\frac{(t-a)(b-s)}{b-a} \geq -\frac{(s-a)(b-s)}{b-a} \geq -\frac{(b-a)}{4}.
\]

Similarly, for $a \leq s \leq t \leq b$,

\[
G(t,s) = v(t,s) = -\frac{(s-a)(b-t)}{b-a} \geq -\frac{(s-a)(b-s)}{b-a} \geq -\frac{(b-a)}{4}.
\]
Next, for \( t \in [a, b] \), consider

\[
\int_a^b |G(t, s)| \, ds = \int_a^t |G(t, s)| \, ds + \int_t^b |G(t, s)| \, ds
\]

\[
= \int_a^t |v(t, s)| \, ds + \int_t^b |u(t, s)| \, ds
\]

\[
= \int_a^t \frac{(s-a)(b-t)}{b-a} \, ds + \int_t^b \frac{(t-a)(b-s)}{b-a} \, ds
\]

\[
= \frac{(b-t)}{b-a} \int_a^t (s-a) \, ds + \frac{(t-a)}{b-a} \int_t^b (b-s) \, ds
\]

\[
= \frac{(b-t)(t-a)^2}{2(b-a)} + \frac{(t-a)(b-t)^2}{2(b-a)}
\]

\[
= \frac{(b-t)(t-a)}{2} \leq \frac{(b-a)^2}{8}.
\]

Finally, for \( t \in [a, b] \), consider

\[
\int_a^b |G'(t, s)| \, ds = \int_a^t |G'(t, s)| \, ds + \int_t^b |G'(t, s)| \, ds
\]

\[
= \int_a^t |v'(t, s)| \, ds + \int_t^b |u'(t, s)| \, ds
\]

\[
= \int_a^t \frac{(s-a)}{b-a} \, ds + \int_t^b \frac{(b-s)}{b-a} \, ds
\]

\[
= \frac{(t-a)^2}{2(b-a)} + \frac{(b-t)^2}{2(b-a)}
\]

\[
\leq \frac{(b-a)}{2}. \quad \square
\]

**Example 5.110** Find the Green’s function for the right focal BVP

\[ (p(t)x')' = 0, \]

\[ x(a) = 0, \quad x'(b) = 0. \]

Note that from Theorem 5.101, this BVP has only the trivial solution.

By Example 5.16 (or Example 5.19) we get that the Cauchy function for \((p(t)x')' = 0\) is

\[ x(t, s) = \int_s^t \frac{1}{p(\tau)} \, d\tau, \]

for \( t, s \in I \). By Theorem 5.102, for each fixed \( s \), \( u(\cdot, s) \) solves the BVP

\[ Lu(t, s) = 0, \quad u(a, s) = 0, \quad u'(b, s) = -x'(b, s). \]
Since for each fixed $s$, $u(\cdot, s)$ is a solution,

$$u(t, s) = c_1(s) + c_2(s) \int_0^t \frac{1}{p(\tau)} \, d\tau.$$ 

But $u(a, s) = 0$ implies that $c_1(s) = 0$, so

$$u(t, s) = c_2(s) \int_0^t \frac{1}{p(\tau)} \, d\tau.$$ 

But $u'(b, s) = -x'(b, s) = -\frac{1}{p(b)}$ implies that $c_2(s) = -1$. Hence

$$G(t, s) = u(t, s) = -\int_a^t \frac{1}{p(\tau)} \, d\tau,$$

for $a \leq t \leq s \leq b$. For $a \leq s \leq t \leq b$,

$$G(t, s) = u(t, s) + x(t, s)$$

$$= -\int_a^t \frac{1}{p(\tau)} \, d\tau + \int_s^t \frac{1}{p(\tau)} \, d\tau$$

$$= -\int_a^s \frac{1}{p(\tau)} \, d\tau.$$

In summary, we have that the Green’s function for the right focal BVP

$$(p(t)x')' = 0,$$

$x(a) = 0$, $x'(b) = 0$

is given by

$$G(t, s) := \begin{cases} 
-\int_a^t \frac{1}{p(\tau)} \, d\tau, & \text{if } a \leq t \leq s \leq b \\
-\int_a^s \frac{1}{p(\tau)} \, d\tau, & \text{if } a \leq s \leq t \leq b. 
\end{cases}$$

Letting $p(t) = 1$ in Example 5.110, we get the following example.

**Example 5.111** The Green’s function for the right focal BVP

$$x'' = 0,$$

$x(a) = 0$, $x'(b) = 0$

is given by

$$G(t, s) := \begin{cases} 
-(t - a), & \text{if } a \leq t \leq s \leq b \\
-(s - a), & \text{if } a \leq s \leq t \leq b. 
\end{cases}$$

**Theorem 5.112** If $Lx = 0$ is disconjugate on $[a, b]$, then the Green’s function for the conjugate BVP

$$Lx = 0, \quad x(a) = 0, \quad x(b) = 0 \quad (5.52)$$

exists and satisfies

$$G(t, s) < 0,$$
for \( t, s \in (a, b) \).

**Proof** Since \( Lx = 0 \) is disconjugate on \([a, b]\), the BVP (5.52) has only the trivial solution. Hence Corollary 5.105 gives the existence of the Green’s function,

\[
G(t, s) := \begin{cases} 
  u(t, s), & \text{if } a \leq t \leq s \leq b \\
  v(t, s), & \text{if } a \leq s \leq t \leq b,
\end{cases}
\]

for the conjugate BVP (5.52). For each fixed \( s \in (a, b) \), \( u(\cdot, s) \) is the solution of \( Lx = 0 \) satisfying

\[
u(a, s) = 0, \quad u(b, s) = -x(b, s) < 0,
\]

where the last inequality is true since \( x(s, s) = 0, \ x'(s, s) > 0 \), and \( Lx = 0 \) is disconjugate on \([a, b]\). But then it follows that

\[
u(t, s) < 0,
\]

for \( t \in (a, b) \). Also, for each fixed \( s \in (a, b) \), \( v(\cdot, s) \) is the solution of \( Lx = 0 \) satisfying

\[
v(b, s) = 0, \quad v(s, s) = u(s, s) + x(s, s) = u(s, s) < 0.
\]

But then it follows that

\[
v(t, s) < 0,
\]

for \( t \in [a, b] \). Therefore,

\[
G(t, s) < 0,
\]

for \( t, s \in (a, b) \).

