Faculteit Wiskunde en Informatica

Lecture notes for courses on

Complex Analysis, Fourier Analysis and Asymptotic Analysis of Integrals

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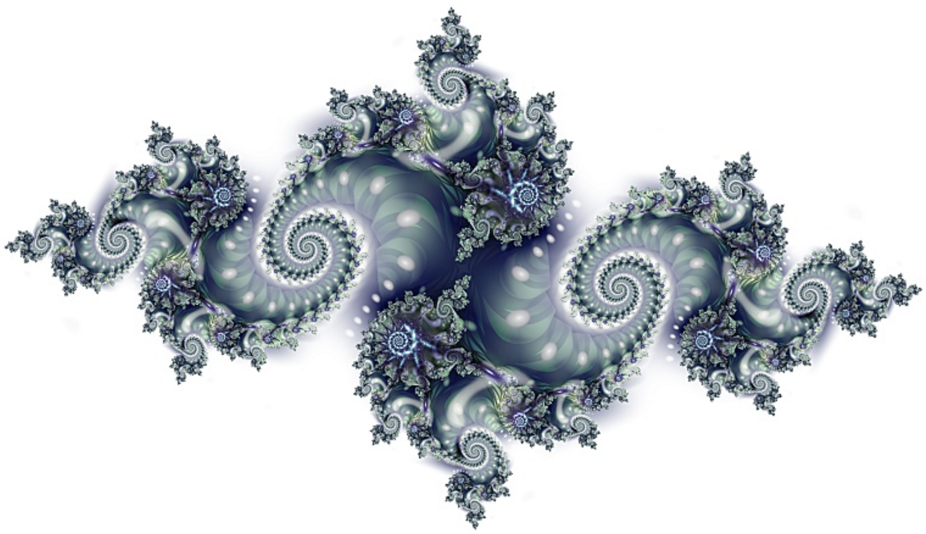

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Contents

Chapter 1

Holomorphic Functions

The subject of chapters 1, 2 and 3 of these lecture notes is "Complex Analysis", and more in particular "Complex Function Theory". This theory examines the properties of functions of complex variables. Complex numbers can be naturally identified with elements of $\mathbb{R}^2$. All definitions and results for two-dimensional vectors are therefore also applicable to complex numbers. For complex numbers extra operations are defined, viz. multiplication and division. These operations imply so many special properties that a very rich and powerful theory may be derived for functions of complex variables.

In this first chapter we will give definitions and basic properties of holomorphic functions. These are defined as functions which are differentiable in an open domain. In the next chapter it will be shown that these functions have especially beautiful properties. In the third chapter we will apply these properties to evaluate some classes of integrals. First we will, however, summarise some general properties of complex numbers.

1.1 Complex numbers

We denote a **complex number** as $a + \mathrm{i}b$, where a and b are real numbers. In a natural way we can identify a complex number $a + \mathrm{i}b$ with an element (a, b) of $\mathbb{R}^2$.

For complex numbers, we define the following operations:

addition $(a + \mathrm{i}b) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d)$,

subtraction $(a + \mathrm{i}b) - (c + \mathrm{i}d) = (a - c) + \mathrm{i}(b - d)$,

multiplication $(a + \mathrm{i}b)(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(ad + bc)$,

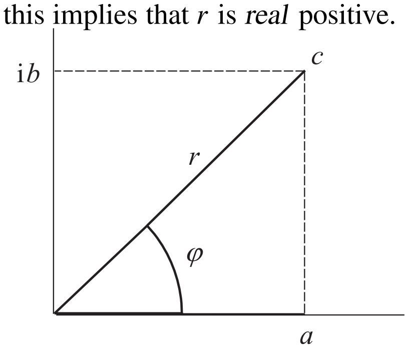
division $\dfrac{a + \mathrm{i}b}{c + \mathrm{i}d} = \dfrac{ac + bd}{c^2 + d^2} + \mathrm{i}\dfrac{bc - ad}{c^2 + d^2}$.

In reality we do not use these definitions for the calculus of complex numbers, but the normal rules known for real numbers and the additional rule $\mathrm{i}^2 = -1$.

We denote the set of complex numbers by $\mathbb{C}$. Since $\mathbb{C}$ is in fact equal to the plane $\mathbb{R}^2$ with an additional property, we call $\mathbb{C}$ also the **complex plane**. For a complex number $c = a + \mathrm{i}b$ we call a the **real part** of c. We write $a = \operatorname{Re} c$. Similarly we call b the **imaginary part** of c. We write $b = \operatorname{Im} c$. If $b = 0$, so $c = a$, we say that c is **real** and we write also $c \in \mathbb{R}$. We call the set of these points in $\mathbb{C}$ the **real axis**. If $a = 0$, so $c = \mathrm{i}b$, we call c **imaginary** or purely imaginary. The collection of such complex numbers is called the **imaginary axis**. The complex number $\overline{c} = a - \mathrm{i}b$ is called the **complex conjugate** of c. We will sometimes use the rules $\operatorname{Re} c = \frac{1}{2}(c + \overline{c})$ and $\operatorname{Im} c = \frac{1}{2\mathrm{i}}(c - \overline{c})$.

For the set $\mathbb{C}$ we have the algebraic operations $(+, -, \times, /)$ with properties as in $\mathbb{R}$. However, there is in $\mathbb{C}$, unlike $\mathbb{R}$, no ordering defined. So we cannot decide for two complex numbers c and d whether $c > d$. The "less than" or "greater than" signs cannot be used in $\mathbb{C}$. We will adopt the convention that if we say that a number r satisfies $r > 0$, this implies that r is *real* positive.

A point c in the complex plane can also be described by polar coordinates r, φ. Here is $r = \sqrt{a^2 + b^2}$ the distance of c to the origin. We also denote this by $|c|$. We call $|c|$ the **modulus** of c. Note that for two complex numbers c_1 and c_2, the expression $|c_1 - c_2|$ represents the distance between c_1 and c_2 in the complex plane.



In addition, φ is the argument of the point $(a, b) \neq 0$ in $\mathbb{R}^2$. We also call φ the **argument** of the complex number $c = a + \mathrm{i}b$, and we write $\varphi = \arg(c)$. Between the cartesian and polar coordinates we have the following relations (with, as we will see later, a natural definition of $\mathrm{e}^{\mathrm{i}\varphi}$)

$$a = r\cos\varphi, \quad b = r\sin\varphi, \quad c = r(\cos\varphi + \mathrm{i}\sin\varphi) = r\,\mathrm{e}^{\mathrm{i}\varphi}\,.$$

We see from these relations that a, b and c are defined by r and φ. The reverse is not true. Although r is completely determined by a and b, this is not true for φ. Of course, φ is not defined if $c = 0$, but if $c \neq 0$, φ is not completely determined by c, as we can always replace φ by $\varphi + 2k\pi$, with k any integer. In order to fully determine φ (for $c \neq 0$), we need to restrict φ, for example by $-\pi < \varphi \leqslant \pi$. There is always exactly one value of the argument of c that satisfies this condition. For this particular choice, $\varphi \in (-\pi, \pi]$, we call φ the **principal value** of the argument, sometimes denoted by $\operatorname{Arg}(c)$.

PROPERTY 1.1.1. *For complex numbers z_1 and z_2 we have*

1. $|z_1 z_2| = |z_1| \cdot |z_2|, \quad \arg(z_1 z_2) \equiv \arg z_1 + \arg z_2 \pmod{2\pi}$,
2. $|z_1/z_2| = |z_1|/|z_2|, \quad \arg(z_1/z_2) \equiv \arg z_1 - \arg z_2 \pmod{2\pi}$.

*For the second line we assume that $z_2 \neq 0$. Furthermore, we have the **triangle inequalities***

3. $\big||z_1| - |z_2|\big| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|$.

For z's complex conjugate we have

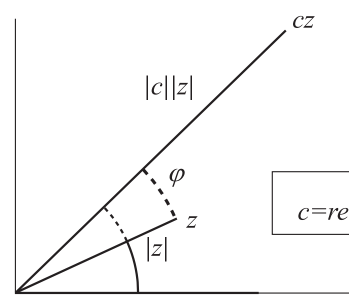
4. $|\overline{z}| = |z|, \quad \arg \overline{z} \equiv -\arg z \pmod{2\pi}, \quad z\overline{z} = |z|^2.$

Note: $\operatorname{Arg}(z_1 z_2^{\pm 1})$ is not always $\operatorname{Arg}(z_1) \pm \operatorname{Arg}(z_2)$, but it is true that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$.

We can interpret the multiplication of $z = x + \mathrm{i}y$ by a complex number $c = a + \mathrm{i}b$ in $\mathbb{C}$ as a linear mapping $L_c\colon \mathbb{R}^2 \to \mathbb{R}^2$, defined by $L_c(x, y)^T \equiv cz$. We have indeed $c(z_1 + z_2) = cz_1 + cz_2$ and $c(\lambda z_1) = \lambda c z_1$ for all $z_1, z_2 \in \mathbb{C}$ and all $\lambda \in \mathbb{R}$. We know from linear algebra that with a given basis a linear mapping can be represented by a matrix. To find this matrix, we apply the linear transformation to the unit vectors $\underline{e}_1 = (1, 0)^T \equiv 1$ and $\underline{e}_2 = (0, 1)^T \equiv \mathrm{i}$. Then we find $L_c\underline{e}_1 \equiv c \cdot 1 = c \equiv (a, b)^T$ and $L_c\underline{e}_2 \equiv c \cdot \mathrm{i} = c\mathrm{i} \equiv (-b, a)^T$, where we identify complex numbers with vectors in $\mathbb{R}^2$, for example $a + \mathrm{i}b$ with $(a, b)^T$. We find that the matrix of L_c is equal to

$$L_c = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Conversely, each linear image with such a matrix can be written as a complex multiplication. Characteristic of the matrix L_c are two properties: columns have the same length $\sqrt{a^2 + b^2}$, and are perpendicular to each other. The matrix is even more transparent if we represent the number $a + \mathrm{i}b$ in polar coordinates $r(\cos\varphi + \mathrm{i}\sin\varphi)$. Then we find



$$L_c = r \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

From this we see that a complex multiplication can always be seen as a **scaling and rotation**, *i.e.* a scaling (by a factor r) and a rotation (by an angle φ). The order of these operations is unimportant.

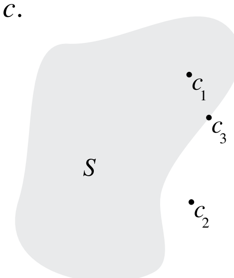
Some geometric figures in the plane are easily expressed as sets of complex numbers. They can often be expressed by means of equations. Examples are

$\operatorname{Re} z = 3$	The straight line through $z = 3$, parallel to the imaginary axis
$\operatorname{Re} z > 3$	The half plane on the right of the line $\operatorname{Re} z = 3$
$\operatorname{Im} z = -2$	The straight line through $z = -2\mathrm{i}$, parallel to the real axis
$\lvert z\rvert = 1$	The circle with centre $z = 0$ and radius 1, so the **unit circle**
$\lvert z\rvert \geqslant 1$	The area outside and including the unit circle
$\lvert z - a\rvert = r$	The circle with centre $z = a$ and radius r.

1.2 Open and closed sets

We introduce some properties of points and sets in $\mathbb{C}$.

- Let c be a complex number and $r > 0$.
  - The set $B_r(c) = \{z \in \mathbb{C} \mid |z - c| < r\}$ is called a **neighbourhood** (or more explicitly, the r**-neighbourhood**) of c.
  - The set $\mathring{B}_r(c) = \{z \in \mathbb{C} \mid 0 < |z - c| < r\}$ is called a **reduced neighbourhood**, **deleted neighbourhood** or **punctured neighbourhood** of c.
- Let c be a complex number and $S \subseteq \mathbb{C}$ a set.
  - c is called an **interior point** of S, if there is a neighbourhood of c which is contained in S.
  - c is called an **exterior point** of S, if there is a neighbourhood of c which is disjoint with S.
  - c is called a **boundary point** of S, if c is no interior and no exterior point of S.



c_1: interior point

c_2: exterior point

c_3: boundary point

Each point of $\mathbb{C}$ is either an interior point, an exterior point or a boundary point of S. It is clear that an interior point always belongs to the set. An exterior point never belongs to the set. A boundary point may or may not belong to the set.

EXAMPLE 1.2.1. *Consider the set* $S_1 = \{z \in \mathbb{C} \mid 1 < |z| \leqslant 2\}$. *The points* z *with* $|z| = 1$ *are boundary points which do not belong to* S_1, *whereas the points* z *with* $|z| = 2$ *are boundary points which do belong to* S_1.

The set $\mathring{S}$ of interior points of $S \subseteq \mathbb{C}$ is called the **interior** of S, the set ∂S of boundary points of S is called the **boundary** of S. Their union, the set $\overline{S}$ consisting of the boundary points and the interior points of S, is called the **closure** of S.

Let $S \subseteq \mathbb{C}$ be a set.

- S is called **open** if $\mathring{S} = S$, *i.e.* no boundary points of S belong to S. Note that a neighbourhood is open.
- S is called **closed** if $\overline{S} = S$, *i.e.* all boundary points belong to S. Note that a (set consisting of a) single point is closed.

EXAMPLE 1.2.2.

1. *The set* $S_2 = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ *is open, the set* $S_3 = \{z \in \mathbb{C} \mid 1 \leqslant |z| \leqslant 2\}$ *is closed, and the set* S_1 *mentioned in the previous example is not open nor closed.*
2. *A curve is always closed.*

If $S \subseteq \mathbb{C}$, then the set $\mathbb{C} \backslash S = \{z \in \mathbb{C} \mid z \notin S\}$ is called the **complement** of S (in $\mathbb{C}$). From the definition it follows immediately:

LEMMA 1.2.3. *$S \subseteq \mathbb{C}$ is closed if and only if $\mathbb{C} \backslash S$ is open.*

The interior of a set is always an open set, the boundary and the closure are always closed.

A set $V \subseteq \mathbb{C}$ is called **bounded** if there is a number $M > 0$ such that $|z| \leqslant M$ for every $z \in V$.

1.3 Limits and continuity

We can introduce convergence and continuity in $\mathbb{C}$ in a similar way as in $\mathbb{R}$ or $\mathbb{R}^2$. The concepts we introduce in this section are identical to the corresponding concepts for $\mathbb{R}$ or $\mathbb{R}^2$.

DEFINITION 1.3.1. *A sequence of complex numbers $z_1, z_2, \ldots$* ***converges*** *to $c \in \mathbb{C}$ if, whatever small neighbourhood of c we choose, it is always possible to find an N such that all z_n, for n beyond this N, are located inside this neighbourhood. Formally:*

$$\forall(\varepsilon > 0)\ \exists(N \in \mathbb{N})\ \forall(n > N)\ |z_n - c| < \varepsilon.$$

In that case we say that c is the **limit** of the sequence z_n. We write this as $z_n \to c \quad (n \to \infty)$ or $\lim_{n\to\infty} z_n = c$. We have

$$\big(z_n \to c \quad (n \to \infty)\big) \quad \Leftrightarrow \quad \left(\left.\begin{array}{l} \operatorname{Re} z_n \to \operatorname{Re} c \\ \operatorname{Im} z_n \to \operatorname{Im} c \end{array} \right\} \quad (n \to \infty) \right). \tag{1.1}$$

The proof is the same as the corresponding proof for sequences in $\mathbb{R}^2$. By using this property, the problem of the convergence of a complex sequence can be reduced to the problem of real sequences.

A **complex series** is a series with complex numbers as terms. The series $a_0 + a_1 + a_2 + \ldots$ with the terms $a_0, a_1, \ldots$ is written as $\sum_{n=0}^{\infty} a_n$. We say that the series **converges** or **is convergent** if the sequence of partial sums $s_N = \sum_{n=0}^{N} a_n$ converges. Otherwise we say that the series **diverges** or **is divergent**. We say that the series $\sum_{n=0}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=0}^{\infty} |a_n|$ is convergent. A series that is convergent but not absolutely convergent, is called **conditionally convergent**.

The following theorem summarises some fundamental results on convergence of series:

THEOREM 1.3.2. *Let the series $\displaystyle\sum_{n=0}^{\infty} a_n$ be given.*

1. *The series converges if and only if both $\sum_{n=0}^{\infty} \operatorname{Re} a_n$ and $\sum_{n=0}^{\infty} \operatorname{Im} a_n$ converge.*

2. *If the series converges, then $a_n \to 0 \quad (n \to \infty)$.*

3. *If the series converges absolutely, then it converges.*

4. *If the series converges absolutely, then the sum is independent of the order of summation.*

PROOF:

1. *This follows directly from (1.1), applied to the partial sums.*
2. *This result is known for real series, and follows then with (1.1).*
3. *Assume that* $\sum_{n=0}^{\infty} |a_n|$ *converges. From the comparison theorem for real series it follows that* $\sum_{n=0}^{\infty} |\operatorname{Re} a_n|$ *and so* $\sum_{n=0}^{\infty} \operatorname{Re} a_n$ *is convergent (note that* $|\operatorname{Re} a_n| \leqslant |a_n|$*). The same is true for* $\sum_{n=0}^{\infty} \operatorname{Im} a_n$*. We can apply now 1.*
4. *The proof is rather technical.*

EXAMPLE 1.3.3. *An important example is the geometric series* $\displaystyle\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$*, absolutely convergent for* $|\rho| < 1$.

The following convergence criteria, known from real analysis, also apply to complex series:

THEOREM 1.3.4. *Let the series* $\sum_{n=0}^{\infty} a_n$ *be given.*

- ***Cauchy** or root test: if*[1] $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ *exists and*
  1. $\rho < 1$*, then the series converges (absolutely).*
  2. $\rho > 1$*, then the series diverges (absolutely).*
- ***d'Alembert** or ratio test: if*[2] $\rho = \lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}$ *exists and*
  1. $\rho < 1$*, then the series converges (absolutely).*
  2. $\rho > 1$*, then the series diverges (absolutely).*

No statement about convergence can be made in case of any of the above limits $\rho = 1$.

Absolute convergence is important for multiplication of series.

THEOREM 1.3.5. *Two absolutely convergent series may be multiplied without ambiguity about the result, by taking the sum over all products of terms. A useful representation of the result is the Cauchy product, given by*

$$\Big(\sum_{n=0}^{\infty} a_n\Big) \cdot \Big(\sum_{m=0}^{\infty} b_m\Big) = \sum_{k=0}^{\infty} c_k, \qquad c_k = \sum_{j=0}^{k} a_j b_{k-j}.$$

[1] More generally: if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \lessgtr 1$.
[2] More generally: if $\limsup_{n\to\infty} |a_{n+1}|/|a_n| < 1$, and $\liminf_{n\to\infty} |a_{n+1}|/|a_n| > 1$, respectively.