**Theorem 5.113** (Comparison Theorem for BVPs) Assume that \( Lx = 0 \) is disconjugate on \([a, b]\). If \( u, v \in \mathbb{D} \) satisfy \( u(a) \geq v(a), \ u(b) \geq v(b) \) and \( Lu(t) \leq Lv(t) \) for \( t \in [a, b] \), then

\[
u(t) \geq v(t),
\]

for \( t \in [a, b] \).

**Proof** Let \( z(t) := u(t) - v(t) \) and let \( h(t) := Lz(t) \), for \( t \in [a, b] \). Then

\[
h(t) = Lz(t) = Lu(t) - Lv(t) \leq 0,
\]

for \( t \in [a, b] \). Also, let

\[
A := z(a) = u(a) - v(a) \geq 0,
\]

and

\[
B := z(b) = u(b) - v(b) \geq 0.
\]

Then \( z \) solves the BVP

\[
Lz = h(t),
\]

\[
z(a) = A, \quad z(b) = B.
\]
By Corollary 5.103,
\[ z(t) = w(t) + \int_a^b G(t,s)h(s) \, ds, \]  
(5.53)

where \( w \) is a solution of \( Lx = 0 \) satisfying \( w(a) = A \geq 0, w(b) = B \geq 0 \). Since \( Lx = 0 \) is disconjugate on \([a,b]\), \( w(t) \geq 0 \) and since by Theorem 5.112, \( G(t,s) \leq 0 \) for \( t,s \in [a,b] \), it follows from (5.53) that
\[ z(t) = u(t) - v(t) \geq 0, \]
on \([a,b]\).

\[ \square \]

**Theorem 5.114** (Liapunov Inequality) If
\[ \int_a^b q^+(t) \, dt \leq \frac{4}{b-a}, \]  
(5.54)

where \( q^+(t) := \max\{q(t),0\} \), then \( x'' + q(t)x = 0 \) is disconjugate on \([a,b]\).

**Proof** Assume the inequality (5.54) holds but \( x'' + q(t)x = 0 \) is not disconjugate on \([a,b]\). Then there is a nontrivial solution \( x \) such that \( x \) has consecutive zeros satisfying \( a \leq t_1 < t_2 \leq b \). Without loss of generality, we can assume that \( x(t) > 0 \) on \((t_1,t_2)\). Since \( x \) satisfies the BVP
\[ x'' = -q(t)x(t), \]
\[ x(t_1) = 0, \quad x(t_2) = 0, \]
we get that
\[ x(t) = \int_{t_1}^{t_2} G(t,s)\left[-q(s)x(s)\right] \, ds, \]
where by Example 5.108 the Green’s function \( G \) is given by
\[ G(t,s) := \begin{cases} \frac{(t-t_1)(t_2-s)}{t_2-t_1}, & \text{if } t_1 \leq t \leq s \leq t_2 \\ \frac{(s-t_2)(t_2-t)}{t_2-t_1}, & \text{if } t_1 \leq s \leq t \leq t_2. \end{cases} \]

Pick \( t_0 \in (t_1,t_2) \) so that
\[ x(t_0) = \max\{x(t) : t_1 \leq t \leq t_2\}. \]

Consider
\[ x(t_0) = \int_{t_1}^{t_2} G(t_0,s)\left[-q(s)x(s)\right] \, ds \]
\[ \leq \int_{t_1}^{t_2} |G(t_0,s)|q^+(s)x(s) \, ds. \]

If \( q^+(t) = 0 \) for \( t \in [t_1,t_2] \), then \( q(t) \leq 0 \) for \( t \in [t_1,t_2] \) and by Exercise 5.26 we have that \( x'' + q(t)x = 0 \) is disconjugate on \([t_1,t_2]\), but this is a
contradiction. Hence it is not true that \( q^+(t) = 0 \) for \( t \in [t_1, t_2] \) and it follows that

\[
x(t_0) < x(t_0) \int_{t_1}^{t_2} |G(t_0, s)|q^+(s) \, ds.
\]

Dividing both sides by \( x(t_0) \), we get the inequality

\[
1 < \int_{t_1}^{t_2} |G(t_0, s)|q^+(s) \, ds.
\]

Using Theorem 5.109, we get the inequality

\[
1 < \frac{t_2 - t_1}{4} \int_{t_1}^{t_2} q^+(t) \, dt.
\]

But this implies that

\[
\int_{a}^{b} q^+(t) \, dt \geq \int_{t_1}^{t_2} q^+(t) \, dt = \frac{4}{t_2 - t_1} \geq \frac{4}{b - a},
\]

which contradicts the inequality (5.54). \( \square \)

**Example 5.115** Use Theorem 5.114 to find \( T \) so that the differential equation

\[
x'' + tx = 0
\]

is disconjugate on \([0, T]\).

By Theorem 5.114 we want to pick \( T \) so that

\[
\int_{0}^{T} t \, dt \leq \frac{4}{T}.
\]

That is, we want

\[
\frac{T^2}{2} \leq \frac{4}{T}.
\]

It follows that \( x'' + tx = 0 \) is disconjugate on \([0, 2]\). \( \triangle \)

**Example 5.116** Use Theorem 5.114 to show that the differential equation

\[
x'' + k \sin tx = 0, \quad k > 0
\]

is disconjugate on \([a, a + 2\pi]\) if \( k \leq \frac{1}{\pi} \).

Since

\[
\int_{a}^{a+2\pi} q^+(t) \, dt = \int_{0}^{\pi} k \sin t \, dt = 2k,
\]

it follows from Theorem 5.114 that if \( k \leq \frac{1}{\pi} \), then \( x'' + k \sin tx = 0 \) is disconjugate on \([a, a + 2\pi]\). It is interesting to note that it is known \([11]\) that every solution of \( x'' + k \sin tx = 0, \quad k > 0 \), is oscillatory (this is not easy to prove). \( \triangle \)
Theorem 5.114 says that if
\[ \int_a^b q^+(t) \, dt \leq \frac{C}{b-a}, \]
where \( C = 4 \), then \( x''+q(t)x = 0 \) is disconjugate on \([a, b]\). The next example shows that this result is sharp in the sense that \( C = 4 \) is the largest constant such that this result is true in general.

**Example 5.117** \((C = 4 \text{ is Sharp})\) Let \( 0 < \delta < \frac{1}{2} \) and choose \( x : [0, 1] \to \mathbb{R} \) so that \( x \) has a continuous second derivative on \([0, 1]\) with
\[ x(t) := \begin{cases} t, & \text{if } 0 \leq t \leq \frac{1}{2} - \delta \\ 1-t, & \text{if } \frac{1}{2} + \delta \leq t \leq 1, \end{cases} \]
and \( x \) satisfies
\[ x''(t) \leq 0 \]
on \([0, 1]\). Next let
\[ q(t) := \begin{cases} -\frac{x''(t)}{x(t)}, & \text{if } 0 < t < 1 \\ 0, & \text{if } t = 0, 1. \end{cases} \]
Note that \( q \) is nonnegative and continuous on \([0, 1]\), and \( x \) is a nontrivial solution of \( x'' + q(t)x = 0 \) with \( x(0) = 0 \), \( x(1) = 0 \). Hence \( x'' + q(t)x = 0 \) is not disconjugate on \([0, 1]\). Also, note that
\[
\int_0^1 q^+(t) \, dt = \int_{\frac{1}{2} - \delta}^{\frac{1}{2} + \delta} \frac{-x''(t)}{x(t)} \, dt \\
\leq \frac{1}{\frac{1}{2} - \delta} \int_{\frac{1}{2} - \delta}^{\frac{1}{2} + \delta} [-x''(t)] \, dt \\
= \frac{2}{1 - 2\delta} \left[ x'(\frac{1}{2} - \delta) - x'(\frac{1}{2} + \delta) \right] \\
= \frac{4}{1 - 2\delta}.
\]
The constant on the right-hand side of this last inequality is larger than 4 and can be made arbitrarily close to 4 by taking \( \delta \) arbitrarily close to zero.

Finally, in this section we consider the nonhomogeneous periodic BVP
\[
Lx = h(t), \quad x(a) = x(b), \quad x'(a) = x'(b). \tag{5.55, 5.56}
\]
The next theorem is the reason why the BVP (5.55), (5.56) is called periodic.

**Theorem 5.118** *In this theorem, in addition to the standard assumptions on \( p, q, \) and \( h \), assume these three functions are periodic on \( \mathbb{R} \) with period*
If the BVP (5.55), (5.56) has a solution \( x \), then it is periodic with period \( b - a \).

**Proof** Assume \( x \) is a solution of the BVP (5.55), (5.56) and let
\[
y(t) := x(t + b - a), \quad t \in \mathbb{R}.
\]
Using the chain rule for differentiation and the periodicity of \( p, q, \) and \( h \), it can be shown that \( y \) is a solution of (5.55). Also,
\[
y(a) = x(b) = x(a),
\]
and
\[
y'(a) = x'(b) = x'(a).
\]
Hence \( x \) and \( y \) are both solutions of the same IVP. It follows that
\[
x(t) = y(t) = x(t + b - a),
\]
for \( t \in \mathbb{R} \). That is, \( x \) is periodic with period \( b - a \). \( \square \)

**Theorem 5.119** Assume that the homogeneous periodic BVP \( Lx = 0 \), (5.56) has only the trivial solution. Then the nonhomogeneous BVP \( Lx = h(t) \),
\[
x(a) - x(b) = A, \quad x'(a) - x'(b) = B, \tag{5.57}
\]
where \( A \) and \( B \) are given constants and \( h \) is a given continuous function on \([a, b]\), has a unique solution.

**Proof** Assume the homogeneous periodic BVP \( Lx = 0 \), (5.56), has only the trivial solution. Let \( x_1, x_2 \) be linearly independent solutions of \( Lx = 0 \); then
\[
x(t) = c_1 x_1(t) + c_2 x_2(t)
\]
defines a general solution of \( Lx = 0 \). Note that \( x \) satisfies the boundary conditions (5.56) iff \( c_1, c_2 \) are constants such that the two equations
\[
c_1 [x_1(a) - x_1(b)] + c_2 [x_2(a) - x_2(b)] = 0 \tag{5.58}
\]
\[
c_1 [x'_1(a) - x'_1(b)] + c_2 [x'_2(a) - x'_2(b)] = 0 \tag{5.59}
\]
hold. Since we are assuming that the BVP \( Lx = 0 \), (5.56) has only the trivial solution, it follows that the only solution of the system (5.58), (5.59) is
\[
c_1 = c_2 = 0.
\]
Therefore, the determinant of the coefficients in the system (5.58), (5.59) is different from zero; that is,
\[
\begin{vmatrix}
x_1(a) - x_1(b) & x_2(a) - x_2(b) \\
x'_1(a) - x'_1(b) & x'_2(a) - x'_2(b)
\end{vmatrix} \neq 0. \tag{5.60}
\]
Now we will show that the BVP \( Lx = h(t) \), (5.57) has a unique solution. Let \( u_0 \) be a fixed solution of \( Lu = h(t) \); then a general solution of \( Lu = h(t) \) is defined by
\[
u(t) = a_1 x_1(t) + a_2 x_2(t) + u_0(t).
\]
It follows that \( u \) satisfies the BCs (5.57) iff \( a_1, a_2 \) are constants satisfying the system of equations

\[
\begin{align*}
a_1[x_1(a) - x_1(b)] + a_2[x_2(a) - x_2(b)] &= A - u_0(a) + u_0(b) \\
a_1[x_1'(a) - x_1'(b)] + a_2[x_2'(a) - x_2'(b)] &= B - u'_0(a) + u'_0(b).
\end{align*}
\]

Since (5.60) holds, the system (5.61), (5.62) has a unique solution \( a_1, a_2 \). This implies that the BVP \( Lx = h(t) \), (5.57) has a unique solution.

\[\square\]