A complex function of a complex variable is a mapping $f\colon V \to \mathbb{C}$, where $V \subseteq \mathbb{C}$ is the **definition set** (sometimes called **domain**[3]) of f. The formula $w = f(z)$ can be written out in its real and imaginary parts. If we write $z = x + \mathrm{i}y$, $w = u + \mathrm{i}v$, u and v can be considered as real functions of the two real variables x and y. We will denote these functions also as u and v. We call $u(x, y)$ and $v(x, y)$ the **component functions** of $f(z)$. We have therefore:

$$f(z) = u(x, y) + \mathrm{i}v(x, y). \tag{1.2}$$

Conversely, we can interpret a couple of functions of two variables $u(x, y)$, $v(x, y)$ as the real and imaginary parts of a complex function f according to formula (1.2).

We now define convergence of a function $f(z)$ to a value L as z approaches a point a. In order to define this concept, it is necessary that there are points z, arbitrarily close to a but not equal to a, where $f(z)$ is defined.

DEFINITION 1.3.6. *Let $V \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. We say that a is a* ***limit point*** *(or accumulation point) of V, if each reduced neighbourhood of a contains points of V.*

DEFINITION 1.3.7. *Let $f\colon V \to \mathbb{C}$ and let a be a limit point of V. We say that $f(z)$* ***converges*** *to a constant L for $z \to a$ if, whatever small neighbourhood $B_\varepsilon(L)$ of L we choose, it is always possible to find a small enough reduced neighbourhood $\mathring{B}_\delta(a)$ of a, such that f maps its image entirely inside $B_\varepsilon(L)$. Formally:*

$$\forall(\varepsilon > 0)\ \exists(\delta > 0)\ \forall(z \in V)\ \big(0 < |z - a| < \delta\ \Rightarrow\ |f(z) - L| < \varepsilon\big)$$

We say that L is the limit of $f(z)$ for $z \to a$. We write $f(z) \to L\ (z \to a)$, or $\lim_{z\to a} f(z) = L$.

Note that in this definition it is not required (but allowed) that $f(z)$ is defined in a. The standard properties of limits, such as $\lim_{z\to a}(f(z)+g(z)) = \lim_{z\to a} f(z)+\lim_{z\to a} g(z)$ if both limits exist, remain valid for complex functions.

DEFINITION 1.3.8. *Let $f\colon V \to \mathbb{C}$ and let $a \in V$. We say that $f(z)$ is* ***continuous*** *in a if its limit for z to a is $f(a)$ itself. Formally:*

$$\forall(\varepsilon > 0)\ \exists(\delta > 0)\ \forall(z \in V)\ \big(|z - a| < \delta\ \Rightarrow\ |f(z) - f(a)| < \varepsilon\big)$$

If $W \subseteq V$, then the function $f(z)$ is ***continuous*** *on W if $f(z)$ is continuous in all $a \in W$.*

REMARK 1.3.9. *For the definition of continuity it is necessary that a belongs to the domain V of $f(z)$. On the other hand, a does not have to be a limit point of V, in contrast to the definition of limit. If both continuity and limit can be defined, so if both a is an element and a limit point of V, then it follows from the definition that f is continuous in a if and only if $\lim_{z\to a} f(z) = f(a)$.*

THEOREM 1.3.10. *If $f(z)$ is a continuous function in $\mathbb{C}$ with component functions $u(x, y)$ and $v(x, y)$, then the functions u and v are continuous in $\mathbb{R}^2$ (and vice versa). In fact, the definition of continuity is exactly the same. Also the functions $z \mapsto \operatorname{Re} f(z)$, $z \mapsto \operatorname{Im} f(z)$, $z \mapsto |f(z)|$ are continuous functions from $\mathbb{C}$ to $\mathbb{R}$.*

[3]Note that *domain* may also refer to a *connected open set*; see page 25.

Note, however, that the principal value of $\arg z$ is discontinuous along the negative real axis.

We also consider sequences $f_n(z)$ of which the elements are complex functions, defined on a domain $V \subseteq \mathbb{C}$. If the sequence converges (definition 1.3.1) for every $z \in V$ to $f(z)$, we say that the sequence (f_n) is **pointwise convergent** to f on V. Although all functions $f_n(z)$ may be smooth, this is not necessarily the case for the (pointwise) limit function $f(z)$. Therefore, the concept of uniform convergence is more useful for us.

DEFINITION 1.3.11. *A sequence of complex functions $f_n(z)$, defined on a set $V \subseteq \mathbb{C}$, is called* ***uniformly convergent*** *on V with limit $f(z)$, if index N only depends on ε and not on z:*

$$\forall(\varepsilon > 0)\ \exists(N \in \mathbb{N})\ \forall(n > N)\ \forall(z \in V)\ |f(z) - f_n(z)| < \varepsilon.$$

Uniform convergence implies pointwise convergence, but the reverse is not true, as may be illustrated by the following

EXAMPLE 1.3.12. *The sequence $f_n(z) = z^n \to 0$ is pointwise but not uniformly convergent on the unit disc $|z| < 1$. (Why?) On the other hand, it is uniformly convergent on any disc $|z| \leqslant r < 1$. (How about z^n/n, and nz^n? Can you prove this?)*

The following result, which is also known from analysis, gives one of the reasons why uniform convergence is important.

THEOREM 1.3.13. *The limit of a uniformly convergent sequence of continuous functions is continuous. So if the sequence $f_n(z)$ converges uniformly to $f(z)$, and the functions $f_n(z)$ are continuous, then $f(z)$ is continuous.*

REMARK 1.3.14. *It can be shown that the limit of a sequence of (Riemann) integrals $\lim_{n\to\infty} \int f_n\, dx$ is the integral of the limit $\int f\, dx$ if the sequence $f_n \to f$ is uniformly convergent, but this is not the case for (real) differentiability. However, we will see later (2.5.13) that the limit of uniformly convergent holomorphic functions is again a holomorphic function.*

We also need uniform convergence of series. A series of functions $\sum_{n=0}^{\infty} f_n(z)$ is called uniformly convergent if the sequence of partial sums converges uniformly. For uniform convergence of series, we have the following important criterion:

THEOREM 1.3.15 (**Criterion of Weierstrass**). *Let $|f_n(z)| \leqslant a_n$ for $n = 1, 2, \ldots$ and $z \in V$, and let $\sum_{n=1}^{\infty} a_n$ be convergent. Then the series $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on V.*

EXAMPLE 1.3.16. *Since $\sum r^n$ and $\sum nr^n$ converge for any $0 \leqslant r < 1$, $\sum z^n$, $\sum z^n/n$ and $\sum nz^n$ are uniformly convergent on any disc $|z| \leqslant r < 1$. Since $\sum 1/n^2$ converges, $\sum z^n/n^2$ is uniformly convergent on $|z| \leqslant 1$.*

1.4 Differentiable functions

The concepts of limit and continuity, as discussed in the previous section, are for complex functions the same as for real functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. The situation is different with differentiability. That is understandable, because differentiability involves the operations multiplication and division, which are defined for complex numbers in $\mathbb{C}$, but not for vectors in $\mathbb{R}^2$. We know the concept of differentiability for real functions of several variables. For functions $f : \mathbb{R} \to \mathbb{R}$, we have the definition:

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable in a if the following limit exists:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

The following equivalent characterisation of differentiability emphasises the essential property of the derivative, namely the linear approximation of a function in the vicinity of a given point:

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable in a if there exist a number $c \in \mathbb{R}$ and a function ρ such that

$$f(a+h) = f(a) + ch + h\rho(h),$$

and $\rho(h) \to 0$ if $h \to 0$.

This latter characterisation is also more easily extended to functions in multi-dimensional space. Thus we say that a function $\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable if it can be approximated locally by a linear function. In other words, the function $\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is called **differentiable** in $\underline{a}$ if there exists a linear mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$\underline{f}(\underline{a} + \underline{h}) = \underline{f}(\underline{a}) + L\underline{h} + |\underline{h}|\underline{\rho}(\underline{h}), \tag{1.3}$$

where $\underline{\rho}(\underline{h}) \to 0$ for $\underline{h} \to \underline{0}$.

If $\underline{f} = (u, v)^T$, the linear mapping L, called the **functional operator**, has the matrix

$$L = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}.$$

Here is u_x the partial derivative of u to x, etc.

Now let $f(z)$ be a complex function. Just like with real functions $\underline{f} : \mathbb{R}^2 \to \mathbb{R}^2$, we say that f is differentiable in a if the function can be approximated by a linear function in the neighbourhood of a. However, rather than a matrix multiplication the linear function is here given by a complex multiplication. So the function $f(z)$ defined on an open set V, is called **differentiable** in $a \in V$, if and only if there exists a number $c \in \mathbb{C}$ such that

$$f(a+h) = f(a) + ch + h\rho(h), \tag{1.4}$$

where ρ is a function with the property $\rho(h) \to 0$ for $h \to 0$. The function $h \mapsto f(a) + ch$ is the linear approximation of $f(a+h)$ for small h, and the term $h\rho(h)$ represents a term of higher order, which is small compared with h.

The following characterisation is equivalent:

DEFINITION 1.4.1. *A function f, defined on an open set V, is* **differentiable** *in $a \in V$ if*

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{1.5}$$

exists. Instead of (1.5) we can of course also write $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$.

The limit (1.5), which we indicate by $f'(a)$, is equal to the coefficient c in (1.4). We call $f'(a)$ the **derivative** of f in a. Note that point a in the definition is an interior point of V (since V is an open set). The point a is therefore a limit point (and also an element) of V. The limit in the definition above is therefore well defined.

If the limit $f'(a)$ exists, it should be the same for any way the limit $h \to 0$ is taken. Therefore

$$\begin{aligned} f'(a) &= \lim_{\substack{h_1\to 0\\ h_2=0}} \frac{u(a_1+h_1,a_2)-u(a_1,a_2)}{h_1} + \mathrm{i}\frac{v(a_1+h_1,a_2)-v(a_1,a_2)}{h_1} = \quad u_x + \mathrm{i}v_x \\ &= \lim_{\substack{h_1=0\\ h_2\to 0}} \frac{u(a_1,a_2+h_2)-u(a_1,a_2)}{\mathrm{i}h_2} + \mathrm{i}\frac{v(a_1,a_2+h_2)-v(a_1,a_2)}{\mathrm{i}h_2} = -\mathrm{i}u_y + v_y \end{aligned}$$

and so a *necessary* condition for differentiability to z is $u_x = v_y$ and $u_y = -v_x$.

It is even a sufficient condition if we know that f is differentiable as a function in $\mathbb{R}^2$. We have seen in section 1.1 that complex multiplication as in the expression ch in (1.4) can also be considered as a linear mapping of a particular type, namely a rotation-scaling, with a matrix of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Compared with differentiation in $\mathbb{R}^2$, differentiability of a complex function requires additionally that the functional matrix L has this form, and thus satisfies the equations $u_x = v_y$, $u_y = -v_x$. We can summarise the result of these considerations as follows:

THEOREM 1.4.2. *A complex function $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is differentiable in the point $a \in \mathbb{C}$ if an only if f is differentiable in $a \in \mathbb{C}$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$, while at the same time the following differential equations are satisfied:*

$$u_x = v_y, \quad u_y = -v_x.$$

These differential equations are called the **equations of Cauchy-Riemann**. As the differentiability of a complex function in an open set has such far-reaching consequences, we use a separate term.

DEFINITION 1.4.3 (**Holomorphic**). *We call a function that is differentiable in an open set $\mathcal{V}$, holomorphic*[4] *(or analytic) in $\mathcal{V}$. We say that a function is holomorphic at a point z_0 if the function is holomorphic in a neighbourhood of z_0.*

A point where the function $f(z)$ is holomorphic is referred to as a **regular point** of the function. Each other point is called a **singular point** of $f(z)$.

[4]From ὁλος = *whole, entire, complete* and μορφη = *form, shape*.

Near a point a where f is holomorphic, we consider the image of $z_\vartheta = a + h\,\mathrm{e}^{\mathrm{i}\vartheta}$ as it rotates around a by angle ϑ. Noting that $\Delta f_\vartheta = f(z_\vartheta) - f(a) \simeq h\,\mathrm{e}^{\mathrm{i}\vartheta}\, f'(a)$, we compare z_{ϑ_2} and z_{ϑ_1} while $f'(a) \neq 0$, to find that $\Delta f_{\vartheta_2}/\Delta f_{\vartheta_1} = \mathrm{e}^{\mathrm{i}\vartheta_2 - \mathrm{i}\vartheta_1}$, or $\arg(\Delta f_{\vartheta_2}) - \arg(\Delta f_{\vartheta_1}) = \vartheta_2 - \vartheta_1$. So the image rotates by the same angle as its argument. In other words, a holomorphic function with non-zero derivative **preserves angles** and is called a **conformal mapping**.

As in the real case, differentiable functions are continuous.

THEOREM 1.4.4. *If f is differentiable in a, then f is continuous in a.*

PROOF: *We can write for $z \neq a$*

$$f(z) - f(a) = \frac{f(z) - f(a)}{z - a}\,(z - a),$$

and so

$$\lim_{z\to a} f(z) - f(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a} \cdot \lim_{z\to a}(z - a) = f'(a) \cdot 0 = 0.$$

Apparently is $\lim_{z\to a} f(z) = f(a)$, and so according to Remark 1.3.9 is f continuous in a.

Already with real functions we saw that the reverse is not true. Continuous functions may have one or more 'kinks'. An example is $f(x) = |x|$. With complex functions we may have similar examples. Yet it is remarkable that there are also functions seemingly perfectly 'smooth' but still not differentiable. This is the case when the equations of Cauchy-Riemann are not satisfied, which may be illustrated by the following observation:

PROPERTY 1.4.5. *A function, which is holomorphic everywhere in $\mathbb{C}$ and assumes only real values, is a constant.*

PROOF: *If $f(z) = u(x, y) + \mathrm{i}v(x, y)$ assumes only real values in $\mathbb{C}$, then the function v is identically equal to zero. Consequently also v_x and v_y are equal to zero. From the equations of Cauchy-Riemann it follows that $u_x = u_y = 0$. The gradient of the function u is therefore equal to zero. This implies that u is a constant.*

EXAMPLE 1.4.6. *The function z is differentiable and holomorphic everywhere with derivative 1. The functions $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\overline{z}$ are nowhere differentiable, and therefore not holomorphic, because they don't satisfy the equations of Cauchy-Riemann. The function $|z|^2$ is differentiable in $z = 0$ but not anywhere else, and is therefore nowhere holomorphic.*

The rules we know for differentiation of real functions and their sums, differences, products and quotients are also applicable here. We have

$$\begin{aligned}
(f \pm g)'(a) &= f'(a) \pm g'(a),\\
(fg)'(a) &= f'(a)g(a) + f(a)g'(a),\\
\left(\frac{f}{g}\right)'(a) &= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}, \quad \text{if } g(a) \neq 0.
\end{aligned}$$

Furthermore, we have also the **chain rule**:

THEOREM 1.4.7. *Let g be differentiable in z_0 and let f be differentiable in $\zeta_0 = g(z_0)$. Then the composite function $h = f \circ g$ (given by $h(z) = f(g(z))$) is differentiable in z_0 with derivative $h'(z_0) = f'(\zeta_0)g'(z_0)$.*

As a consequence of these rules we can understand that the following functions are differentiable everywhere, where they are defined, and we can determine the derivatives.

$f(z)$	$f'(z)$
C (constant)	0
z	1
z^n	nz^{n-1}
z^{-n}	$-nz^{-n-1}$ if $z \neq 0$.
$\sum_{k=0}^{n} a_k z^k$ (polynomial)	$\sum_{k=1}^{n} k a_k z^{k-1}$

1.5 The Equations of Cauchy-Riemann

Let $f(z)$ be a holomorphic function defined on an open set $V \subseteq \mathbb{C}$, with real and imaginary parts $u(x, y)$ and $v(x, y)$. We have seen in the previous paragraph that u and v have partial derivatives u_x, u_y, v_x, v_y which satisfy the equations of Cauchy-Riemann:

$$u_x = v_y, u_y = -v_x.$$

We will further investigate these equations here. If we assume that u and v are **continuously differentiable**, that means differentiable with continuous derivatives, then there is also the reverse result:

THEOREM 1.5.1. *Let the real functions $u(x, y)$ and $v(x, y)$ have continuously differentiable partial derivatives in an open set $V \subseteq \mathbb{C}$ which satisfy $u_x = v_y, u_y = -v_x$, the equations of Cauchy-Riemann. Then the function $f(z) = u(x, y) + \mathrm{i}v(x, y)$ is holomorphic in V.*

Continuity of u_x, u_y, v_x and v_y is important, as is illustrated by the following counter-example.

EXAMPLE 1.5.2. *Consider the function $f(z) = z^5/|z|^4 = u(x, y) + \mathrm{i}v(x, y)$ with $f(0) = 0$. Although f does satisfy the Cauchy-Riemann equations in $z = 0$,*

$$u_x(0, 0) = \lim_{h \to 0} \frac{h^5}{h^4 h} = 1 = v_y(0, 0), \quad \text{etc.}$$

it is not differentiable in $z = 0$, as follows from

$$\frac{f(h) - f(0)}{h} = \frac{h^5}{|h|^4 h} = \frac{h^4}{|h|^4} \not\to$$

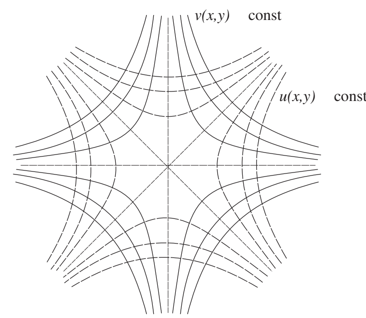
Indeed, the partial derivatives of u and v are not continuous in 0, as follows (with $z = r\,\mathrm{e}^{i\varphi}$) from

$$\begin{aligned} u_x &= -2\cos 6\varphi + 3\cos 4\varphi, & u_y &= -2\sin 6\varphi - 3\sin 4\varphi, \\ v_x &= -2\sin 6\varphi + 3\sin 4\varphi, & v_y &= 2\cos 6\varphi + 3\cos 4\varphi. \end{aligned}$$

We recall some well-known properties of scalar functions in $\mathbb{R}^n$. We can portray a function $u : \mathbb{R}^2 \to \mathbb{R}$ by means of **level curves** or contour lines, *i.e.* sets defined by the equation $u(x, y) = c$, where c is a constant. The vector $\nabla u = (u_x, u_y)$ is called the **gradient** of u, and depends on $\underline{x} = (x, y)$. Let $u(\underline{x}_0) = c$. Then for every point $\underline{x}$ on the level curve near $\underline{x}_0$ we have $u - c = (\underline{x} - \underline{x}_0) \cdot \nabla u + \ldots = 0$. So the gradient at any point on a level curve is perpendicular to the corresponding (tangent of the) level curve: $\nabla u \perp \{u = c\}$. As a result, there is exactly one level curve through every point where $\nabla u \neq 0$.