**Theorem 5.120** (Green’s Function for Periodic BVP) Assume the homogeneous BVP \( Lx = 0 \), (5.56) has only the trivial solution. For each fixed \( s \in [a, b] \), let \( u(\cdot, s) \) be the solution of the BVP

\[
\begin{align*}
Lu &= 0, \\
u(a, s) - u(b, s) &= x(b, s), \\
u'(a, s) - u'(b, s) &= x'(b, s),
\end{align*}
\]

where \( x(\cdot, \cdot) \) is the Cauchy function for \( Lx = 0 \). Define the Green’s function \( G \) for the BVP \( Lx = 0 \), (5.56) by

\[
G(t, s) := \begin{cases} 
  u(t, s), & \text{if } a \leq t \leq s \leq b \\
  v(t, s), & \text{if } a \leq s \leq t \leq b,
\end{cases}
\]

where \( v(t, s) := u(t, s) + x(t, s) \). Assume \( h \) is continuous on \([a, b] \); then

\[
x(t) := \int_a^b G(t, s)h(s) \, ds,
\]

for \( t \in [a, b] \), defines the unique solution \( x \) of the nonhomogeneous periodic BVP \( Lx = h(t) \), (5.56). Furthermore, for each fixed \( s \in [a, b] \), \( v(\cdot, s) \) is a solution of \( Lx = 0 \) and \( u(a, s) = v(b, s) \), \( u'(a, s) = v'(b, s) \).

**Proof** The existence and uniqueness of \( u(\cdot, \cdot) \) is guaranteed by Theorem 5.119. Since \( v(t, s) := u(t, s) + x(t, s) \), we have for each fixed \( s \) that \( v(\cdot, s) \) is a solution of \( Lx = 0 \). Using the boundary conditions (5.64), (5.65), it is easy to see that for each fixed \( s \), \( u(a, s) = v(b, s) \), \( u'(a, s) = v'(b, s) \). Let
$G(t, s)$ be as in the statement of this theorem and consider

\[
x(t) = \int_a^b G(t, s)h(s) \, ds \\
= \int_a^t G(t, s)h(s) \, ds + \int_t^b G(t, s)h(s) \, ds \\
= \int_a^t v(t, s)h(s) \, ds + \int_t^b u(t, s)h(s) \, ds \\
= \int_a^t [u(t, s) + x(t, s)]h(s) \, ds + \int_t^b u(t, s)h(s) \, ds \\
= \int_a^b u(t, s)h(s) \, ds + \int_a^t x(t, s)h(s) \, ds \\
= \int_a^b u(t, s)h(s) \, ds + z(t),
\]

where, by the variation of constants formula (Theorem 5.22), $z(t) := \int_a^t x(t, s)h(s) \, ds$ defines the solution of the IVP

\[
Lz = h(t), \quad z(a) = 0, \quad z'(a) = 0.
\]

Hence

\[
Lx(t) = \int_a^b Lu(t, s)h(s) \, ds + Lz(t) \\
= \int_a^b Lu(t, s)h(s) \, ds + h(t) \\
= h(t).
\]

Thus $x$ is a solution of $Lx = h(t)$. Note that

\[
x(a) = \int_a^b G(a, s)h(s) \, ds \\
= \int_a^b u(a, s)h(s) \, ds \\
= \int_a^b v(b, s)h(s) \, ds \\
= \int_a^b G(b, s)h(s) \, ds \\
= x(b).
\]
Similarly,

\[ x'(a) = \int_a^b G'(a, s)h(s) \, ds \]
\[ = \int_a^b u'(a, s)h(s) \, ds \]
\[ = \int_a^b v'(b, s)h(s) \, ds \]
\[ = \int_a^b G'(b, s)h(s) \, ds \]
\[ = x'(b). \]

Hence \( x \) satisfies the periodic boundary conditions (5.56). \( \square \)

**Example 5.121** Using Theorem 5.120, we will solve the periodic BVP

\[ x'' + x = \sin(2t), \quad (5.67) \]
\[ x(0) = x(\pi), \quad x'(0) = x' (\pi). \quad (5.68) \]

It is easy to show that the homogeneous BVP

\[ x'' + x = 0, \quad x(0) = x(\pi), \quad x'(0) = x'(\pi) \]

has only the trivial solution and hence we can use Theorem 5.120 to solve the BVP (5.67), (5.68). The Cauchy function for \( x'' + x = 0 \) is given [see Exercise 5.10, part (v)] by

\[ x(t, s) = \sin(t - s). \]

By Theorem 5.120, the Green’s function \( G \) is given by

\[ G(t, s) := \begin{cases} u(t, s), & \text{if } 0 \leq t \leq s \leq \pi \\ v(t, s), & \text{if } 0 \leq s \leq t \leq \pi, \end{cases} \quad (5.69) \]

where for each fixed \( s \in [0, \pi] \), \( u(\cdot, s) \) is the solution of the BVP

\[ u'' + u = 0, \quad (5.70) \]
\[ u(0, s) - u(\pi, s) = x(\pi, s), \quad (5.71) \]
\[ u'(0, s) - u'(\pi, s) = x'(\pi, s), \quad (5.72) \]

where \( x(\cdot, \cdot) \) is the Cauchy function for \( Lx = 0 \) and \( v(t, s) := u(t, s) + x(t, s) \).

Using (5.70), we get

\[ u(t, s) = A(s) \cos t + B(s) \sin t. \]

From the boundary conditions (5.71), (5.72), we get

\[ 2A(s) = \sin(\pi - s) = \sin s, \]
\[ 2B(s) = \cos(\pi - s) = -\cos s. \]

It follows that

\[ u(t, s) = \frac{1}{2} \sin s \cos t - \frac{1}{2} \cos s \sin t = -\frac{1}{2} \sin(t - s). \]
Therefore,
\[ v(t, s) = u(t, s) + x(t, s) = \frac{1}{2} \sin(t - s). \]
Hence by Theorem 5.120 the solution of the BVP (5.67), (5.68) is given by

\[
x(t) = \int_{a}^{b} G(t, s) h(s) \, ds
\]
\[ = \int_{0}^{\pi} G(t, s) \sin(2s) \, ds \]
\[ = \int_{0}^{t} v(t, s) \sin(2s) \, ds + \int_{t}^{\pi} u(t, s) \sin(2s) \, ds \]
\[ = \frac{1}{2} \int_{0}^{t} \sin(t - s) \sin(2s) \, ds - \frac{1}{2} \int_{t}^{\pi} \sin(t - s) \sin(2s) \, ds \]
\[ = \frac{1}{2} \sin t \int_{0}^{t} \cos s \sin(2s) \, ds - \frac{1}{2} \cos t \int_{0}^{t} \sin(s) \sin(2s) \, ds \]
\[ - \frac{1}{2} \sin t \int_{t}^{\pi} \cos s \sin(2s) \, ds + \frac{1}{2} \cos t \int_{t}^{\pi} \sin(s) \sin(2s) \, ds \]
\[ = \frac{1}{2} \sin t \left\{-\frac{1}{3} \sin(2t) \sin t - \frac{2}{3} \cos(2t) \cos t + \frac{2}{3}\right\} \]
\[ - \frac{1}{2} \cos t \left\{\frac{1}{3} \cos t \sin(2t) - \frac{2}{3} \sin t \cos(2t)\right\} \]
\[ - \frac{1}{2} \sin t \left\{\frac{2}{3} \sin(2t) \sin t + \frac{3}{3} \cos(2t) \cos t\right\} \]
\[ + \frac{1}{2} \cos t \left\{-\frac{1}{3} \cos t \sin(2t) + \frac{2}{3} \sin t \cos(2t)\right\} \]
\[ = -\frac{1}{3} \sin^2 t \sin(2t) - \frac{2}{3} \sin t \cos t \cos(2t) \]
\[ - \frac{1}{3} \cos^2 t \sin(2t) + \frac{2}{3} \sin t \cos t \cos(2t) \]
\[ = -\frac{1}{3} \sin(2t). \]

The fact that this solution is periodic with period \(\pi\) as guaranteed by Theorem 5.118. (Note that this simple problem could be solved by the annihilator method).

\[\triangle\]

### 5.10 Exercises

5.1 Write each of the following differential equations in self-adjoint form:

(i) (Legendre’s Equation) \((1 - t^2)x'' - 2tx' + n(n + 1)x = 0, \quad t \in I := (-1, 1)\)

(ii) (Chebychev’s Equation) \((1 - t^2)x'' - tx' + n^2x = 0, \quad t \in I := (-1, 1)\)

(iii) (Laquerre’s Equation) \(tx'' + (1-t)x' + ax = 0, \quad t \in I := (0, \infty)\)
(iv) (Hermite’s Equation) \( x'' - 2tx' + 2nx = 0, \quad t \in I := (-\infty, \infty) \)
(v) \( x'' + \frac{2}{n+1}x' + \frac{\lambda}{(n+1)^2}x = 0, \quad t \in I := (-3, \infty) \)

5.2 Find a homogeneous self-adjoint equation that has the function \( x \) defined by \( x(t) = \sin(\frac{1}{t^2}), \quad t \in I := (0, \infty) \), where \( n \) is a positive integer as a solution.

5.3 Prove Abel’s formula directly by taking the derivative of
\[
p(t)w[x(t), y(t)].
\]

5.4 Let \( a \in I \) and let \( \tau(t) := \int_{a}^{t} \frac{1}{p(s)}ds, \quad t \in I \). Show that this defines \( t = t(\tau) \). Then show that if \( x \) is a solution of \( Lx = 0 \), then \( y(\tau) := x(t(\tau)) \) defines a solution of \( y'' + Q(\tau)y = 0 \), where \( Q(\tau) = p(t(\tau))q(t(t)). \)

5.5 Complete the proof of Corollary 5.10.

5.6 Show that if \( x_1, x_2, \ldots, x_k \) are functions that have \( k-1 \) derivatives on an interval \( I \) and if
\[
w[x_1, x_2, \ldots, x_k](t_0) \neq 0,
\]
where \( t_0 \in I \), then \( x_1, x_2, \ldots, x_k \) are linearly independent functions on \( I \).

5.7 Let \( x_1(t) = t^2, x_2(t) = t|t|, \quad t \in I := \mathbb{R} \). Show that \( x_1, x_2 \) are linearly independent on \( \mathbb{R} \) but \( w[x_1, x_2](0) = 0 \). Are these two functions \( x_1, x_2 \) solutions of the same self-adjoint equation \( Lx = 0 \) on \( \mathbb{R} \)? Are \( x_1, x_2 \) linearly dependent or linearly independent on \( \mathbb{R}^+ := [0, \infty) \)?

5.8 Show that if \( x, \ y \in \mathbb{D} \) and they satisfy the boundary conditions \( \alpha z(a) - \beta z'(a) = 0, \gamma z(b) + \delta z'(b) = 0 \), where \( \alpha^2 + \beta^2 > 0, \quad \gamma^2 + \delta^2 > 0 \), then \( <Lx, y> = <x, Ly> \).

5.9 Define the functions \( ch \) and \( sh \) to be the solutions of the differential equation \( x'' - x = 0 \) satisfying the initial conditions \( ch(0) = 1, \ ch'(0) = 0 \) and \( sh(0) = 0, \ sh'(0) = 1 \), respectively. State and prove a theorem like Theorem 5.13 giving properties of \( ch \) and \( sh \).

5.10 Use the definition of the Cauchy function to find the Cauchy function for each of the following differential equations:
(i) \( (e^{-3t}x')' + 2e^{-3t}x = 0, \quad I = (-\infty, \infty) \)
(ii) \( (e^{-10t}x')' + 25e^{-10t}x = 0, \quad I = (-\infty, \infty) \)
(iii) \( (\frac{1}{t^2}x')' + \frac{6}{t}x = 0, \quad I = (0, \infty) \)
(iv) \( (e^{-2t}x')' = 0, \quad I = (-\infty, \infty) \)
(v) \( x'' + x = 0, \quad I = (-\infty, \infty) \)
In (v) write your answer as a single term.