Associated to a holomorphic function $f(z)$ are the real and imaginary parts $u(x, y)$ and $v(x, y)$. Through every point where $f(z)$ is defined and $f'(z) \neq 0$, there are two level curves: one of u and one of v. The gradients of the functions u and v are equal to $\nabla u = (u_x, u_y)$ and $\nabla v = (v_x, v_y)$ respectively. From the equations of Cauchy-Riemann it follows that the inner product $\nabla u \cdot \nabla v = u_x v_x + u_y v_y$ is equal to 0, so $\nabla u \perp \nabla v$. We conclude that the level curves of the functions u and v are perpendicular to each other. Furthermore, in each point we have $|\nabla u| = |\nabla v|$.

EXAMPLE 1.5.3. *The function $f(z) = z^2$ has the component functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. The level curves of u form a family hyperbolas with axes $x = \pm y$. The level curves of v form a family hyperbolas with the axes the x-axis and the y-axis.*



An important result from complex function theory, that we will prove later, is that a holomorphic function is arbitrarily many times differentiable. This implies in particular that the component functions u and v are arbitrarily many times differentiable. If we differentiate u two times to x, then we find with the equations of Cauchy-Riemann

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = -(u_y)_y = -u_{yy},$$

where we interchanged the derivatives to x and to y. We see that u satisfies the **potential** or **Laplace equation**[5] $\nabla^2 u = 0$. In the same way we see that v satisfies the equation $\nabla^2 v = 0$. Solutions of the potential equation are called **harmonic functions**. We see that the real and imaginary parts of a holomorphic function are harmonic functions.[6]

Conversely, we can also start with a harmonic function u in $\mathbb{R}^2$ and wonder if u is the real part of a holomorphic function. For this we need to find a function v such that u and v satisfy the equations of Cauchy-Riemann, so

$$v_x = -u_y, \quad v_y = u_x.$$

In the first equation we consider $-u_y$ as a given function of x (with y as parameter). Of this function we select arbitrarily an x-primitive function, *i.e.* a function $V(x, y)$ such that $V_x = -u_y$. Then the unknown function v satisfies $v_x(x, y) = V_x(x, y)$. We conclude that $v(x, y) = V(x, y) + \Phi(x, y)$, where $\Phi_x = 0$. This means that Φ is a function of y. The first equation of Cauchy-Riemann has the general solution $v(x, y) = V(x, y) + \Phi(y)$, where we determined V and where

[5] The Laplace operator or Laplacian is given by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Also common is the notation Δ.

[6] So the Cauchy-Riemann relations are necessary for differentiability, and harmonicity is necessary for holomorphy.

Φ is an arbitrary, for the moment unknown, function of y. We determine this function by means of the second equation of Cauchy-Riemann. We see that $v_y = V_y + \Phi' = u_x$, so

$$\Phi' = \varphi = u_x - V_y. \tag{1.6}$$

To make sure that this equation has solutions that depend on y only, φ has to be independent of x. This is indeed the case, since we have

$$\varphi_x = u_{xx} - V_{xy} = u_{xx} + u_{yy} = 0.$$

With the second equation we used the definition of V, and with the third equation we used the fact that u is harmonic. We see that we can determine Φ by a primitive of the function φ. This primitive is known up to a constant. So we arrive at the following result:

THEOREM 1.5.4. *For every harmonic function u there exists a holomorphic function $f(z)$ such that $u(x, y) = \operatorname{Re} f(z)$. This holomorphic function is uniquely defined up to a (purely imaginary) constant.*

Of course, any harmonic u also defines a holomorphic $g(z)$ such that $u = \operatorname{Im} g(z)$, simply because $\operatorname{Im} g(z) = \operatorname{Re}(-\mathrm{i}g(z))$. The harmonic function $v(x, y)$, which together with $u(x, y)$ constitutes the holomorphic function $f(z)$, is called the **associated harmonic function** of u.

EXAMPLE 1.5.5. *Let $u(x, y) = x^3 - 3xy^2 + x^2 - y^2$. We determine successively:*

$$\begin{aligned} u_x &= 3x^2 - 3y^2 + 2x, \\ u_{xx} &= 6x + 2, \\ u_y &= -6xy - 2y, \\ u_{yy} &= -6x - 2. \end{aligned}$$

From this it follows that $\nabla^2 u = u_{xx} + u_{yy} = 0$, hence u is harmonic. We determine the associated harmonic function following the previous procedure:

$$\begin{aligned} v_x(x, y) &= -u_y = 6xy + 2y, \\ v(x, y) &= 3x^2 y + 2xy + \Phi(y), \\ v_y(x, y) &= 3x^2 + 2x + \Phi' = u_x = 3x^2 + 2x - 3y^2, \\ \Phi'(y) &= -3y^2, \\ \Phi(y) &= -y^3 + C, \\ v(x, y) &= 3x^2 y + 2xy - y^3 + C. \end{aligned}$$

It is not really necessary to verify in advance if the function is indeed harmonic. If this is not the case, the right-hand side of equation (1.6) would not have been a function dependent of y only. If this is a function which depends also on x, the function u would not be harmonic (or we made an error in the calculus). The sought function $f(z)$ is now given by

$$f(z) = x^3 - 3xy^2 + x^2 - y^2 + \mathrm{i}(3x^2 y + 2xy - y^3 + C).$$

We want to write this function explicitly as a function of z. Since u and v are both polynomials of degree 3 in x and y, we expect that f is a polynomial in z is of degree 3. (From the generalised theorem of Liouville (2.6.6) we will be able to conclude that this is true in general.) If we note that

$$z^2 = (x^2 - y^2) + \mathrm{i}(2xy),$$
$$z^3 = (x^3 - 3xy^2) + \mathrm{i}(3x^2y - y^3),$$

we see easily that $f(z) = z^3 + z^2 + \mathrm{i}C$, where C is an arbitrary real constant.

1.6 Power series

A series of the form $\sum_{n=0}^{\infty} a_n(z-q)^n$ with $z, q \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$ is called a **complex power series** in the variable z with **centre** q, or, formulated more concisely, **around** q. By introducing the new variable $\zeta = z - q$, we obtain a power series in the variable ζ with centre 0, namely $\sum_{n=0}^{\infty} a_n\zeta^n$. In this way we can generalise results found for power series with centre 0 to power series with arbitrary centre. Therefore, we restrict ourselves for the moment to power series of the form $\sum_{n=0}^{\infty} a_n z^n$.

We are interested in the region of convergence of the series. We have the following auxiliary result:

LEMMA 1.6.1. *Let the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ converge for some z_0 and let ρ be a positive number with[7] $\rho < |z_0|$.*

1. *Let $\alpha = \rho/|z_0| < 1$. There exists a number $M > 0$ such that $|a_n z^n| \leqslant M\alpha^n$ for all $n \in \mathbb{N}$ and all z with $|z| \leqslant \rho$.*
2. *The series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges absolutely and[8] uniformly in $\{z \in \mathbb{C} \big| |z| \leqslant \rho\}$.*
3. *The series $\sum\limits_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely and uniformly in $\{z \in \mathbb{C} \mid |z| \leqslant \rho\}$.*

PROOF:

1. *As $\sum_{n=0}^{\infty} a_n z_0^n$ converges, we have $a_n z_0^n \to 0 \quad (n \to \infty)$. In particular the sequence $(a_n z_0^n)$ is bounded, say $|a_n z_0^n| \leqslant M$ for $n = 0, 1, 2, \ldots$. Then it follows that $|a_n z^n| = |a_n z_0^n|\,|z/z_0|^n \leqslant M\alpha^n$ for $|z| \leqslant \rho$.*
2. *The terms of the series $\sum_{n=0}^{\infty} |a_n z^n|$ are bounded from above by the terms of the convergent series $\sum_{n=0}^{\infty} M\alpha^n$. So we can apply the criterion of Weierstrass.*
3. *The terms of the series $\sum_{n=1}^{\infty} |n a_n z^{n-1}|$ are bounded from above by the terms of the convergent series $\sum_{n=1}^{\infty} Mn\alpha^{n-1}$.*

An important consequence is

[7] If $\sum a_n z_0^n$ converges absolutely, we may take $\rho = |z_0|$, since $|a_n||z|^n \leqslant |a_n||z_0|^n$ and Theorem 1.3.15.

[8] Absolute and uniform convergence go hand in hand in the case of power series. This is not generally true.

THEOREM 1.6.2. *For the region of convergence exactly one of the following statements apply:*

1. *The power series converges for* $z = 0$ *and diverges for all* $z \neq 0$.
2. *There exists a number* $R > 0$ *such that the power series converges absolutely for* $|z| < R$ *and diverges for* $|z| > R$.
3. *The power series converges for all* $z \in \mathbb{C}$.

PROOF: *If there exists a* $z_0 \neq 0$ *where the series converges and a* z_1 *where the series diverges, then we have to show that there exists an* $R > 0$ *with the properties indicated in 2. Define* V *as the set of positive numbers* r *with the property that the power series converges for* $|z| < r$. *Since the series converges for* $z = z_0$, *it follows that* $|z_0| \in V$. *So the set* V *is not empty. Since the series diverges for* $z = z_1$, *it follows that numbers* $r > |z_1|$ *are no elements of* V. *So the set* V *is bounded. It is possible to prove*[9] *that* V *has a maximum* R. *If* $|z| < R$, *the series apparently converges, and if* $|z| > R$ *it doesn't, according to Lemma 1.6.1.*

The number R in the theorem is called the **radius of convergence** of the power series. In case 1 we say that $R = 0$, in case 3 that $R = \infty$. Inside the **disc of convergence** $|z| < R$ the series converges everywhere, but *at* the **circle of convergence** $|z| = R$ the series may or may not converge. Every power series has a radius of convergence[10]. Sometimes we can use the convergence criterion of Cauchy or d'Alembert (1.3.4) to determine it. For example, if the following limit exists

$$\lim_{n\to\infty} \frac{|a_{n+1}z^{n+1}|}{|a_n z^n|} = |z| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |z|L,$$

with $0 \leqslant L \leqslant \infty$, then $|z|L < 1$ (*i*) for $|z| < L^{-1}$ if $L \in (0, \infty)$ and the corresponding radius of convergence $R = L^{-1}$; (*ii*) for all z if $L = 0$ and $R = \infty$; and (*iii*) for $z = 0$ if $L = \infty$ and $R = 0$.

EXAMPLE 1.6.3.

1. *The series* $\sum_0^\infty n!z^n$ *has radius of convergence* 0.
2. *The series* $\sum_1^\infty z^n n!/n^n$ *has radius of convergence* e.
3. *The series* $\sum_0^\infty z^n/n!$ *has radius of convergence* ∞.
4. *The series* $\sum_0^\infty z^n$, $\sum_0^\infty nz^n$, $\sum_1^\infty z^n/n^2$, $\sum_1^\infty z^n/n$ *have radius of convergence 1.*

REMARK 1.6.4. *It is difficult to give a general statement about convergence at the circle of convergence, unless the series is absolutely convergent there or the terms do not converge to zero. A partial result is the following. Assume* $\{a_n\}$ *is a real sequence, monotonically decreasing to zero. Then for any* $z \neq R$, $|z| = R$, *the series* $\sum_{n=0}^\infty a_n(z/R)^n$ *converges. (Check this for example 1.6.3.)*

THEOREM 1.6.5 (**A convergent power series is holomorphic**). *If the power series* $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ *converges for* $|z| < R$, $R > 0$, *the sum*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

[9] For the mathematics students: first note that $R = \sup V$ exists. Then we can prove that R itself also belongs to V.

[10] in general given by $R^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$ (Cauchy-Hadamard theorem)

is for $|z| < R$ a holomorphic function. The derivative is found by term-by-term differentiation, i.e.

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

This series is also convergent for $|z| < R$.

PROOF: *Select an arbitrary positive number $\rho < R$. Then the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ converge uniformly and absolutely on the closed disc $D_\rho = \{z \mid |z| \leqslant \rho\}$ (Lemma 1.6.1). Select z_0 with $|z_0| < \rho$. The closed disc $D_\zeta = \{z \mid |z - z_0| \leqslant \rho - |z_0|\}$ around z_0 is contained in D_ρ. Define on D_ζ, with $\zeta = z - z_0$, the auxiliary function*

$$F(\zeta) = \begin{cases} \big(f(z_0+\zeta) - f(z_0)\big)/\zeta & (\zeta \neq 0), \\ g(z_0) & (\zeta = 0). \end{cases}$$

Since the difference of convergent series is again a convergent series, we write $F = \sum_{n=0}^{\infty} a_n F_n$, with

$$F_n(\zeta) = \begin{cases} \big((z_0+\zeta)^n - z_0^n\big)/\zeta & (\zeta \neq 0), \\ n z_0^{n-1} & (\zeta = 0), \end{cases}$$

(i) The functions $F_n(\zeta)$ are continuous on D_ζ because $n z_0^{n-1}$ is the derivative of z^n in z_0.
(ii) From the factorisation $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$ it follows that

$$F_n(\zeta) = (z_0+\zeta)^{n-1} + (z_0+\zeta)^{n-2} z_0 + \cdots + z_0^{n-1}.$$

Since $|z_0| < \rho$ and $|z_0 + \zeta| \leqslant \rho$ on D_ζ, it follows with the triangle inequality that

$$|F_n(\zeta)| \leqslant |z_0+\zeta|^{n-1} + |z_0+\zeta|^{n-2}|z_0| + \cdots + |z_0|^{n-1} \leqslant n\rho^{n-1}.$$

Since $\sum_{n=0}^{\infty} n a_n \rho^{n-1}$ converges, it follows with Weierstrass' criterion that $\sum_{n=0}^{\infty} a_n F_n(z)$ converges uniformly on D_ζ. From (i) and (ii) and Theorem 1.3.13 it follows that $F(\zeta)$ is continuous in $\zeta = 0$, so $f'(z_0) = \lim_{\zeta \to 0} F(\zeta) = g(z_0)$.

The formula for the derivative is also a power series. This is again differentiable within the same circle of convergence. In this way we can continue and establish that a function in the form of a power series is arbitrarily differentiable within the circle of convergence. Furthermore, the derivatives are found by "term-by-term differentiation". For the 2nd, 3rd, resp. kth derivatives we have

$$\begin{aligned} f''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} &&= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n, \\ f^{(3)}(z) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n z^{n-3} &&= \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} z^n, \\ f^{(k)}(z) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k} &&= \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1) a_{n+k} z^n. \end{aligned}$$

When we substitute $z = 0$ in the last of these equations, we find $f^{(k)}(0) = k! a_k$. For a power series around the point a, so for $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$, we find the corresponding

$$c_k = \frac{f^{(k)}(a)}{k!} \quad (k = 0, 1, \dots). \tag{1.7}$$

We will discuss some important functions that are defined by a power series.

The exponential function

For real x we know the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

DEFINITION 1.6.6. *In a similar way we define for complex z the* ***exponential function***

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \cdots .$$

For example from the criterion of d'Alembert it follows that this series has a radius of convergence ∞. The function exp is therefore holomorphic for all $z \in \mathbb{C}$. The following theorem gives some properties of this function:

THEOREM 1.6.7.

1. *For all $z \in \mathbb{C}$ we have:* $(e^z)' = e^z$.
2. *For all $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$ we have:* $e^{z_1+z_2} = e^{z_1} e^{z_2}$.
3. e^z *has no zeros.*
4. *For a purely imaginary $z = iy$ we have* ***Euler's Formula*** $e^{iy} = \cos y + i \sin y$,
   yielding ***Euler's Identity*** $e^{i\pi} + 1 = 0$.
5. *If $z = x + iy$, then* $e^z = e^x(\cos y + i \sin y)$.
6. $|e^z| = e^{\operatorname{Re} z}$ *for all $z \in \mathbb{C}$.*
7. e^z *is $2\pi i$-periodic, i.e.* $e^{z+2\pi i} = e^z$ *for all $z \in \mathbb{C}$.*

PROOF:

1. $(e^z)' = \sum_{n=0}^{\infty} \frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
2. *For arbitrary $a \in \mathbb{C}$ the function $F(z) = e^z e^{a-z}$ has derivative $F'(z) = e^z e^{a-z} - e^z e^{a-z} = 0$ for each $z \in \mathbb{C}$. From this it follows that $F(z)$ is a constant, so $F(z) = F(0) = e^a$. If we apply, for given $z_1, z_2 \in \mathbb{C}$, the substitution $z = z_1$, $a = z_1 + z_2$ in the identity $e^z e^{a-z} = e^a$, we find the sought identity.*
3. *This follows from $e^z e^{-z} = 1$, as e^z cannot be zero if e^{-z} exists everywhere.*
4. *For purely imaginary $z = iy$ we have*
   $$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{m=0}^{\infty} \frac{(-1)^m y^{2m}}{(2m)!} + \sum_{m=0}^{\infty} \frac{i(-1)^m y^{2m+1}}{(2m+1)!} = \cos y + i \sin y.$$
5. *This follows from 2. and 4.*
6. *This follows from 5.*
7. *This follows from 5.*

The trigonometric functions

DEFINITION 1.6.8. *These are defined for $z \in \mathbb{C}$ by*

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

Also these series have radius of convergence ∞. The functions $\sin z$ and $\cos z$ (not $\tan z$ or $\cot z$) are therefore holomorphic in the whole complex plane. For real values of z they are identically equal to the known (real) functions.

PROPERTY 1.6.9.

1. $(\cos z)' = -\sin z$, $\quad (\sin z)' = \cos z$.
2. $e^{\pm iz} = \cos z \pm i \sin z$, $\quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.
3. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$, $\quad \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

From the above it follows easily that most other known properties remain valid for complex numbers, such as $\sin^2 z + \cos^2 z = 1$, $\sin(2z) = 2 \sin z \cos z$. Note, however, that inequalities are not preserved. We know that $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$ for real x. For complex numbers this is not the case. For example, $\cos(iy) = \frac{1}{2}(e^{-y} + e^{y}) = \cosh y \to \infty$ for $y \to \infty$.

The hyperbolic functions

We give briefly the definitions and some properties:

DEFINITION 1.6.10.

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

PROPERTY 1.6.11.

1. $(\cosh z)' = \sinh z$, $\quad (\sinh z') = \cosh z$,
2. $e^{\pm z} = \cosh z \pm \sinh z$, $\quad \cosh z = \frac{1}{2}(e^{z} + e^{-z})$, $\quad \sinh z = \frac{1}{2}(e^{z} - e^{-z})$,
3. $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$, $\quad \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.
4. $\cos(iz) = \cosh z$, $\quad \sin(iz) = i \sinh z$,

1.7 Applications

1.7.1 Ohmic heating at a corner.

We consider the following model for the edge singularity of the time-dependent temperature field generated in a homogeneous and isotropic conductor by an electric field.

The electric current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$ satisfy Ohm's law $\boldsymbol{J} = \sigma \boldsymbol{E}$, where σ is the electric conductivity, *i.e.* the inverse of the specific electric resistance. For an effectively stationary current flow the conservation of electric charge leads to a vanishing divergence of the electric current density, $\nabla \cdot \boldsymbol{J} = 0$. The electric field $\boldsymbol{E}$ satisfies $\nabla \times \boldsymbol{E} = \boldsymbol{0}$, and therefore has a potential φ, with $\boldsymbol{E} = -\nabla\varphi$, satisfying $\nabla \cdot (\sigma \nabla\varphi) = 0$. The electric conductivity σ is a material parameter which is quite strongly dependent on temperature but here we will assume it a constant σ, independent of T. This, then, leads to the Laplace equation for φ

$$\nabla^2\varphi = 0. \tag{1.8}$$

The heat dissipated as a result of the work done by the field per unit time and volume (Ohmic heating) is given by Joule's law $\boldsymbol{J} \cdot \boldsymbol{E}$, and leads to the heat-source distribution

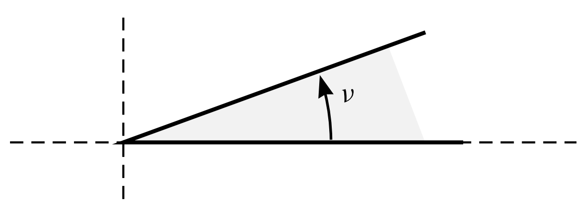
$$\boldsymbol{J} \cdot \boldsymbol{E} = \sigma \boldsymbol{E} \cdot \boldsymbol{E} = \sigma |\nabla\varphi|^2. \tag{1.9}$$

Since energy is conserved, the net rate of heat conduction and the rate of increase of internal energy are balanced by the heat source, which yields the equation for temperature T

$$\rho C \frac{\partial T}{\partial t} = \kappa \nabla^2 T + \sigma |\nabla\varphi|^2. \tag{1.10}$$

The thermal conductivity κ, the density ρ and the specific heat of the material C are mildly dependent on temperature, but we assume these parameters constant.

Since we are interested in the rôle of the edge only, the conductor is modelled, in cylindrical (r, ϑ)-coordinates, as a both electrically **and** thermally isolated infinite wedge-shaped two-dimensional region $0 \leqslant \vartheta \leqslant \nu$.



Questions

i. Formulate the boundary conditions for φ and T along $\vartheta = 0$ and $\vartheta = \nu$.

ii. Show that solutions φ exist of equation (1.8) and satisfying the boundary conditions of question [i], as follows: φ can be written as the real part of pz^α for $z = x + \mathrm{i}y$ and certain values of p and α. Determine α from the boundary conditions along $\vartheta = 0$ and $= \nu$. (Many solutions are possible.)

iii. Assume that φ is not singular in $r = 0$ (a singularity of φ would amount to a linesource there) and assume that the electric field $\boldsymbol{E}$ remains bounded for $r \to \infty$. What is the solution?

iv. Try to find similarity solutions of (1.10), with boundary conditions as found in question [i.], for T, in $t > 0$, of the form

$$T(r, \vartheta, t) = qr^{\mu} h(sr^{\beta}/t).$$

Find expressions for q, μ and β.

Answers

Because of electric isolation, we need the boundary conditions $\frac{\partial}{\partial \vartheta}\varphi = 0$ along $\vartheta = 0$ and $\vartheta = \nu$. This is obtained with $\alpha = n\pi/\nu$, $n = \ldots, -2, -1, 1, 2, \ldots$ and

$$\varphi(r, \vartheta) = \alpha^{-1} A\, r^{\alpha} \cos(\alpha\vartheta).$$

However, $n < 0$ is not likely because it amounts to a point source at $r = 0$ which isn't there. For an arbitrary driving field in $r \to \infty$, we may expect contributions for all $n \geqslant 1$. So we take the leading order term (the biggest near $r = 0$), which is $n = 1$. Then we have for the temperature field

$$\rho C \frac{\partial T}{\partial t} = \kappa \nabla^2 T + \sigma A^2 r^{2\alpha - 2}, \qquad \alpha = \frac{\pi}{\nu},$$

with boundary and initial conditions

$$\frac{\partial T}{\partial \vartheta} = 0 \text{ at } \vartheta = 0,\ \vartheta = \nu, \qquad T(x, y, 0) \equiv 0.$$

Since there are no other (point) sources at $r = 0$, we have the additional condition of a finite field at the origin: $0 \leqslant T(0, 0, t) < \infty$. Boundary conditions and the symmetric source imply that T is a function of t and r only, so that equation reduces to

$$\rho C \frac{\partial T}{\partial t} = \kappa \Big(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \Big) + \sigma A^2 r^{2\alpha - 2}.$$

Owing to the homogeneous initial and boundary conditions, the infinite geometry, and the fact that the source is a monomial in r, homogeneous of the order $2\alpha - 2$, there is no length scale in the problem other than $\sqrt{\big(\frac{\kappa t}{\rho C}\big)}$, while the temperature T can only scale on $(\sigma A^2/\kappa) r^{2\alpha}$. This indicates that a similarity solution is possible of the following form

$$T(r, t) = \frac{\sigma}{4\kappa} A^2 r^{2\alpha} h(X), \quad X = \frac{\rho C r^2}{4\kappa t},$$

where X is a similarity variable, reducing the equation to

$$X^2 h'' + X(1 + 2\alpha + X)h' + \alpha^2 h = -1.$$

We can determine the corresponding boundary conditions and solve the equation as follows. The boundary conditions, corresponding to the behaviour near $r = 0$ and $t = 0$, are

$$0 \leqslant X^{\alpha} h(X) < \infty \quad \text{if} \quad X \downarrow 0, \qquad h(X) \to 0 \quad \text{if} \quad X \to \infty.$$

The equation has the (constant) particular solution $-\alpha^{-2}$, while the homogeneous equation happens to be related to the confluent hypergeometric equation in $-X$. The general solution with the required behaviour in $X = 0$ is then given by

$$h(X) = \text{constant} \cdot X^{-\alpha} M(-\alpha; 1; -X) - \alpha^{-2}$$

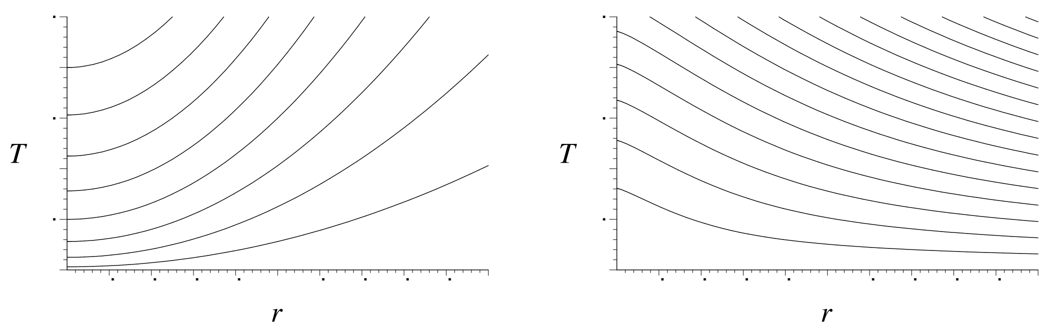
where $M(a; b; z)$ is Kummer's function or the regular confluent hypergeometric function, and the unknown constant is to be determined. From the asymptotic expansion[11] of $M(-\alpha; 1; -X)$

$$M(-\alpha; 1; -X) = \frac{X^\alpha}{\alpha!}\big(1 + O(X^{-1})\big) \quad (X \to \infty)$$

and the condition $h(X) \to 0$ for $X \to \infty$, the constant is found to be $\alpha!/\alpha^2$. Putting everything together, we have the solution

$$T(r,t) = \frac{\sigma A^2}{4\alpha^2\kappa} r^{2\alpha} \left[\alpha!\Big(\frac{\rho C r^2}{4\kappa t}\Big)^{-\alpha} M\Big(-\alpha; 1; -\frac{\rho C r^2}{4\kappa t}\Big) - 1\right].$$

An interesting conclusion that can be drawn from this result is that in a corner with acute angle ($\nu < \pi$) the temperature lags behind the rest of the field, while in a corner with obtuse angle ($\nu > \pi$) it is ahead. See the figure below.



Radial temperature distribution in wedge of $\nu = \frac{1}{2}\pi$ (left) and $\nu = \frac{3}{2}\pi$ (right) for $\sigma A^2/4\kappa = 1$ and $4\kappa t/\rho C = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$.

[11] $z! = \Gamma(z+1)$, where Γ is the Gamma function, section 6.2

1.7.2 Biharmonic equation

Questions

1. Small deformations $\boldsymbol{u}$ of a linear elastic continuum are described by the **Navier equation**

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} + \rho \boldsymbol{f}.$$

Show that in steady equilibrium and no body forces ($\boldsymbol{f} = 0$) (forces act only via the surface) $\boldsymbol{u}$ satisfies the biharmonic equation

$$\nabla^4 \boldsymbol{u} = 0.$$

2. **Stokes flow** (highly viscous flow with negligible inertia effects) is described by

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \nabla p = \mu \nabla^2 \boldsymbol{v}.$$

According to **Helmholtz' decomposition theorem**, the velocity field can be written under rather general conditions as $\boldsymbol{v} = \nabla\varphi + \nabla \times \boldsymbol{A}$, with a scalar potential φ and a vector potential $\boldsymbol{A}$. Show that in 2D $\boldsymbol{A}$ simplifies to $\boldsymbol{A} = (0, 0, \psi)$, while ψ (the stream function) satisfies the biharmonic equation

$$\nabla^4 \psi = 0.$$

3. Show that the solution to the 2D biharmonic equation $\nabla^4 w = 0$ can be written (under rather general conditions) as

$$w(x, y) = \mathrm{Re}(\overline{z} f(z) + g(z))$$

(or the imaginary part) with analytic functions f and g.

Answers

1. Apply the divergence to the Navier equation to show that $\nabla^2(\nabla \cdot \boldsymbol{u}) = 0$. Then apply the Laplacian ∇^2 again to the Navier equation.

$$\nabla \cdot ((\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u}) = (\lambda + \mu)\nabla^2(\nabla \cdot \boldsymbol{u}) + \mu \nabla^2(\nabla \cdot \boldsymbol{u}) = (\lambda + 2\mu)\nabla^2(\nabla \cdot \boldsymbol{u}) = 0$$

$$(\lambda + \mu)\nabla\big(\nabla^2(\nabla \cdot \boldsymbol{u})\big) + \mu \nabla^4 \boldsymbol{u} = \mu \nabla^4 \boldsymbol{u} = 0$$

2. Apply the curl to the momentum equation.

$$\nabla \times \nabla p = \mu \nabla^2(\nabla \times \boldsymbol{v}) = -\mu \nabla^4 \psi = 0$$

3. For analytic $f = f_1 + \mathrm{i} f_2$ and $g = g_1 + \mathrm{i} g_2$, the real and imaginary parts f_1, f_2, g_1, g_2 are harmonic, and we have

$$\nabla^2 w = \nabla^2(x f_1 + y f_2 + g_1) = 2(f_{1,x} + f_{2,y}) + x\nabla^2 f_1 + y \nabla^2 f_2 = 2(f_{1,x} + f_{2,y})$$

Since $f_{1,x}$ and $f_{2,y}$ are also harmonic, the result follows.

Chapter 2

Complex integration

In this chapter we will discuss the definition and properties of the integral of a complex function $f(z)$ along a curve in the complex plane. We derive then for the integral of a holomorphic function the integral theorem of Cauchy. With this fundamental result we can show that holomorphic functions have particularly beautiful properties. We start with a general discussion on curves in $\mathbb{C}$.

2.1 Arcs, curves and contours

Complex integration as considered here is only along arcs, curves and contours. For this we will introduce some concepts. A **parameter representation**, or parametrisation, is given by a function ζ from an interval $[a, b]$ to $\mathbb{C}$. We will always assume here that the function ζ is continuously differentiable. A **singular point** of the parameter representation is a point where $\zeta'(t) = 0$. An **arc** $\mathcal{K}$ is the image in $\mathbb{C}$ of a continuously differentiable parameter representation without singular points, provided with an **orientation** imposed by the parametrisation. In $\mathbb{R}^2$ we could have described such a representation by $\underline{\zeta}(t) = (\xi(t), \eta(t)) \quad (a \leqslant t \leqslant b)$. In $\mathbb{C}$ we write $z = \zeta(t) = \xi(t) + \mathrm{i}\eta(t) \quad (a \leqslant t \leqslant b)$. The functions $\xi(t)$ and $\eta(t)$ are continuously differentiable along the interval $[a, b]$. The variable t is called the **parameter**. The point $z_1 = \zeta(a)$ is called the **initial point**, and the point $z_2 = \zeta(b)$ is called the **final point** of the arc. z_1 and z_2 are the end points of the arc.

EXAMPLE 2.1.1. $z = \mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t \quad (0 \leqslant t \leqslant \pi)$ *is a parametrisation of the semi-circle* $|z| = 1, \operatorname{Im} z \geqslant 0$. *The initial point is* $z = 1$, *the final point* $z = -1$.

From analysis we have the following result:

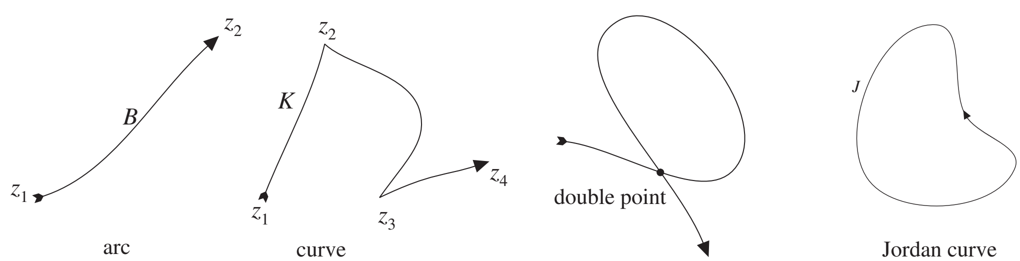
THEOREM 2.1.2. *Let $\mathcal{K}$ be an arc with parametrisation $z = \zeta(t) = \xi(t) + \mathrm{i}\eta(t) \quad (a \leqslant t \leqslant b)$. The length of $\mathcal{K}$ is given by*

$$\int_a^b \sqrt{\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2}\, dt = \int_a^b \left|\frac{d\zeta}{dt}\right| dt = \int_a^b |\zeta'(t)|\, dt.$$

We write this expression often as $\int_{\mathcal{K}} ds$, where we call ds a line element, which symbolises the expression $|\zeta'(t)|\,dt$. More generally one defines the **line integral** of a complex function $f(z)$ along an arc $\mathcal{K}$ as

$$\int_{\mathcal{K}} f(z)\,ds = \int_a^b f(\zeta(t))|\zeta'(t)|\,dt.$$

A **curve** $\mathcal{C} = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n$ is a succession of a finite number of arcs $\mathcal{K}_j$, such that the final point of arc $\mathcal{K}_j$, $j < n$, is the initial point of successor arc $\mathcal{K}_{j+1}$. The curve's inherited **orientation** corresponds with the initial point of the first arc $\mathcal{K}_1$ being the initial point of $\mathcal{C}$, and the final point of the last arc $\mathcal{K}_n$ being the final point of $\mathcal{C}$. Intuitively a curve is an arc with possibly a number of kinks. A **path** or **contour** is a similar succession of arcs, but with possibly an infinite number of arcs. (In order to avoid for now irrelevant problems we assume that the length of each constituting arc is larger than a positive number.) The real axis is an example of a path. We say that a point c is a **double point** of the arc $\mathcal{K}$ with parameter representation $\zeta(t)$, $(a \leqslant t \leqslant b)$, if there exist distinct parameter values t_1 and t_2, such that $\zeta(t_1) = \zeta(t_2) = c$. We say that c is a double point of a curve $\mathcal{C}$ if c is a double point of one of the arcs of $\mathcal{C}$, or if c is located on more than one arc of $\mathcal{C}$, not counting the trivial cases of the final point of arc $\mathcal{K}_j$ being equal to the initial point of successor arc $\mathcal{K}_{j+1}$. A **simple curve** is a curve without double points. A **closed curve** is a curve with coinciding initial and final points. A **simple closed curve** is a curve without double points, except for the initial point that coincides with the final point. A simple closed curve is also known as a **Jordan curve**.



A set $S \subseteq \mathbb{C}$ is called (path-)**connected** if each pair of points in S can be connected by a curve within S; in other words, if there exists for every pair of points in S a curve in S such that one point is the initial point and the other is the final point of the curve. If a set is not connected, it is often possible to write the set as the union of a number of mutually disjoint, connected subsets. These subsets are called the **connected components** of S.

Furthermore, we define a **domain**[1] as a non-empty open connected set in $\mathbb{C}$. A **region** is a domain together with all, some, or none of its boundary points.

The following theorem, which is called **Jordan's Theorem**, is very important, intuitively evident but difficult to prove.

THEOREM 2.1.3 (**Jordan's Theorem**). *The complement of a Jordan curve $\mathcal{J}$ in $\mathbb{C}$ has two connected components, one bounded and one unbounded. Both sets are open, with boundary $\mathcal{J}$.*

[1]Note that *domain* may also refer to a *definition set*; see page 7.

We call the bounded component the **inner domain** of $\mathcal{J}$, and the unbounded component the **outer domain**. Two points in the outer domain can be connected by a curve that has no points in common with $\mathcal{J}$. The same is true for two points in the inner domain. But it is not possible to construct a curve from one point inside $\mathcal{J}$ to one point outside $\mathcal{J}$ without crossing $\mathcal{J}$.

Along a Jordan curve we define an **orientation** by the way the curve is traversed. If the curve is followed in the direction where the inner domain is on the left, we say that the curve has a **positive orientation**. This is the counter-clockwise direction. Otherwise, with the clockwise direction, the curve has a **negative orientation**.