5.11 Use your answers in Exercise 5.10 and the variation of constants formula to solve each of the following initial value problems:
(i) \( (e^{-3t}x')' + 2e^{-3t}x = e^{-t}, \quad x(0) = 0, \quad x'(0) = 0 \)
(ii) \( (e^{-10t}x')' + 25e^{-10t}x = e^{-5t}, \quad x(0) = 0, \quad x'(0) = 0 \)
(iii) \( (\frac{1}{t^2}x')' + \frac{6}{t}x = t, \quad x(1) = 0, \quad x'(1) = 0 \)
Use Theorem 5.18 to find the Cauchy function for each of the differential equations in Exercise 5.10.

Use Theorem 5.18 to find the Cauchy function for the differential equation $(\frac{t}{2}x')' + \frac{2}{\pi}x = 0$, $t > 0$.

Use Corollary 5.23 to solve the following IVPs:

(i) $(\frac{t}{2}x')' + \frac{12}{\pi}x = \frac{2}{\pi}$, $x(1) = 1$, $x'(1) = 2$
(ii) $x'' + 9x = 1$, $x(0) = 2$, $x'(0) = 0$

Find the Cauchy function for the differential equation

$$\left(\frac{x'}{1+t}\right)' = 0.$$ 

Use this Cauchy function and an appropriate variation of constants formula to solve the IVP

$$\left(\frac{x'}{1+t}\right)' = t^2, \quad x(0) = x'(0) = 0.$$ 

Find eigenpairs for the following Sturm-Liouville problems:

(i) $x'' = -\lambda x$, $x(0) = 0$, $x(\frac{\pi}{2}) = 0$
(ii) $x'' = -\lambda x$, $x'(0) = 0$, $x(\frac{\pi}{2}) = 0$
(iii) $(tx')' - \frac{4}{t}x$, $x(1) = 0$, $x(e) = 0$
(iv) $x'' = -\lambda x$, $x'(-\pi) = 0$, $x'(-\pi) = 0$
(v) $(t3x')' + \lambda tx = 0$, $x(1) = 0$, $x(e) = 0$

Find a constant $\alpha$ so that $x(t) = t$, $y(t) = t + \alpha$ are orthogonal with respect to $r(t) = t + 1$ on $[0, 1]$.

Prove that if $\lambda_0, x_0$ is an eigenpair for the SLP (5.5)–(5.7), then $\overline{\lambda_0, x_0}$ is also an eigenpair.

Show that if $x_1$, $x_2$ satisfy the boundary conditions (5.6), then $w[x_1(t), x_2(t)](a) = 0$.

Find all eigenpairs for the periodic Sturm-Liouville problem:

$$x'' + \lambda x = 0, \quad x\left(-\frac{\pi}{2}\right) = x\left(\frac{\pi}{2}\right), \quad x'\left(-\frac{\pi}{2}\right) = x'\left(\frac{\pi}{2}\right).$$

Find all eigenpairs for the periodic Sturm-Liouville problem:

$$x'' + \lambda x = 0, \quad x(-3) = x(3), \quad x'(-3) = x'(3).$$

What do you get if you cross a hurricane with the Kentucky Derby?

Use separation of variables to find solutions of each of the following partial differential equations:

(i) $u_t = ku_{xx}$
Exercises

(ii) \( y_{tt} = ky_{xx} \)

(iii) \( u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \)

5.24 (Periodic Boundary Conditions) Show that if, in addition to the standard assumptions on the coefficient functions \( p, q \), we assume that these coefficient functions are periodic on \( \mathbb{R} \) with period \( b - a \), then any solution of \( Lx = 0 \) satisfying the boundary conditions \( x(a) = x(b), \ x'(a) = x'(b) \) is periodic with period \( b - a \). Because of this fact we say that the boundary conditions \( x(a) = x(b), \ x'(a) = x'(b) \) are periodic boundary conditions.

5.25 (Sturm-Liouville Problem) Assume that \( I = (a, b] \), and, in addition to the standard assumptions on the coefficient functions \( p, q, \) and \( r \), that \( \lim_{t \to a^+} p(t) = 0 \). Show that eigenfunctions corresponding to distinct eigenvalues for the singular Sturm-Liouville problem
\[
Lx = -\lambda r(t)x, \\
x, \ x' \text{ are bounded on } I, \\
\gamma x(b) + \delta x'(b) = 0
\]
are orthogonal with respect to the weight function \( r \) on \( I \).

5.26 Show that if \( q(t) \leq 0 \) on an interval \( I \), then \( x'' + q(t)x = 0 \) is disconjugate on \( I \).

5.27 Find a self-adjoint equation \( Lx = 0 \) and an interval \( I \) such that \( Lx = 0 \) is disconjugate on \( I \) but there is no positive solution on \( I \) (Hint: See Theorem 5.46.)

5.28 In the proof of Theorem 5.50, prove
\[
\lim_{t \to b^-} \left\{ \frac{u(t)}{v(t)} \right\} \left[ p_1(t)u'(t)v(t) - p_2(t)u(t)v'(t) \right] = 0.
\]

5.29 Show that the Euler–Cauchy differential equation
\[
t^2 x'' + 2tx' + bx = 0
\]
is oscillatory on \([1, \infty)\) if \( b > \frac{1}{4} \) and disconjugate on \([1, \infty)\) if \( b \leq \frac{1}{4} \).

5.30 Complete the proof of Theorem 5.51.

5.31 Find a Polya factorization for each of the following differential equations on the given intervals \( J \):

(i) \( (e^{-5t}x')' + 6e^{-5t}x = 0, \ J = (-\infty, \infty) \)
(ii) \( x'' - x = 0, \ J = (-\infty, \infty) \)
(iii) \( t^2x'' - 5tx' + 8x = 0, \ J = (0, \infty) \)
(iv) \( x'' - 6x' + 9x = 0, \ J = (-\infty, \infty) \)

5.32 Find a Trench factorization for each of the differential equations in Exercise 5.31 on the given intervals \( J \).
5.33 For each of the following show that the given function \( u \) is a solution on the given interval \( J \) and solve the given differential equation on that interval:

(i) \( \left( \frac{e^{2t}}{1+2t} x' \right)' - \frac{4e^{2t}}{(1+2t)^2} x = 0 \), \( J = \left(-\frac{1}{2}, \infty\right) \), \( u(t) = e^{-2t} \)

(ii) \( \left( \frac{e^t}{1+t} x' \right)' - \frac{e^t}{(1+t)^2} x = 0 \), \( J = (0, \infty) \), \( u(t) = t \)

(iii) \( \left( \frac{1}{(t^2+2t-1)} x' \right)' + \frac{2}{(t^2+2t-1)^2} x = 0 \), \( J = [2, \infty) \), \( u(t) = 1 + t \)

(iv) \( tx'' - (1 + t)x' + x = 0 \), \( J = (0, \infty) \), \( u(t) = e^t \)

5.34 Show that \( u(t) = \frac{\sin t}{\sqrt{t}} \) defines a nonzero solution of

\[
(tx')' + \left( t - \frac{1}{4t} \right) x = 0,
\]
on \( J = (0, \pi) \), and then solve this differential equation.