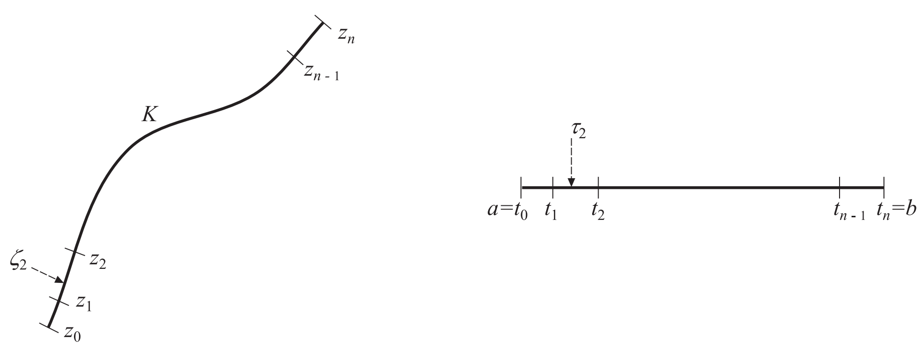
Finally we say that a domain S is **simply connected** if the inner domain of every Jordan curve in S is a subset of S. The annular domain S_2 (see example 1.2.2) is connected but not simply connected.

2.2 The definition of the integral

Let $\mathcal{K}$ be an arc with parametrisation $z = \zeta(t)$ for $a \leqslant t \leqslant b$, and let $f(z)$ be a continuous complex function on $\mathcal{K}$. We define the integral of f along $\mathcal{K}$ as follows:

We divide the interval $[a, b]$ in subintervals $[t_{i-1}, t_i]$ by means of the partition points $t_0, \dots, t_n$, which satisfy $a = t_0 < t_1 < \cdots < t_n = b$. These partition points correspond with points $z_i = \zeta(t_i)$ of $\mathcal{K}$. So the partition of the parameter interval produces a partition of the arc. We choose numbers $\tau_i \in [t_{i-1}, t_i]$ and we denote $\zeta(\tau_i)$ by ζ_i for $i = 1, \dots, n$. Then ζ_i is a point of $\mathcal{K}$ between z_{i-1} and z_i. With these quantities we define the so-called Riemann sum

$$S = \sum_{i=1}^{n} f(\zeta_i)(z_i - z_{i-1}) = \sum_{i=1}^{n} f(\zeta(\tau_i))[\zeta(t_i) - \zeta(t_{i-1})] \tag{2.1}$$



It can be proved that the Riemann sum has a limit if $n \to \infty$, independent of the partitions, provided the stepsize $\max(t_i - t_{i-1})$ tends to zero. We call this limit the integral of $f(z)$ along the arc $\mathcal{K}$. We use the notation $\int_{\mathcal{K}} f(z)\, dz$.

By means of the parameter representation we can write the integral as the integral of a complex function along a real interval. From $\zeta(t_{i-1}) = \zeta(t_i) - \zeta'(t_i)(t_i - t_{i-1}) + o(t_i - t_{i-1})$ the Riemann sum (2.1) can be approximated by

$$S' = \sum_{i=1}^{n} f(\zeta(\tau_i))[\zeta'(t_i)(t_i - t_{i-1})]. \tag{2.2}$$

If we choose here $\tau_i = t_i$, we find for S' a Riemann sum for the integral $\int_a^b f(\zeta(t))\zeta'(t)\,dt$. As we can make this approximation arbitrarily accurate by taking the stepsize small enough, we find

$$\int_{\mathcal{K}} f(z)\,dz = \int_a^b f(\zeta(t))\zeta'(t)\,dt. \tag{2.3}$$

There are always many parametrisations of an arc. From the definition it follows that the value of $\int_a^b f(\zeta(t))\zeta'(t)\,dt$ is independent of the chosen parameter representation. Only the orientation or integration direction is important, *i.e.* the direction the arc is traversed for increasing parameter value. If we reverse the integration direction, the integral changes sign. This can be seen from (2.1) as $z_i - z_{i-1}$ changes sign. We will indicate the curve $\mathcal{K}$, followed in opposite direction, by $-\mathcal{K}$. If for example $\mathcal{K}$ is given by the representation $\zeta(t)$ for $a \leqslant t \leqslant b$, then $-\mathcal{K}$ is given by $\zeta(a+b-t)$ for $a \leqslant t \leqslant b$. We have then

$$\int_{-\mathcal{K}} f(z)\,dz = -\int_{\mathcal{K}} f(z)\,dz$$

A curve is a chain of consecutive arcs. We define the integral of a continuous function along a curve as the sum of the integrals of the function along the separate arcs. One can prove that the integral is independent of the way the curve is divided into arcs. In the following we will restrict ourselves for proofs involving integrals along curves to the case of a curve that is an arc, and therefore has a parameter representation. The proofs are easily generalised for more general curves. A curve, consisting of a number of consecutive arcs $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$, can be written as $\mathcal{K}_1 + \mathcal{K}_2 + \cdots + \mathcal{K}_n$. If one of the arcs has originally a reversed parametrisation, we can use a minus sign. So if we have points z_1, z_2, z_3, z_4 and arcs $\mathcal{K}_1$ from z_1 to z_2, $\mathcal{K}_2$ from z_3 to z_2, and $\mathcal{K}_3$ from z_3 to z_4, then $\mathcal{C} = \mathcal{K}_1 - \mathcal{K}_2 + \mathcal{K}_3$ is a curve from z_1 to z_4. In that case we have

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{K}_1} f(z)\,dz - \int_{\mathcal{K}_2} f(z)\,dz + \int_{\mathcal{K}_3} f(z)\,dz$$

With the representation (2.3) of the complex integral we can determine this in simple cases.

EXAMPLE 2.2.1. *Let $f(z) = z$ and let $\mathcal{K}$ be an arc with parametrisation $\zeta(t)$ for $a \leqslant t \leqslant b$. Then $A = \zeta(a)$ is the initial point and $B = \zeta(b)$ is the final point. We find then*

$$\int_{\mathcal{K}} f(z)\,dz = \int_a^b \zeta(t)\zeta'(t)\,dt = \left[\tfrac{1}{2}\zeta^2(t)\right]_{t=a}^{t=b} = \tfrac{1}{2}(B^2 - A^2).$$

This example is a special case of the following general result:

THEOREM 2.2.2. *Let the function f be continuous in a domain G, and let F be a primitive function of f, i.e. a function such that $F'(z) = f(z)$ in G. If $\mathcal{K}$ is a curve inside G with initial point A and final point B, then*

$$\int_{\mathcal{K}} f(z)\,dz = F(B) - F(A).$$

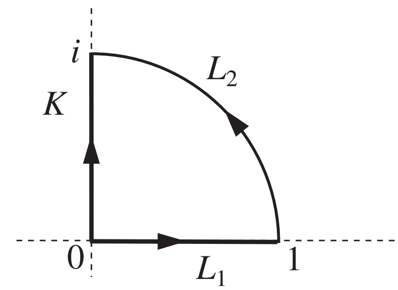
PROOF: *Let $\zeta(t)$ with $a \leqslant t \leqslant b$ be a parametrisation of $\mathcal{K}$. Then $\zeta(a) = A$ and $\zeta(b) = B$. We find by means of the chain rule that*

$$\int_{\mathcal{K}} f(z)dz = \int_a^b f(\zeta(t))\zeta'(t)\,dt = \int_a^b F'(\zeta(t))\zeta'(t)\,dt = \big[F(\zeta(t))\big]_{t=a}^{t=b} = F(B) - F(A).$$

So if we can find a primitive of the integrand, we can easily determine the integral, just like in the real case. Remarkably, the integral does not depend on the integration path, only on the end points. This property is not always true.

EXAMPLE 2.2.3. *Let $f(z) = |z|$. We determine the integral of f from 0 to i along two curves, namely*

- *K, the straight line segment from 0 to i, which can be parametrised by $z = \mathrm{i}t$, $0 \leqslant t \leqslant 1$,*
- *$L = L_1 + L_2$, a curve consisting of two arcs, L_1, the line segment from 0 to 1 (parameter representation $z = t$, $0 \leqslant t \leqslant 1$), and L_2, the quarter part of unit circle between 1 and i. Here we have a parameter representation $z = \mathrm{e}^{\mathrm{i}t}$, $0 \leqslant t \leqslant \frac{1}{2}\pi$.*



We find

$$\int_K |z|\,dz = \int_0^1 |it|i\,dt = \tfrac{1}{2}\mathrm{i},$$

and

$$\int_L |z|\,dz = \int_{L_1} |z|\,dz + \int_{L_2} |z|\,dz = \int_0^1 |t|\,dt + \int_0^{\frac{1}{2}\pi} |\,\mathrm{e}^{it}\,|\mathrm{i}\,\mathrm{e}^{it}\;dt = \tfrac{1}{2} + \mathrm{i} - 1 = \mathrm{i} - \tfrac{1}{2}.$$

We see that the integrals differ. According to the above it means that the function $f(z) = |z|$ has no primitive function.

The following inequality is important for the development of the theory and for applications (in particular the calculation of integrals by residue calculus in Chapter 3).

LEMMA 2.2.4. *(**ML-lemma**) Let the curve $\mathcal{C}$ have length L and assume that f is a continuous function satisfying $|f(z)| \leqslant M$ for $z \in \mathcal{C}$. Then we have*

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| \leqslant \int_a^b |f(\zeta)||\zeta'|\,dt \leqslant ML.$$

PROOF: *It is no restriction to assume that $\mathcal{C}$ is an arc, parametrised by $z = \zeta(t)$ for $a \leqslant t \leqslant b$. If $\vartheta = \arg\left[\int_{\mathcal{C}} f(z)\,dz\right]$ and $\varphi(t) = \arg\left[f(\zeta)\zeta'\right]$, we have*

$$\left|\int_{\mathcal{C}} f(z)\,dz\right| = \left|\int_a^b f(\zeta(t))\zeta'(t)\,dt\right| = \mathrm{e}^{-\mathrm{i}\vartheta}\int_a^b f(\zeta)\zeta'\,dt = \int_a^b |f(\zeta)\zeta'|\,\mathrm{e}^{\mathrm{i}\varphi - \mathrm{i}\vartheta}\;dt =$$

$$\int_a^b |f(\zeta)\zeta'|\cos(\varphi - \vartheta)\,dt \leqslant \int_a^b |f(\zeta)||\zeta'|\,dt \leqslant M\int_a^b |\zeta'|\,dt = ML.$$

We used the identity $|z| = \mathrm{e}^{-\mathrm{i}\arg z}\, z$, *a standard inequality for real integrals, and the formula for the length of a curve, as given by Theorem 2.1.2.*

The following result will be important for the development of the theory.

THEOREM 2.2.5. *Let $a \in \mathbb{C}$, $r > 0$, $m \in \mathbb{Z}$ be given. Let C_a be the circle with centre a and radius r and a positive orientation. Then we have*

$$I_m = \frac{1}{2\pi \mathrm{i}} \int_{C_a} (z-a)^m \, dz = \begin{cases} 1 & \text{if} \quad m = -1, \\ 0 & \text{if} \quad m \neq -1. \end{cases}$$

PROOF: *C_a can be parametrised by $z = a + r\,\mathrm{e}^{\mathrm{i}t}$ with $0 \leqslant t \leqslant 2\pi$. With this we find*

$$I_m = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} r^m \, \mathrm{e}^{\mathrm{i}mt} \, \mathrm{i}r\, \mathrm{e}^{\mathrm{i}t} \, dt = \frac{r^{m+1}}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m+1)t} \, dt = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} 1 \, dt & = 1 \quad \text{if } m = -1, \\ \dfrac{r^{m+1}}{2\pi} \left[\dfrac{\mathrm{e}^{\mathrm{i}(m+1)t}}{\mathrm{i}(m+1)} \right]_0^{2\pi} & = 0 \quad \text{if } m \neq -1. \end{cases}$$

2.3 Cauchy's Integral Theorem

We have seen a number of cases where the integral of a function along a curve is not dependent of the path, only of the end points. In fact this is always true if the integrand is a holomorphic function. The following result, so important that we call it the fundamental theorem of complex analysis, will allow us to prove this.

THEOREM 2.3.1 (**Cauchy's Integral Theorem**). *Let the function f be holomorphic in a domain G. Let K be a Jordan curve which together with its inner domain belongs to G. Then*

$$\int_K f(z)\, dz = 0.$$

PROOF: *We give the proof under the assumption that $f(z)$ is continuously differentiable. We denote the inner domain of K by G'. Then $G' \subseteq G$. We assume that K is given by a parametrisation $\zeta(t) = \xi(t) + \mathrm{i}\eta(t)$ for $a \leqslant t \leqslant b$. We use the theorem of Green, which is Stokes' theorem for a surface in $\mathbb{R}^2$:*

GREEN:

> *Let $p(x, y)$ and $q(x, y)$ be continuously differentiable on a Jordan curve K and in G'. Then*
>
> $$\iint_{G'} (q_x - p_y)\, dx\, dy = \oint_K (p, q) \cdot d\boldsymbol{r},$$
>
> *where $d\boldsymbol{r}$ denotes a vectorial line element along K. With parametrisation $(x, y) = (\xi(t), \eta(t))$ we can also write*
>
> $$\iint_{G'} (q_x - p_y)\, dx\, dy = \int_a^b [p(\xi(t), \eta(t))\xi'(t) + q(\xi(t), \eta(t))\eta'(t)]\, dt. \quad (2.4)$$

In our equivalent case in the complex plane we have

$$\int_K f(z)\,dz = \int_a^b f(\zeta(t))\zeta'(t)\,dt = \int_a^b (u(\xi,\eta) + \mathrm{i}v(\xi,\eta))(\xi' + \mathrm{i}\eta')\,dt$$
$$= \int_a^b (u\xi' - v\eta')\,dt + \mathrm{i}\int_a^b (v\xi' + u\eta')\,dt.$$

We apply now (2.4) with $p = u, q = -v$, and $p = v, q = u$ respectively, to find

$$\int_a^b (u\xi' - v\eta')\,dt = \iint_{G'} (-v_x - u_y)\,dx\,dy = 0,$$
$$\int_a^b (v\xi' + u\eta')\,dt = \iint_{G'} (u_x - v_y)\,dx\,dy = 0,$$

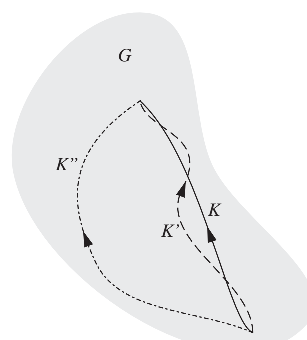
since u and v satisfy the equations of Cauchy-Riemann. From this it follows that $\int_K f(z)\,dz = 0$.

THEOREM 2.3.2. *Let the function f be holomorphic in a simply connected domain G. Let K and K' be two curves in G with the same initial and final points. Then we have*

$$\int_{K'} f(z)\,dz = \int_K f(z)\,dz.$$

PROOF: *If the two curves are simple and have no points in common, except the end points, then they form together a Jordan curve $K_1 = K - K'$, if we assume that K' is at the left of K. Since the domain is simply connected, not only K_1 but also the inner domain of K_1 is in G. So we have*

$$0 = \int_{K_1} f(z)\,dz = \int_K f(z)\,dz - \int_{K'} f(z)\,dz.$$



If the curves K and K' do have points in common apart from the end points, we can find a third curve K'', located in G, with the same end points as K and K', and without a point in common with K or K' except the end points. This is possible because G is open. Then we find

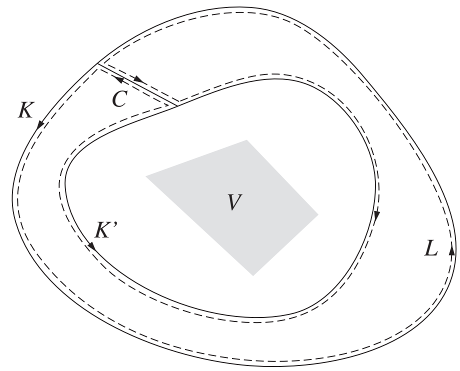
$$\int_{K'} f(z)\,dz = \int_{K''} f(z)\,dz = \int_K f(z)\,dz.$$

In the following we allow that G is not simply connected. We consider the values of an integral along two closed curves K and K', with the property that the region inside the curves where the function is not holomorphic, is in both cases the same. We conclude that the integrals are equal to each other.

THEOREM 2.3.3. *Let G be a simply connected domain and V a closed subset of G. Assume that the function $f(z)$ is holomorphic in the domain $\tilde{G} = G \backslash V$. Let K and K' be closed curves in G such that V is both part of the inner domain of K and the inner domain of K'. Then we have*

$$\int_{K'} f(z)\,dz = \int_K f(z)\,dz.$$

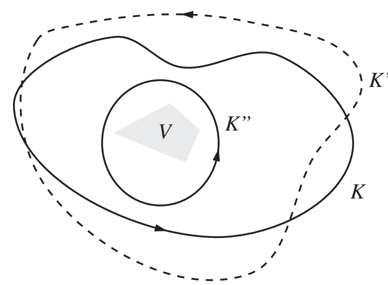
PROOF:
First we assume that one of the curves, say K', is entirely inside the other curve K. The domain inside K and outside K' is inside G, but is disjoint with V. We connect the curves K and K' by the curve C, from K to K'. We open the curves K and K' at their points of intersection with C, and denote the result K_1 and K_1'. We combine K_1, K_1' and C in such a way that the curve $L = K_1 + C - K_1' - C$ is a Jordan curve.



The inner domain of this curve is in $\tilde{G}$. The function $f(z)$ is there holomorphic. Due to Cauchy's integral theorem the integral $\int_L f(z)\,dz$ is zero. Writing out the constituting integrals (and suppressing for clarity the expression $f(z)\,dz$) yields:

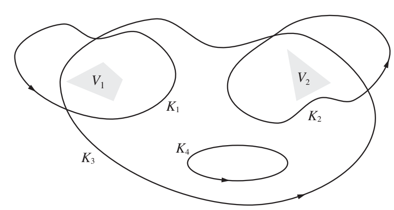
$$0 = \int_L = \int_{K_1} + \int_C - \int_{K_1'} - \int_C = \int_{K_1} - \int_{K_1'} = \int_K - \int_{K'} .$$

If one of the curves is not inside the other, we can, in a similar way as with the proof of Theorem 2.3.2, introduce a third curve K'' with no points in common with K or K'. Then we obtain our result from $\int_{K'} = \int_{K''} = \int_K$.



Consider a simply connected domain G with two disjoint subsets V_1 and V_2, such that the function $f(z)$ is holomorphic inside G but outside $V_1 \cup V_2$. It is possible to draw four types of Jordan curves in G and outside $V_1 \cup V_2$:

- A curve K_1 enclosing only V_1.
- A curve K_2 enclosing only V_2.
- A curve K_3 enclosing both V_1 and V_2.
- A curve K_4 enclosing none of the sets V_1 or V_2.