5.35 Find a recessive and dominant solution on \( J \) for each of the following equations and for these two solutions show directly that the conclusions of Theorem 5.59 are true:

(i) \( x'' - 5x' + 6x = 0 \), \( J = [0, \infty) \)

(ii) \( t^2x'' - 5tx' + 9x = 0 \), \( J = [1, \infty) \)

(iii) \( x'' - 4x' + 4x = 0 \), \( J = [0, \infty) \)

5.36 Prove the last statement in Theorem 5.59.

5.37 Find a Mammana factorization on \( J \) for each of the problems in Exercise 5.31.

5.38 Show that the Riccati operator \( R : C^1(I) \to C(I) \), which is defined in Section 5.7, is not a linear operator (hence is a nonlinear operator).

5.39 Find a solution of some IVP for the Riccati equation \( Rz = z' + z^2 = 0 \) whose maximal interval of existence is not the whole interval \( I = \mathbb{R} \). Give the maximal interval of existence for the solution you found.

5.40 Prove Corollary 5.75.

5.41 Solve the following Riccati equations:

(i) \( z' - 3e^{2t} + e^{-2t}z^2 = 0 \)

(ii) \( z' + 4e^{-4t} + e^{4t}z^2 = 0 \)

(iii) \( z' - \frac{1}{t} + \frac{1}{t}z^2 = 0 \)

(iv) \( z' + 4 + z^2 = 0 \)

(v) \( z' = -3e^{-4t} - e^{4t}z^2 \)

(vi) \( z' + 16e^{-8t} + e^{8t}z^2 = 0 \)

5.42 For the Riccati equations in Exercises 5.41 (i), (ii), (iii), (v), and (vi), find the minimum solution that exists for all \( t \) sufficiently large. For the Riccati equation in (iv), find the minimum solution that exists for all \( t < \frac{\pi}{2} \), sufficiently close.
5.43 Show if the following equations are oscillatory or not:

(i) \((t \log t x')' + \frac{1}{t \log t} x = 0\)

(ii) \(x'' + \frac{1}{t^2} x = 0\)

5.44 Show that if \(I = [2, \infty)\), and

\[ \int_2^\infty \frac{t}{\log^{1+\beta} t} q(t) \, dt = \infty, \]

where \(\beta > 0\), then \(x'' + q(t)x = 0\) is oscillatory on \([2, \infty)\). [Hint: Use Theorem 5.81 with \(u(t) = \frac{t^{\frac{\beta}{1+\beta}}}{\log t} \).]

5.45 Assume that for each of the following, \(I\) has a global minimum at \(x_0\) and \(x_0\) is twice continuously differentiable. Find \(x_0(t)\).

(i) \(I[x] = \int_0^\pi \{[x'(t)]^2 - x^2(t)\} \, dt\), \(x(0) = 1\), \(x(\pi) = 0\)

(ii) \(I[x] = \int_1^e \{e^{-3t} [x'(t)]^2 - 2e^{-3t} x^2(t)\} \, dt\), \(x(0) = 1\), \(x(1) = 2\)

(iii) \(I[x] = \int_1^e \left\{ \frac{1}{t} [x'(t)]^2 - 2 e^{-\frac{1}{\pi^2}} x^2(t) \right\} \, dt\), \(x(1) = 0\), \(x(e) = 1\)

5.46 Do the proof of the case where \(g(t_0) < 0\) in the proof of Lemma 5.87.

5.47 Assume that the calculus of variations problem in Example 5.83 has a global minimum at \(x_0\) and \(x_0 \in C^2[a, b]\). Find \(x_0(t)\) by solving the appropriate Euler-Lagrange differential equation.

5.48 Assume that \(I\) has a global minimum at \(x_0\) and \(x_0\) is twice continuously differentiable. Find \(x_0(t)\), given that

\[ I[x] = \int_0^1 \left[ e^{-5t} (x'(t))^2 - 6e^{-5t} x^2(t) \right] \, dt, \quad x(0) = 0, \quad x(1) = 1. \]

5.49 Show that the function \(x_1\) defined by \(x_1(t) := 1 - t\), \(0 \leq t \leq 1\), is in the set \(D\) for the calculus of variations problem considered in Example 5.96 and show directly that \(Q[x_1] > -1\).

5.50 Using Theorem 5.92 (Legendre’s necessary condition), state what you can about the existence of local extrema for each of the simplest problems of the calculus of variations, where \(I\) is given by

(i) \(I[x] = \int_0^1 \{e^{-t} x(t) + x^2(t) - t^2 x(t) x'(t) - (1+t^2)(x'(t))^2\} \, dt\), \(x \in D\)

(ii) \(I[x] = \int_1^2 \{e^{-t} \sin(x(t)) - e^{-2t} x(t) x'(t) + (t-1)(x'(t))^2\} \, dt\), \(x \in D\)

(iii) \(I[x] = \int_0^\pi \{e^{-t} x'(t) - t^2 x(t) x'(t) - \sin(2t)(x'(t))^2\} \, dt\), \(x \in D\)

5.51 Show that if \(f(t, u, v) = f(v)\), then The Euler–Lagrange equation reduces to

\[ f_{uu}(x') x'' = 0. \]

Also show if \(f(t, u, v) = f(t, v)\), then the Euler–Lagrange equation leads to the differential equation

\[ f_v(t, x') = C \]

where \(C\) is a constant.
5.52 Solve the Euler–Lagrange equation given that
(i) \( f(t, u, v) = 2u^2 - v^2 \)
(ii) \( f(t, u, v) = \frac{\sqrt{1+u^2}}{u} \).