In none of these cases we allow the sets V_1 and V_2 to have a point in common with the curves. From Theorem 2.3.3 it follows that the value of the integral $\int_K f(z)\,dz$ along any curve of type K_1 is the same, say I_1. In the same way integrals along curves of type K_2 are the same, say I_2. Now it is possible to see, in the same way as in the proof of Theorem 2.3.3, that integrals along curves of type K_3 are equal to $I_1 + I_2$. Finally, the integral along a curve of type K_4 is equal to zero.

In this way it is possible to reduce complicated integrals to a number of integrals of simpler type.

2.4 The residue

Let G be a domain, a a point in G, and let $f(z)$ be a function which is holomorphic in a subset of G that contains a reduced neighbourhood $\mathring{B}_\delta(a) = \{z \in \mathbb{C} \mid 0 < |z-a| < \delta\}$ of a, but not necessarily in point a itself. We say that a is an **isolated singular point** of $f(z)$. Due to Theorem 2.3.3 the integral of $f(z)$ along a Jordan curve in $\mathring{B}_\delta(a)$, with a in its inner domain, is independent of the chosen curve.

DEFINITION 2.4.1. *Let a be an isolated singular point of the complex function $f(z)$, and assume this function is holomorphic in a reduced neighbourhood $\mathring{B}_\delta(a)$ of a. The **residue** of $f(z)$ in a is*

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{2\pi\mathrm{i}} \int_K f(z)\,dz,$$

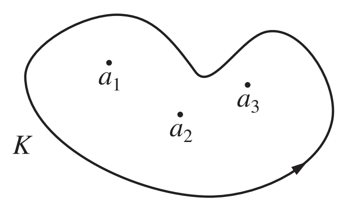
where K is an arbitrary positively oriented Jordan curve in $\mathring{B}_\delta(a)$, with a in its inner domain.

We apply this definition also if $f(z)$ is holomorphic in a. In that case $\operatorname{Res}_{z=a} f(z) = 0$.

EXAMPLE 2.4.2. *From Theorem 2.2.5 it follows that*

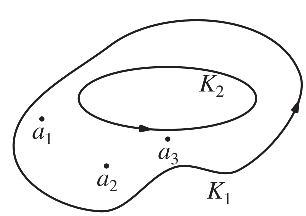
$$\operatorname{Res}_{z=a}(z-a)^m = \begin{cases} 1 & \text{if } m = -1, \\ 0 & \text{if } m \neq -1. \end{cases}$$

It is for this simple result that we added the factor $\frac{1}{2\pi\mathrm{i}}$ to the definition of residue.



Assume that domain G contains a number of isolated singular points, say $a_1, \ldots, a_n$. Further we assume that K is a curve which encloses all these points. According to the remarks at the end of the previous section we can replace the integral by n integrals along curves K_m for $m = 1, \ldots, n$, each of which enclose one point, namely a_m. Consequently we find

$$\int_K f(z)\,dz = 2\pi\mathrm{i} \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z). \tag{2.5}$$



In a similar way we can consider a situation of two Jordan curves K_1 and K_2 with the property that K_2 is inside the inner domain of K_1. If we further assume that $f(z)$ is holomorphic in a domain G, that contains K_1 and K_2 and the domain G' between K_1 and K_2, except for a finite number of points $a_1, \ldots, a_n$ in G', then we have

$$\int_{K_1} f(z)\,dz - \int_{K_2} f(z)\,dz = 2\pi\mathrm{i} \sum_{k=1}^{n} \operatorname{Res}_{z=a_k} f(z). \tag{2.6}$$

We can use these results to determine all sorts of integrals. For this it is necessary that we are able to determine the residues. Starting point here is example 2.4.2. Furthermore, the residue operator is linear: if both f and g are holomorphic in a reduced neighbourhood of a, and α and β are complex numbers, then

$$\operatorname{Res}_{z=a}(\alpha f(z) + \beta g(z)) = \alpha \operatorname{Res}_{z=a} f(z) + \beta \operatorname{Res}_{z=a} g(z).$$

This is a direct consequence of the definition of residue. So if $f(z)$ has the following form

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{(z-a)^k} + h(z),$$

where $h(z)$ is holomorphic in a, then

$$\operatorname{Res}_{z=a} f(z) = \sum_{k=1}^{n} \operatorname{Res}_{z=a} \frac{A_k}{(z-a)^k} + \operatorname{Res}_{z=a} h(z) = A_1. \tag{2.7}$$

The residue is therefore equal to the coefficient of $(z-a)^{-1}$.

EXAMPLE 2.4.3. *Let the following function be given*

$$f(z) = \frac{z^2+1}{z^2(z+1)}.$$

We expand in partial fraction

$$f(z) = -\frac{1}{z} + \frac{1}{z^2} + \frac{2}{z+1}.$$

The singular points are $z=0$ *and* $z=-1$. *In order to determine the residue in* $z=-1$, *we write*

$$f(z) = \frac{2}{z+1} + h(z),$$

where $h(z) = -\frac{1}{z} + \frac{1}{z^2}$ *is holomorphic in a neighbourhood of* $z=-1$. *So* $\operatorname{Res}_{z=-1} f(z) = 2$. *Similarly we write* $f(z) = -\frac{1}{z} + \frac{1}{z^2} + h_0(z)$, *where* $h_0(z) = 2/(z+1)$ *is holomorphic in* $z=0$. *So* $\operatorname{Res}_{z=0} f(z) = -1$.

With formula (2.5) we find

$$\int_{|z|=\frac{1}{2}} f(z)\,dz = 2\pi \mathrm{i} \operatorname{Res}_{z=0} f(z) = -2\pi \mathrm{i},$$

and

$$\int_{|z|=2} f(z)\,dz = 2\pi \mathrm{i}\big[\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=-1} f(z)\big] = 2\pi \mathrm{i}(-1+2) = 2\pi \mathrm{i}.$$

EXAMPLE 2.4.4. *The function* $f(z) = \sin z/z^4$ *is everywhere holomorphic except in* $z = 0$. *For the sine function we know the power series*

$$\sin z = z - \tfrac{1}{3!}z^3 + \tfrac{1}{5!}z^5 - \tfrac{1}{7!}z^7 + \cdots .$$

Therefore

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{1}{5!}z - \frac{1}{7!}z^3 + \cdots = \frac{1}{z^3} - \frac{1}{6z} + h(z).$$

The function $h(z)$ *is given by the power series*

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{(2n+3)!} z^{2n-1} = \frac{1}{5!}z - \frac{1}{7!}z^3 + \cdots .$$

This power series is everywhere convergent, and so the function $h(z)$ *is holomorphic in* $z = 0$. *We find*

$$\mathrm{Res}_{z=0} \frac{\sin z}{z^4} = \mathrm{Res}_{z=0} \frac{1}{z^3} - \frac{1}{6} \mathrm{Res}_{z=0} \frac{1}{z} + \mathrm{Res}_{z=0}\, h(z) = -\frac{1}{6}.$$

From this we have, for example,

$$\int_{|z|=1} \frac{\sin z}{z^4}\, dz = -\frac{\pi \mathrm{i}}{3}.$$

Other methods to determine residues will be given in section 3.1.

2.5 Cauchy's Integral Formula, Taylor series, analyticity

The following result plays an important role:

LEMMA 2.5.1. *Let function* f *be holomorphic in* a. *Then*

$$\mathrm{Res}_{z=a} \frac{f(z)}{z-a} = f(a).$$

PROOF: *We choose for* K *a circle with centre* a *and radius* $\rho > 0$ *small enough for* f *to be holomorphic on and inside* K. *Then*

$$\begin{aligned} A = \mathrm{Res}_{z=a} \frac{f(z)}{z-a} &= \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(z)}{z-a}\, dz = \frac{f(a)}{2\pi \mathrm{i}} \int_K \frac{1}{z-a}\, dz + \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(z)-f(a)}{z-a}\, dz \\ &= f(a) + \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(z)-f(a)}{z-a}\, dz. \end{aligned}$$

Function f *is holomorphic, and so continuous. For a given* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(z) - f(a)| < \varepsilon$ *for* $|z-a| < \delta$. *We choose* $\rho < \delta$, *such that* $|z-a| < \delta$ *for all* $z \in K$. *Then also* $|f(z) - f(a)| < \varepsilon$ *for all* $z \in K$. *From the ML-lemma we have*

$$|A - f(a)| = \left| \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(z)-f(a)}{z-a}\, dz \right| \leqslant \frac{1}{2\pi} \frac{\varepsilon}{\rho} 2\pi\rho = \varepsilon.$$

Since this is true for all $\varepsilon > 0$, *we conclude that* $A = f(a)$.

A direct consequence is

THEOREM 2.5.2 (**Cauchy's Integral Formula**). *Let function f be holomorphic inside and on the Jordan curve K, which is followed in positive direction. Then we have*

$$\frac{1}{2\pi \mathrm{i}} \int_K \frac{f(z)}{z-a}\, dz = \begin{cases} f(a), & \text{if } a \text{ is inside } K, \\ 0, & \text{if } a \text{ is outside } K. \end{cases}$$

PROOF: *If a is inside K, then a is the only point where the integrand is not holomorphic. From Lemma 2.5.1 it follows that the integral is equal to $f(a)$. If a is outside K, then the integrand is on and inside K holomorphic. According to Cauchy's Theorem the integral is then equal to* 0.

A remarkable property of holomorphic functions follows from this result: *The value of function f inside K is entirely determined by its values along K.* Expressed in other variables, we can write this out as follows. *Let function f be holomorphic inside and on the curve K. Inside the curve K the function can be represented by the following integral representation:*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_K \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{2.8}$$

We replaced here z by ζ and a by z. In this integral ζ is the integration variable and z is the free variable. The representation reveals that f is completely determined inside K if f is given along K, but also that the way $f(z)$ depends on z (inside K) has a particularly simple form, namely $p/(q-z)$, where p and q do not depend on z. So we can view any holomorphic function as a superposition of such simple functions. This will imply that a holomorphic function behaves very well. We will see that a holomorphic function is not just once, but actually arbitrarily many times differentiable. We will see that a function can be expressed in a power series in the neighbourhood of any point where it is holomorphic. For this we need a result about integrals of uniformly convergent sequences.

LEMMA 2.5.3.

1. *Let f_n be a sequence of continuous functions that converge uniformly on a curve K to a function f. Then we have*

$$\int_K f_n(z)\, dz \to \int_K f(z)\, dz \quad (n \to \infty).$$

2. *Let $\sum_{n=0}^{\infty} f_n$ be a series that converges uniformly on a curve K, where the functions f_n are continuous. Then we have*

$$\int_K \left[\sum_{n=0}^{\infty} f_n(z) \right] dz = \sum_{n=0}^{\infty} \left[\int_K f_n(z) \right] dz.$$

PROOF:

1. *From the uniform convergence it follows that* $\mu_n = \max_{z\in K} |f_n(z) - f(z)| \to 0$ *for* $n \to \infty$. *From the* ML*-lemma it follows therefore*

$$\left|\int_K f_n(z)\,dz - \int_K f(z)\,dz\right| \leqslant \int_K |f_n(z) - f(z)|\,dz \leqslant L\mu_n \to 0 \quad (n \to \infty)$$

2. *We apply the previous result to the partial sums* $s_n(z) = \sum_{k=0}^{n} f_k(z)$.

LEMMA 2.5.4. *Let* K *be a curve, and* φ *a continuous function defined on* K. *We define the function* f *by*

$$f(z) = \int_K \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta$$

for points z *not on* K. *Let* $a \notin K$. *Then* f *can be expanded in a power series around* a *with radius of convergence equal to*

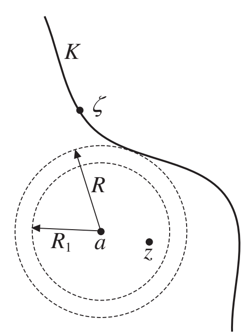
$$R = \min\{|a - \zeta| \;\big|\; \zeta \in K\}.$$

This means that there exists a sequence of numbers $c_0, c_1, c_2, \ldots$ *such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad (|z - a| < R).$$

The numbers c_n *are given by*

$$c_n = \int_K \frac{\varphi(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta.$$



PROOF:

We choose a positive number $R_1 < R$. *For* $|z - a| < R_1$ *and* $\zeta \in K$, *such that* $|\zeta - a| \geqslant R$, *we have*

$$\frac{|z - a|}{|\zeta - a|} < \rho = \frac{R_1}{R} < 1. \tag{2.9}$$

From this the convergence of the following series follows:

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a) - (z - a)} = \frac{1}{\zeta - a}\left(1 - \frac{z - a}{\zeta - a}\right)^{-1} = \frac{1}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z - a}{\zeta - a}\right)^n.$$

We conclude that

$$f(z) = \int_K \left[\sum_{n=0}^{\infty} \frac{\varphi(\zeta)}{\zeta - a}\left(\frac{z - a}{\zeta - a}\right)^n\right] d\zeta.$$

If the maximum van $|\varphi(\zeta)|$ *is denoted by* M, *then*

$$\left|\frac{\varphi(\zeta)}{\zeta - a}\left(\frac{z - a}{\zeta - a}\right)^n\right| \leqslant \frac{M}{R}\rho^n,$$

because of (2.9) and $|\zeta - a| \geqslant R$. From the Criterion of Weierstrass (1.3.15) it follows that the series in the integral converges uniformly for $\zeta \in K$. So we can apply Lemma 2.5.3,2:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n,$$

with c_n as defined. This argument is valid for all $R_1 < R$, so the formula is valid for all z with $|z - a| < R$.

If Lemma 2.5.4 is applied to the integral in (2.8) with $\varphi(z) = f(z)/(2\pi \mathrm{i})$, we find Taylor's theorem:

THEOREM 2.5.5 (**Taylor's Theorem**). *Let function f be holomorphic in a domain G. Let point a have a distance R to the boundary of G. Then $f(z)$ can be expanded in a power series around a. We have*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{with } c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta = \frac{f^n(a)}{n!}, \tag{2.10}$$

where C is a circle with centre a and a radius smaller than R. The radius of convergence of this power series is $\geqslant R$.

The second formula for c_n is given by (1.7). We call the above power series expansion a **Taylor series** or **Taylor expansion**. The existence of a power series representation on $|z - a| < R$, where $R > 0$, proves (see Theorem 1.6.5) that

THEOREM 2.5.6. *If f is holomorphic in a domain G, then all its derivatives exist. Its Taylor series around a point $a \in G$ can be differentiated term-by-term, and we have*

$$\frac{d^k}{dz^k} f(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n (z-a)^{n-k} = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} c_{n+k} (z-a)^n, \tag{2.11}$$

with the same radius of convergence for all k.

DEFINITION 2.5.7. *A function $f(z)$ defined in a domain G is called* ***analytic***[2] *if f can be expanded in a power series about a, convergent in a neighbourhood of a, for each $a \in G$.*

From Theorem 1.6.5 it follows that every analytic function is holomorphic. Theorem 2.5.5 says the reverse: every holomorphic function is analytic. From now on we will therefore consider the terms "analytic" and "holomorphic" as identical and use them interchangeably. Furthermore:

COROLLARY 2.5.8. *The* ***radius of convergence*** *of the Taylor series around a is equal to the distance from a to the nearest point where $f(z)$ cannot be defined in a way that makes it holomorphic.*

From Theorem 2.5.5 it follows that for complex functions it is much clearer what the Taylor series' radius of convergence is than for real functions.

[2]From ἀναλυω = *resolve into its elements*, where ἀνα = *up, throughout* and λυω = *unbind, solve, resolve, dissolve*.

EXAMPLE 2.5.9. *The function* $f(x) = (1+x^2)^{-1}$ *is well defined and analytic everywhere on* $\mathbb{R}$. *One could expect that the Taylor series around* $x = 0$ *of this function converges for all* x. *This, however, is not the case. The expansion,* $f(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, *has radius of convergence* 1. *In the complex plane it is clear why the radius of convergence of* $(1+z^2)^{-1} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ *is not greater than* 1. *The function is indeed not defined in the points* $z = \pm\,\mathrm{i}$, *which are at a distance* 1 *from the origin.*

For a given function $f(z)$ we can determine the Taylor series by using formula (2.10). In many cases, however, it is possible to find the Taylor series more easily from known properties of the function $f(z)$.

EXAMPLE 2.5.10. *Let* $a \in \mathbb{C}$. *The function* $\cos z$ *can be expanded in a power series around* $z = a$. *In order to realise this, we write* $w = z - a$ *and*

$$\cos(a+w) = \cos a \cos w - \sin a \sin w = \cos a \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!} - \sin a \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!},$$

so

$$\cos z = \cos(a+w) = \sum_{m=0}^{\infty} c_m w^m = \sum_{m=0}^{\infty} c_m (z-a)^m \quad \textit{for all } z \in \mathbb{C}$$

where

$$c_m = \begin{cases} \dfrac{(-1)^n}{(2n)!}\cos a & \textit{if } m = 2n, \\ \dfrac{(-1)^{n+1}}{(2n+1)!}\sin a & \textit{if } m = 2n+1. \end{cases}$$

EXAMPLE 2.5.11. *The function*

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

is analytic in $\mathbb{C}$ *except for the points* $z = 1$ *and* $z = 2$. *Via the derivatives*

$$f^{(n)}(z) = \frac{(-1)^n n!}{(z-2)^{n+1}} - \frac{(-1)^n n!}{(z-1)^{n+1}} \quad \textit{and so} \quad f^{(n)}(0) = n!\left(1 - 2^{-n-1}\right)$$

we find the Taylor series of f *around* $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} (1 - 2^{-n-1}) z^n.$$

Another way to obtain this result is by using the well-known geometric series:

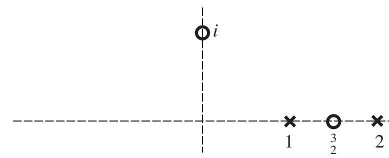
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

We find then

$$f(z) = \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{1}{2}z} = \sum_{n=0}^{\infty} (1 - 2^{-n-1}) z^n.$$

with radius of convergence 1.

Without actually determining the series we can conclude from Theorem 2.5.5 that the Taylor series of f around $z = \mathrm{i}$ and $z = \frac{3}{2}$ have radii of convergence equal to $\sqrt{2}$ and $\frac{1}{2}$, respectively.



Knowing that a holomorphic function is analytic, we can now prove a theorem that is useful to prove analyticity of infinite series of functions, other than powers.

THEOREM 2.5.12 (**Morera's theorem**). *A continuous complex-valued function φ, defined on a connected open domain G, that satisfies for every Jordan curve $\mathcal{C} \subset G$*

$$\int_{\mathcal{C}} \varphi(z)\,dz = 0, \tag{2.12}$$

is holomorphic in G.

PROOF: *We show that φ has a differentiable primitive (also known as: antiderivative). Define for any $z \in G$ and a point $z_0 \in G$ the function*

$$F(z) = \int_{z_0}^{z} \varphi(\zeta)\,d\zeta.$$

$F(z)$ is well-defined and does not depend on the chosen integration curve that connects z to z_0, since for any two curves γ_1 and γ_2 between z_0 and z we have

$$\int_{\gamma_1} \varphi(\zeta)\,d\zeta - \int_{\gamma_2} \varphi(\zeta)\,d\zeta = \int_{\gamma_1} \varphi(\zeta)\,d\zeta + \int_{-\gamma_2} \varphi(\zeta)\,d\zeta = 0.$$

Since φ is continuous, $\rho(h) = \max(|\varphi(\zeta) - \varphi(z)|) \to 0$ for $\zeta \in [z, z+h]$ (the straight arc from z to $z+h$) and $h \to 0$. Hence

$$\Big|\frac{F(z+h) - F(z)}{h} - \varphi(z)\Big| = \Big|\frac{1}{h}\int_{z}^{z+h} \varphi(\zeta) - \varphi(z)\,d\zeta\Big| \leqslant \rho(h) \to 0.$$

In other words, $F'(z)$ exists and is equal to $\varphi(z)$. So $F(z)$, and therefore $F'(z)$, is holomorphic, and so is $\varphi(z)$.

THEOREM 2.5.13. *If $w_n(z)$ are functions, holomorphic on domain G, such that $f(z) = \sum_{n=0}^{\infty} w_n(z)$ converge uniformly on G, then f is also holomorphic.*

PROOF: *Since w_n are holomorphic, they are continuous. Since the series converges uniformly, f is also continuous. For any Jordan curve $\mathcal{C} \subset G$ is $\int_{\mathcal{C}} w_n(z)\,dz = 0$. Therefore, using uniform convergence, we have*

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} \left[\sum_{n=0}^{\infty} w_n(z)\right] dz = \sum_{n=0}^{\infty} \left[\int_{\mathcal{C}} w_n(z)\,dz\right] = \sum_{n=0}^{\infty} 0 = 0,$$

and with Theorem 2.5.12 f is holomorphic.

Composite series

In order to determine the leading order terms of series which are constructed from elementary series, we use the following

LEMMA 2.5.14. *Let F and g be analytic functions with $g(0) = 0$ and*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < K; \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |z| < L,$$

where $0 < K, L \leqslant \infty$. Then there are $R > 0$ and $\{c_n\}$ such that the composite function

$$F(g(z)) = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < R.$$

PROOF: *An analytic function, with domain $\mathcal{D}$, applied to an analytic function, with range $\mathcal{R} \subset \mathcal{D}$, defines a function that is again analytic (Thm. 1.4.7). So if* $\operatorname{range}(g) \subset \operatorname{dom}(F)$ *(or* $\operatorname{dom}(g) \ni 0$ *can be restricted to that effect), $F(g(z))$ is analytic in $z = 0$ and the proposed series exists and is unique. Indeed, since $g(0) = 0$ and g is continuous, there is an $M > 0$ such that $|g(z)| < K$ on $|z| < M$. Take $R = \min(M, L)$, then*

$$F(g(z)) = \sum_{n=0}^{\infty} a_n \big(g(z)\big)^n = \sum_{n=0}^{\infty} a_n \left(\sum_{k=1}^{\infty} b_k z^k\right)^n = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < R.$$

REMARK 2.5.15. *Inside its radius of convergence a power series is absolutely convergent, so we may take the summation in any order. Hence, for the first few terms of $F(g(z))$ we find*

$$a_0 + a_1(b_1 z + b_2 z^2 + \dots) + a_2(b_1 z + b_2 z^2 + \dots)^2 + \cdots = a_0 + a_1 b_1 z + (a_1 b_2 + a_2 b_1^2) z^2 + \dots$$

EXAMPLE 2.5.16. *By using the geometric series $(1 - r)^{-1} = 1 + r + r^2 + \dots$ we may find*

$$\begin{aligned} \frac{1}{\cos z} &= \frac{1}{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \dots} = 1 + (\tfrac{1}{2}z^2 - \tfrac{1}{24}z^4 + \dots) + (\tfrac{1}{2}z^2 - \tfrac{1}{24}z^4 + \dots)^2 + \dots \\ &= 1 + \tfrac{1}{2}z^2 + \tfrac{5}{24}z^4 + \dots \\ \tanh z &= \frac{z + \frac{1}{6}z^3 + \dots}{1 + \frac{1}{2}z^2 + \dots} = (z + \tfrac{1}{6}z^3 + \dots)(1 - \tfrac{1}{2}z^2 + \dots) = z - \tfrac{1}{3}z^3 + \dots \end{aligned}$$

2.6 Zeros, entire functions, Liouville's Theorem

DEFINITION 2.6.1. *Let $a \in \mathbb{C}$, $k \in \mathbb{N}$ and the function $f(z)$ analytic in a.*

1. *The point a is called a* ***zero*** *of f if $f(a) = 0$.*
2. *The point a is called a* ***zero of multiplicity*** *k (or order k) of f if*

   $$f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0 \text{ and } f^{(k)}(a) \neq 0.$$

   A zero of multiplicity 1 is called a ***simple zero****.*

If f has a zero of multiplicity k in a, the Taylor series of $f(z)$ around a starts with the term $c_k(z-a)^k$, where $c_k = f^{(k)}(a)/k! \neq 0$. So $f(z)$ can be represented in a neighbourhood of a by

$$f(z) = (z-a)^k[c_k + c_{k+1}(z-a) + c_{k-2}(z-a)^2 + \cdots] = (z-a)^k g(z),$$

where $g(z)$ is a function, analytic in a, with $g(a) = c_k \neq 0$. If, conversely, the function f can be represented in this way, then f has a zero of order k in a.

EXAMPLE 2.6.2.

1. *The function* $f(z) = 1 - \cos z$ *has zeros where* $\cos z = 1$. *This equation can be rewritten as* $\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z} = 2$. *If we introduce the variable* $w = \mathrm{e}^{\mathrm{i}z}$, *we find* $w + 1/w = 2$, *so* $(w-1)^2 = w^2 - 2w + 1 = 0$. *The only solution is* $w = 1$. *For* z *we find* $\mathrm{e}^{\mathrm{i}z} = 1 = \mathrm{e}^{2k\pi\mathrm{i}}$ *and so* $z = 2k\pi$ *for* $k \in \mathbb{Z}$. *The equation* $\cos z = 1$ *has thus only real solutions. Next we determine the multiplicity of these zeros. We have* $f'(2k\pi) = -\sin(2k\pi) = 0$, $f''(2k\pi) = -\cos(2k\pi) = 1$. *So the zeros have order* 2.

2. *The function* $f(z) = (z - z^2)\sin(\pi z)\big(\mathrm{e}^{z^2} - 1\big)$ *has a zero in* $z = 0$. *We determine the multiplicity of this zero by using the Taylor series of the functions* e^z *and* $\sin z$. *We have*

$$\sin(\pi z) = z\pi\left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \cdots\right), \quad \mathrm{e}^{z^2} - 1 = z^2\left(1 + \frac{z^2}{2!} + \frac{z^4}{3!} + \cdots\right).$$

Since $z - z^2 = z(1-z)$, *it follows that* $f(z)$ *can be written as* $f(z) = z^4 g(z)$, *where* $g(z)$ *is analytic in* $z = 0$, *and* $g(0) \neq 0$. *So* $z = 0$ *is a zero of order* 4.

DEFINITION 2.6.3. *A function* f *is called* ***entire*** *if* f *is analytic in the entire complex plane. An equivalent formulation is: the Taylor series of* $f(z)$ *around any point* $a \in \mathbb{C}$ *has radius of convergence* ∞.

Examples of entire functions are polynomials and the functions e^z, $\cos z$, $\sin z$.

The following result, known as Liouville's theorem, is very powerful.

THEOREM 2.6.4 (**Liouville's Theorem**). *A bounded entire function is constant.*

PROOF: *Let* $f(z)$ *be a bounded entire function. Then* $f(z)$ *can be written as a power series that converges in the whole complex plane (equation (2.10) with* $a = 0$*):*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \text{ with } c_n = \frac{1}{2\pi\mathrm{i}} \int_{C_R} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta, \tag{2.13}$$

where C_R *is the circle* $\{z \in \mathbb{C} \mid |z| = R\}$ *and* R *any large number.*

The function f *is bounded, so there is an* M *with* $|f(z)| \leqslant M$ *for all* $z \in \mathbb{C}$. *Then it follows from the ML-lemma (2.2.4) that*

$$|c_n| = \left|\frac{1}{2\pi\mathrm{i}} \int_{C_R} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta\right| \leqslant \frac{1}{2\pi}\, 2\pi R\, \frac{M}{R^{n+1}} = \frac{M}{R^n}$$

Since $|c_n| \leqslant M/R^n$ *for arbitrarily large* R, *we find that* $c_n = 0$ *for* $n \geqslant 1$. *It follows that* $f(z) = c_0$ *for all* z.

CONSEQUENCE 2.6.5. *If $f(z)$ is entire and $\lim_{z\to\infty} f(z) = c$, then $f(z) = c$ everywhere.*

We prove this by noting that since $f(z) - c$ is continuous everywhere, it is bounded on any disc $|z| \leqslant R$. Then, from the limit, it follows that for any $\varepsilon > 0$ there is an R_ε such that $|f(z) - c| < \varepsilon$ on $|z| \geqslant R_\varepsilon$. Take $R \geqslant R_\varepsilon$, then $f(z) - c$ is entire and bounded, and so according to (2.6.4) equal to a constant. Since ε is arbitrary this constant can only be zero, and $f(z) = c$.

Liouville's theorem can be generalised as follows:

THEOREM 2.6.6 (**Generalised Theorem of Liouville**). *Let $f(z)$ be an entire function. Assume that there are real numbers $M > 0$, $q \geqslant 0$ and $R_0 > 0$ such that $|f(z)| \leqslant M|z|^q$ for $|z| \geqslant R_0$. Then $f(z)$ is a polynomial of degree $\leqslant q$.*

PROOF: *We follow the same argument as before, but replace the inequality $|f(z)| \leqslant M$ by $|f(z)| \leqslant MR^q$ on C_R, where we assume that $R \geqslant R_0$. Then it follows:*

$$|c_n| \leqslant \frac{1}{2\pi} 2\pi R \frac{MR^q}{R^{n+1}} = \frac{M}{R^{n-q}}$$

As R may be arbitrarily large, it follows that $c_n = 0$ for $n > q$, and so $f(z) = \sum_{n=0}^{\lfloor q \rfloor} c_n z^n$.

CONSEQUENCE 2.6.7. *If $f(z)$ is entire, while there is a polynomial $p_n(z)$ of order n such that*

$$\lim_{z\to\infty} \frac{f(z)}{p_n(z)} = c,$$

then $f(z)$ is a polynomial of exactly order n if $c \neq 0$, and at most order $n-1$ if $c = 0$.

We prove this as follows. Suppose $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots a_0$. Since

$$\lim_{z\to\infty} \frac{f(z)}{z^n} = \lim_{z\to\infty} \frac{f(z)}{p_n(z)} \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right) = ca_n,$$

it follows that for any $\varepsilon > 0$ there is an R_1 such that $|f(z) - a_n c z^n| < \varepsilon |z|^n$ on $|z| \geqslant R_1$. So according to (2.6.6) $f(z) - a_n c z^n$ is a polynomial of at most order n, or

$$f(z) = \sum_{k=0}^{n} b_k z^k.$$

Since ε is arbitrary, $b_n = a_n c$.

An important application of Liouville's Theorem is the following result:

THEOREM 2.6.8 (**Fundamental Theorem of Algebra**). *Every polynomial equation of degree $n > 0$ has a root. In other words: a non-constant polynomial has a zero.*

PROOF: *We prove by contradiction. Consider the n-th degree polynomial*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $n > 0$ and $a_n \neq 0$. We assume that $p(z)$ has no zeros. Then the function $f(z) = 1/p(z)$ is entire. For $z \to \infty$ the function $f(z)$ tends to zero. This follows from the inequality

$$|p(z)| \geqslant |a_n||z|^n \left\{1 - (|c_{n-1}z^{-1}| + \cdots + |c_0 z^{-n}|)\right\},$$

where $c_k = a_k/a_0$. If $|z| \to \infty$, the expression between curly brackets converges to 1, and so $f(z) \to 0$. We see that $f(z)$ is a bounded entire function and so equal to a constant. Since $f(z) \to 0$ for $|z| \to \infty$, the constant must be zero. This is not possible and the assumption that $p(z)$ has no zeros leads to a contradiction. We conclude that $p(z)$ does have zeros.

CONSEQUENCE 2.6.9. *A polynomial of degree n has exactly n zeros, if we count multiplicity. In other words, a polynomial of degree n can be factorised in n linear factors:*

$$p(z) = a_n z^n + \cdots + a_0 = c(z - z_1)\cdots(z - z_n),$$

with constant c.

PROOF: *From the Fundamental Theorem of Algebra we conclude that $p(z)$ has a zero z_n. From Taylor's Theorem (2.5.5) we can expand $p(z)$ in powers of $z - z_n$*

$$p(z) = p(z_n) + (z - z_n)p'(z_n) + \tfrac{1}{2}(z - z_n)^2 p''(z_n) + \cdots + \tfrac{1}{n!}(z - z_n)^n p^{(n)}(z_n).$$

The series ends at the n-th term because $p^{(n)}(z) = n!a_n$ is a constant and all higher derivatives vanish. Since $p(z_n) = 0$, one factor $(z - z_n)$ can be taken out and $p(z)$ can be factorised into

$$p(z) = (z - z_n)q(z)$$

where $q(z)$ is a polynomial of degree $(n - 1)$. By the same argument, q has a zero z_{n-1} and also can be factorised. The argument can be repeated until a polynomial of degree 0 (a constant c). Hence $p(z)$ can be factorised in n linear factors

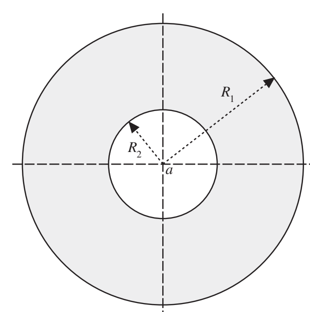
$$p(z) = c(z - z_1)\cdots(z - z_n).$$

2.7 Laurent series, isolated singular points

A **Laurent series** around $a \in \mathbb{C}$ is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n},$$

so the sum of a power series in powers of $z - a$ and a power series in powers of $(z - a)^{-1}$. If we denote the radius of convergence of the first series by R_1 and the second by $1/R_2$, we have



- $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges absolutely for $|z-a| < R_1$ and diverges for $|z-a| > R_1$.
- $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converges absolutely for $|z-a|^{-1} < R_2^{-1}$, so for $|z-a| > R_2$, and diverges for $|z-a| < R_2$.

The Laurent series itself converges if both $|z-a| < R_1$ and $|z-a| > R_2$. This is only possible if $R_2 < R_1$. In that case the region of (absolute) convergence of the Laurent series is the annular domain $\{z \in \mathbb{C} \,\big|\, R_2 < |z-a| < R_1\}$. In this domain the function is also holomorphic. The following theorem states that every function, holomorphic on such a domain, has a representation in the form of a Laurent series.

THEOREM 2.7.1 (**Laurent's Theorem**). *Let the function $f(z)$ be analytic in the annular domain $G = \{z \,\big|\, r < |z-a| < R\}$, where $a \in \mathbb{C}$, and r, R are numbers with $0 \leqslant r < R \leqslant \infty$. Then $f(z)$ can be expanded in a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ around a, convergent in G, with*

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta, \tag{2.14}$$

where C is a Jordan curve in G encircling a in positive direction.

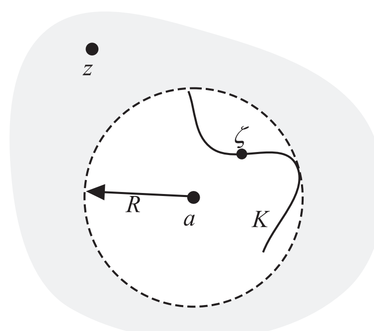
PROOF: *We use Lemma 2.5.4 and the following analogue of it:*

LEMMA 2.7.2.

Let K be a curve and φ a continuous function defined on K. We define the function f by

$$f(z) = \int_K \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta$$

for points z not on K. Let $a \in \mathbb{C}$. Then f can be expanded in a power series in the variable $(z-a)^{-1}$ around a which converges for



$$|z-a| > R = \max\{|a-\zeta| \,\big|\, \zeta \in K\}.$$

This means that there exists a sequence of numbers $c_{-1}, c_{-2}, \ldots$ such that

$$f(z) = -\sum_{n=0}^{\infty} c_{-n-1}(z-a)^{-n-1} = -\sum_{n=-\infty}^{-1} c_n(z-a)^n \quad (|z-a| > R).$$

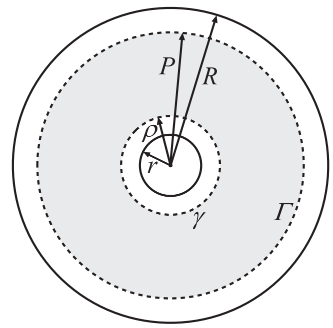
These numbers are given by

$$c_{-n-1} = \int_K \frac{\varphi(\zeta)}{(\zeta - a)^{-n}}\, d\zeta = \int_K \varphi(\zeta)(\zeta - a)^n\, d\zeta.$$

The proof is analogous to the proof of Lemma 2.5.4.

We choose two positive numbers ρ and P with $r < \rho < P < R$. The circles $\gamma : |z-a| = \rho$ and $\Gamma : |z-a| = P$, and the domain $G_1 : \rho < |z-a| < P$ between these circles, are subsets of G.

For $z \in G_1$ we have according to formula (2.6)



$$f(z) = f_\Gamma(z) - f_\gamma(z)$$

where

$$f_\Gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$
$$f_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

According to Lemma 2.5.4, $f_\Gamma(z)$ can be expanded in a power series with radius of convergence at least equal to P. This means that there exists a sequence of numbers $c_0, c_1, \ldots$ such that

$$f_\Gamma(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (|z-a| < P).$$

These numbers c_n are given by

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta.$$

According to Lemma 2.7.2, f_γ can be expanded in a power series in the variable $(z-a)^{-1}$, at least convergent for $|z-a| > \rho$:

$$f_\gamma(z) = -\sum_{n=0}^{\infty} c_{-n-1}(z-a)^{-n-1}.$$

In the annular domain G_1 both series converge. There we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n - \sum_{n=0}^{\infty} c_{-n-1}(z-a)^{-n-1} = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

For all n the coefficients c_n are given by equation (2.14).