5.53 Find a possible extremum for the problem:
\[
I[x] = \int_0^4 [tx'(t) - (x'(t))^2] \, dt
\]
\[
x(0) = 0, \quad x(4) = 3.
\]

5.54 Prove Theorem 5.92 for the local maximum case.

5.55 Prove Theorem 5.101.

5.58 (Boundary Conditions) Define \( M : C^1[a, b] \to \mathbb{R} \) by
\[
Mx = c_1 x(a) + c_2 x'(a) + c_3 x(b) + c_4 x'(b),
\]
where \( c_i, 1 \leq i \leq 4 \), are given constants. Show that \( M \) is a linear operator. Because of this any boundary condition of the form \( Mx = A \), where \( A \) is a given constant, is called a linear nonhomogeneous boundary condition, whereas any boundary condition of the form \( Mx = 0 \) is called a linear homogeneous boundary condition. If a boundary condition is not linear it is called a nonlinear boundary condition. Classify each of the following boundary conditions:
(i) \( 2x(a) - 3x'(a) = 0 \)
(ii) \( x(a) = x(b) \)
(iii) \( x(a) = 0 \)
(iv) \( x(a) + 6 = x(b) \)
(v) \( x(a) + 2x^2(b) = 0 \)

5.59 Use Theorem 5.102 to find the Green’s function for the left focal BVP
\[
(p(t)x')' = 0,
\]
\[
x'(a) = 0, \quad x(b) = 0.
\]
5.60 Show that if the boundary value problem (5.32) – (5.34) has only the trivial solution, then the Green’s function for the BVP (5.32) – (5.34) satisfies \( G(s+, s) = G(s-, s) \) and satisfies the *jump condition* \[ G'(s+, s) - G'(s-, s) = \frac{1}{p(s)}, \]
for \( s \in (a, b) \). Here \( G(s+, s) := \lim_{t \to s^+} G(t, s) \).

5.61 Use Theorem 5.102 to find the Green’s function for the BVP \( x'' = 0, \ x(0) - x'(0) = 0, \ x(1) + x'(1) = 0. \)

5.62 For each of the following, find an appropriate Green’s function and solve the given BVP:

(i) \( x'' = t^2, \ x(0) = 0, \ x(1) = 0 \)

(ii) \( (e^{2t}x')' = e^{3t}, \ x(0) = 0, \ x(\log(2)) = 0 \)

(iii) \( (e^{-5t}x')' + 6e^{-5t}x = e^{3t}, \ x(0) = 0, \ x(\log(2)) = 0 \)

5.63 Prove Corollary 5.103.

5.64 Use Corollary 5.103 to solve the BVPs

(i) \( x'' - 3x' + 2x = e^{3t}, \ x(0) = 1, \ x'(\log(2)) = 2 \)

(ii) \( x'' + 4x = 5e^t, \ x(0) = 1, \ x'(\frac{\pi}{8}) = 0 \)

(iii) \( t^2x'' - 6tx' + 12x = 2t^5, \ x(1) = 0, \ x'(2) = 88 \)

5.65 Using an appropriate Green’s function solve the BVP \( x'' = t, \ x(0) = 0, \ x(1) = 1. \)

5.66 Let \( G(\cdot, \cdot) \) be the Green’s function for the right focal BVP \( x'' = 0, \ x(a) = x'(b) = 0. \)

Show that \( -(b - a) \leq G(t, s) \leq 0, \) for \( a \leq t, s \leq b, \)
\[ \int_a^b |G(t, s)| \, ds \leq \frac{(b - a)^2}{2}, \] for \( a \leq t \leq b, \)

and
\[ \int_a^b |G'(t, s)| \, ds \leq b - a, \] for \( a \leq t \leq b. \)

5.67 Find the Green’s function for the BVP
\[ \left( \frac{x'}{1 + t} \right)' = 0, \ x(0) = 0, \ x(1) = 0. \]

Use this Green’s function and an appropriate variation of constants formula to solve the BVP
\[ \left( \frac{x'}{1 + t} \right)' = t, \ x(0) = 0, \ x'(1) = 1. \)

5.68 In each of the following use Theorem 5.114 to find \( T \) as large as possible so that the given differential equation is disconjugate on \([0, T]\).
5. The Self-Adjoint Second-Order Differential Equation

(i) \( x'' + t^2 x = 0 \)  
(ii) \( x'' + (t^2 + \frac{1}{3}) x = 0 \)  
(iii) \( x'' + (\frac{5}{3} + 3t^2) x = 0 \)  
(iv) \( x'' + \frac{3}{2} t^2 x = 0 \)  
(v) \( x'' + 4(t - 1)x = 0 \) 

In part (iii) use your calculator to solve the inequality in \( T \) that you got.

5.69 Use Theorem 5.114 to prove that if \( q \) is a continuous function on \([a, \infty)\), then there is a \( \lambda_0, 0 \leq \lambda_0 \leq \infty \) such that \( x'' + \lambda q(t)x = 0 \) is oscillatory on \([a, \infty)\) if \( \lambda > \lambda_0 \) and nonoscillatory on \([a, \infty)\) if \( 0 \leq \lambda < \lambda_0 \). Also show that if \( \int_a^\infty q(t) \, dt = \infty \), then \( \lambda_0 = 0 \).

5.70 For each of the following, find an appropriate Green’s function and solve the given periodic BVP (note that by Theorem 5.118 the solution you will find is periodic with period \( b - a)\):

(i) \( x'' + x = 4 \), \( x(0) = x(\pi), \quad x'(0) = x'(\pi) \)  
(ii) \( x'' + x = 2 \), \( x(0) = x(\frac{\pi}{2}), \quad x'(0) = x'(\frac{\pi}{2}) \)  
(iii) \( x'' + x = \cos(4t) \), \( x(0) = x(\frac{\pi}{2}), \quad x'(0) = x'(\frac{\pi}{2}) \)  
(iv) \( x'' - x = \sin t \), \( x(0) = x(\pi), \quad x'(0) = x'(\pi) \)