From the above it follows that a function $f(z)$, analytic in an annular domain of the form $G = \{z \,\big|\, r < |z-a| < R\}$, can be written as the sum of two functions $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is analytic in the domain $G_1 = \{z \,\big|\, |z-a| < R\}$ and $f_2(z)$ in $G_2 = \{z \,\big|\, r < |z-a|\}$. The series $f_1(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ is called the **positive part** of the Laurent series. The other series, $f_2(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n$, is called the **negative part** or the **principal part**.

If the inner radius of convergence r is larger than zero, the Laurent series gives no information about any possible singular behaviour of f in center point a. For this, we need to have $r = 0$, while a is an **isolated singular point** (see 2.4). Therefore, we consider now a function $f(z)$ which is analytic in a **reduced neighbourhood** of a point a, say in $G = \{z \,\big|\, 0 < |z-a| < R\}$. From the theorem of Laurent it follows that this function can be expanded in a Laurent series that converges in G. We distinguish three cases:

1. **The principal part is zero.**

   In that case $f(z)$ can be defined in a by its limit $f(a) = \mathrm{im}_{z\to a} f(z) = c_0$, such that $f(z)$ is analytic in $|z-a| < R$. We say that $f(z)$ has a **removable singularity** in point a.

   EXAMPLE 2.7.3. *The function* $\sin z/z$ *is analytic for* $z \neq 0$*. The point* $z = 0$ *is an isolated singularity. The Laurent expansion follows directly from the Taylor expansion of the sine function:*

   $$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots .$$

   It follows that the singularity is removable.

   We have the following result:

   THEOREM 2.7.4. *The singularity in* a *is removable if and only if the function is bounded in a neighbourhood of* a*.*

   PROOF: *It is clear that the function is bounded if the singularity is removable. If we, conversely, assume that* $f(z)$ *is bounded in a neighbourhood of* a*, say* $|f(z)| \leqslant M$ *for* $|z-a| \leqslant r < R$*, then we have for* $n = -m$*, where* $m > 0$*:*

   $$|c_{-m}| \leqslant \frac{1}{2\pi}\left|\int_{|z-a|=r} f(z)(z-a)^{m-1}\,dz\right| \leqslant \frac{1}{2\pi}\,2\pi r M\, r^{m-1} \leqslant M r^m \ \to\ 0 \quad (r \to 0).$$

   Apparently is $c_n = 0$ *for* $n < 0$*.*

2. **The principal part consists of a finite number of terms.**

   Then there is a $k \in \mathbb{N}$ such that $f(z)$ is in G of the form

   $$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n + \frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \cdots + \frac{c_{-k}}{(z-a)^k}, \quad c_{-k} \neq 0.$$

   We say that $f(z)$ has a **pole**, *viz.* a **pole of order** k, in $z = a$. A **simple pole** if $k = 1$. $1/f(z)$ defines a holomorphic function with a zero of order k in a, as follows from

   THEOREM 2.7.5. *Define* $g(z) = (z-a)^k f(z)$*. The following statements are equivalent:*

   (a) $f(z)$ *has a pole of order* k *in* a*.*
   (b) $g(z)$ *has a limit* $\neq 0$ *for* $z \to a$*.*
   (c) $g(z)$ *has a removable singularity in* $z = a$*.*

   EXAMPLE 2.7.6. *The function* $\tan z$ *has a simple pole in* $z = \frac{1}{2}\pi$*. Indeed*

   $$\lim_{z\to\frac{\pi}{2}} \left(z - \frac{\pi}{2}\right)\tan z = \lim_{z\to\frac{\pi}{2}} \left(\frac{z - \frac{\pi}{2}}{\cos z}\,\sin z\right) = -1.$$

   EXAMPLE 2.7.7. *Using the technique of Lemma 2.5.14 we find that*

   $$\frac{1}{(\sin z)^3} = \frac{1}{z^3(1 - \frac{1}{6}z^2 + \dots)^3} = \frac{1}{z^3}\left(1 + \tfrac{1}{2}z^2 + \dots\right) = \frac{1}{z^3} + \frac{1}{2z} + \dots$$

   has a pole of order 3 in $z = 0$*, with a residue equal to* $\frac{1}{2}$*.*

3. **The principal part has infinitely many terms.**
   We say that $f(z)$ has an **essential singularity** in a.
   Note: this is *not* a pole. There is no m such that $(z-a)^m f(z)$ has a removable singularity.

   REMARK 2.7.8. *Remember that it is essential for the inner radius of convergence to be zero. Otherwise any possible infinite number of terms of the principal part conveys no information. Compare the series $\sum_{n=0}^{\infty} z^{-n}$, convergent for $|z|>1$, in which case $z=0$ is certainly not a singularity.*

   EXAMPLE 2.7.9. *The point $z=0$ is an essential singularity of $f(z)=\sin(z^{-1})$. Indeed*
   $$\sin(z^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-(2n+1)}}{(2n+1)!} = \frac{1}{z} - \frac{1}{3!z^3} + \cdots \quad \textit{for any} \quad |z|>0.$$
   Note, on the other hand, that $z=0$ is not an essential singularity of $(\sin z^{-1})^{-1}$. It is a singularity but not an isolated one. In some sense it is worse than an essential singularity.

   REMARK 2.7.10. *The behaviour of a holomorphic functions near an essential singularity is particularly odd. A main result (which will not be discussed here further) is that in* every *neighbourhood of an essential singularity the function takes on* every *complex value, except possibly one,* infinitely *many times. Therefore, an essential singularity is not to be confused with a pole.*

Technically speaking, a complex function $f(z)$ has a singularity at $z=\infty$ by virtue of not being defined there. We can classify this singularity in a natural way by considering the function $f(1/z)$ in $z=0$. First we define

DEFINITION 2.7.11. *The point at infinity $z=\infty$ is called an **isolated singularity at infinity** if there is an $R>0$ such that $f(z)$ is holomorphic for $|z|>R$.*

Then we have

DEFINITION 2.7.12. *Let $z=\infty$ be an isolated singularity of $f(z)$. Then*
- *(a) $f(z)$ has a removable singularity at ∞ if $f(1/z)$ has a removable singularity at 0.*
- *(b) $f(z)$ has a pole of order $m \geqslant 1$ at ∞ if $f(1/z)$ has a pole of order $m \geqslant 1$ at 0.*
- *(c) $f(z)$ has an essential singularity at ∞ if $f(1/z)$ has an essential singularity at 0.*

By application of Liouville's (Generalised) Theorem, we can prove the following

THEOREM 2.7.13. *Let $f(z)$ be an entire function.*
- *(a) If $z=\infty$ is a removable singularity of f, then f is a constant.*
- *(b) If $z=\infty$ is a pole of order $m \geqslant 1$ of f, then f is a polynomial of degree m.*

EXAMPLE 2.7.14. $(z+2)/(z+1)$ *has a removable singularity,* $(z^2+1)/(z+1)$ *has simple pole, and* $\sin z$ *has an essential singularity at $z=\infty$.*

2.8 Meromorphic functions

A complex function f defined on open domain V is called **meromorphic** if it is holomorphic on all V except for a set of isolated singularities, being the poles of the function. Every quotient g/h of holomorphic functions g and $h \not\equiv 0$, with no common zeros, defines a meromorphic function. Every meromorphic function (defined on V) can be expressed as the quotient g/h of two holomorphic functions (defined on V) with no common zeros. Any pole must coincide with a zero of the denominator, while around each pole the function has a Laurent series.

EXAMPLE 2.8.1. *Examples are rational functions (quotients of polynomials),* $\tan(z)$, $1/\sin(z)$. *Not meromorphic are for example* $1/\sin(1/z)$, $\mathrm{e}^{1/z}$ *or* $\log(z)$ *(to be defined in 2.16a).*

2.9 Logarithms and non-integer powers

Non-uniqueness of the logarithm as inverse of the exponential

The complex logarithm $\log(z)$ is an inverse of the exponential function and therefore a solution w of the equation

$$\mathrm{e}^{w} = z = r\,\mathrm{e}^{\mathrm{i}\arg(z)} = r\,\mathrm{e}^{\mathrm{i}\arg(z)+2k\pi\mathrm{i}}, \qquad k \in \mathbb{Z},$$

For any k and a choice of arg we have a logarithm $w(z) = \log(z) = \ln(r) + \mathrm{i}\arg(z) + 2k\pi\mathrm{i}$. For example, $\log(1)$ is not necessarily 0 but may be any integer multiple of $2\pi\mathrm{i}$. So we can associate to z infinitely many values $\log(z)$. To do this in a way that leads to a holomorphic log-function, we define the logarithm as an integral of the derivative $w'(z) = 1/z$. This derivative follows immediately from the derivative of the defining equation $(\mathrm{e}^{w} - z)' = w'\,\mathrm{e}^{w} - 1 = w'z - 1 = 0$.

We recall an earlier result.

Definition of the logarithm

An immediate consequence of Morera's Theorem 2.5.12 is

THEOREM 2.9.1. *If the integral of a continuous function $f(z)$, from a point z_0 to an arbitrary point z, is independent of the integration contour that runs inside a domain G from z_0 to z, then we can define a primitive*

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta.$$

This function is holomorphic and $F'(z) = f(z)$.

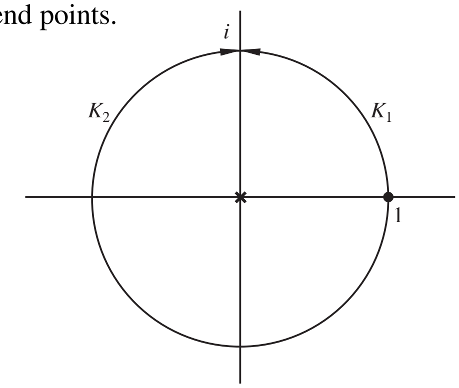
If the function $f(z)$ is holomorphic in a simply connected domain, then we know from Cauchy's Theorem that the integral is independent of the integration path. As a result, Theorem 2.9.1 is applicable, and $f(z)$ has a holomorphic primitive.

This is (in general) not the case if the domain is not simply connected. For example, if $f(z)$ is holomorphic in a domain, everywhere except for a pole with nonzero residue, any integral along a contour that encircles the pole a number of times adds a multiple of the residue to the result. Clearly the primitive depends on the chosen contour and cannot be defined uniquely. One way to cure this, is to restrict the domain to a simply connected one.

We will use the above considerations to define the complex **logarithm** as the primitive of the function $1/z$, *i.e.* the integral

$$\log z = \int_K \frac{d\zeta}{\zeta} + \log(1), \tag{2.15}$$

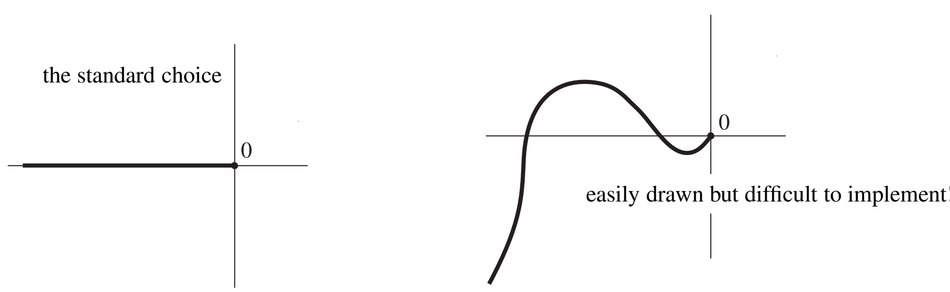
where K is a curve that runs from 1 to z not via the origin, and $\log(1)$ is a possible value, *i.e.* an integer multiple of $2\pi\mathrm{i}$. For a proper definition it is necessary that the integral is independent of the actually chosen curve and only depends on the given end points.



Due to the pole in $\zeta = 0$, this is not yet the case. We can, for example, determine $\log \mathrm{i}$ in two ways, namely via a quarter circle K_1 in the first quadrant, given by the parametrisation $z = \mathrm{e}^{\mathrm{i}t}$ for $0 \leqslant t \leqslant \frac{1}{2}\pi$, or via a three-quarter circle K_2 in the lower half plane and the second quadrant, given by the parametrisation $z = \mathrm{e}^{\mathrm{i}t}$ for $-\frac{3}{2}\pi\mathrm{i} \leqslant t \leqslant 0$. A standard calculation reveals that in the first case $\log \mathrm{i} = \frac{1}{2}\pi\mathrm{i} + \log(1)$, and in the second case $\log \mathrm{i} = -\frac{3}{2}\pi\mathrm{i} + \log(1)$.

The different values correspond with different **branches** of the logarithm, generated by the **branch point** $z = 0$. If we add n loops (in positive direction if $n > 0$, in negative direction if $n < 0$) around the origin, the found value of the logarithm is increased by $2n\pi\mathrm{i}$.

To achieve a unique result we restrict the domain $\mathbb{C}\setminus\{0\}$ to a simply connected one, by making a cut S from (and including) branch point 0 to infinity and exclude this cut from the domain. The new domain $G = \mathbb{C}\setminus S$ is then simply connected. This cut is called a **branch cut** for the logarithm. It can be chosen freely as long as $0 \in S$ and G is simply connected. The choice of S only depends on, and is part of, the preferred or required definition of the logarithm.



Examples of branch cuts from branch point $z = 0$.

The most common choice, corresponding with the principal value $\mathrm{Arg}(z)$ of the argument, is a cut along the negative real axis:

$$S_0 = \{z \in \mathbb{C} \mid z = -t, t \geqslant 0\}.$$

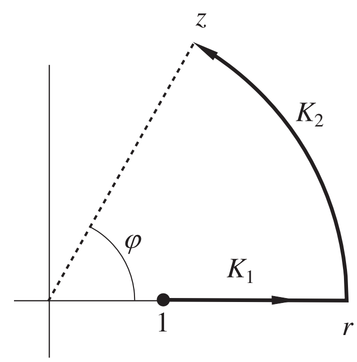
We require now that the curve K that connects 1 with z will be inside the domain G. Then any two of these curves will give for the integral $\int_K d\zeta/\zeta$ the same value. This value will be called $\log z$. In order to include the other branches we can add a constant $2n\pi\mathrm{i}$. So a complex log is **defined** by its domain given by branch cut S and its branch given by its value $\log(1) = 2n\pi\mathrm{i}$. We have then[3]

$$\log z = 2n\pi\mathrm{i} + \int_K \frac{1}{\zeta}\, d\zeta \tag{2.16a}$$

and an integration contour K from 1 to z that does not cross branch cut S. The logarithm corresponding with $n = 0$ and $S = S_0$ is called the **principal value logarithm**, and is sometimes denoted by $\operatorname{Log} z$. It is the standard definition of the complex logarithm in computer software.

If $\log z$ is to be holomorphic on open domain G, the above definition of logarithm is only applicable if $z \notin S$. In order to define $\log z$ also on S, we can take a suitable limit. For example, to define $\log(-1)$ for the principal value logarithm with branch cut S_0, we usually agree to take for any $a > 0$:

$$\log(-a) = \log(-a + \mathrm{i}0) = \lim_{y \downarrow 0} \log(-a + \mathrm{i}y).$$



We will see that this limit indeed exists. For this we will calculate $\log z$ for $z \in G$. We choose $\log(1) = 0$ and the curve K consisting of two arcs, namely the line segment K_1 from 1 to $r = |z|$ and the circular arc K_2 from r to z. We have for this the parametrisation $z = 1 + (r-1)t$ with $0 \leqslant t \leqslant 1$, and $z = \mathrm{e}^{\mathrm{i}t}$ with t running from 0 tot $\varphi = \arg z$, respectively. For $\arg(z)$ we take the principal value $\operatorname{Arg}(z)$. Then we have

$$\log z = \left(\int_{K_1} + \int_{K_2}\right)\frac{d\zeta}{\zeta} = \int_1^r \frac{dt}{t} + \int_0^{\varphi} \frac{1}{\mathrm{e}^{\mathrm{i}t}}\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}t}\, dt = \ln r + \mathrm{i}\varphi.$$

So we find that the **principal value logarithm**

$$\operatorname{Log} z = \ln|z| + \mathrm{i}\operatorname{Arg} z, \tag{2.16b}$$

extended with the convention that for negative real numbers we have $\operatorname{Log}(-a) = \ln a + \pi\mathrm{i}$.

EXAMPLE 2.9.2. *Suppose we want to construct a logarithm $L_\alpha(z)$ with the properties that $L_\alpha(1) = 0$ and a branch cut along the half-line $z = \lambda\,\mathrm{e}^{\mathrm{i}\alpha}$, $\lambda \geqslant 0$. A simple recipe to construct one, by means of the principal value logarithm* Log, *is*[4]

$$L_\alpha(z) = \operatorname{Log}(-\,\mathrm{e}^{-\mathrm{i}\alpha} z) - \operatorname{Log}(-\,\mathrm{e}^{-\mathrm{i}\alpha}).$$

Be careful with further simplification. For example, $\operatorname{Log}(-\,\mathrm{e}^{-\mathrm{i}\alpha}) = -\mathrm{i}\alpha + \mathrm{i}\pi$ *if* $\alpha \in (0, 2\pi)$, *but* $= -\mathrm{i}\alpha - \mathrm{i}\pi$ *if* $\alpha \in (-2\pi, 0)$.

[3] We choose 1 because $\log(1)$ is known, but equivalent is $\log z = \log z_0 + \int_{z_0}^{z} \zeta^{-1}\, d\zeta$ with $\log z_0$ a possible branch.
[4] Note that $(\log(az) + b)' = 1/z$ for any constants a and b.

EXAMPLE 2.9.3. *By integration of the geometric series we may find the following Taylor series, valid for any definition of* $\log(1-z)$ *that has a branch cut not intersecting the unit disc.*

$$\log(1-z) = \log(1) - \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1.$$

Properties

We consider some properties of the complex logarithm. First we have (if we write $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$):

$$\mathrm{e}^{\log z} = \mathrm{e}^{\ln r + \mathrm{i}\varphi + 2k\pi\mathrm{i}} = \mathrm{e}^{\ln r}\ \mathrm{e}^{\mathrm{i}\varphi} = r\,\mathrm{e}^{\mathrm{i}\varphi} = z,$$

and (if we use the notation $z = x + \mathrm{i}y$, note that $\mathrm{e}^z = \mathrm{e}^x\,\mathrm{e}^{\mathrm{i}y}$, and assume that $\log(1) = 2k\pi\mathrm{i}$):

$$\log(\mathrm{e}^z) = \ln|\,\mathrm{e}^z\,| + \mathrm{i}\arg\mathrm{e}^z = \ln\mathrm{e}^x + \mathrm{i}y + 2k\pi\mathrm{i} = z + 2k\pi\mathrm{i}.$$

It would be nice if we could extend the well-known product rule of real logarithms, $\ln(xy) = \ln x + \ln y$, to the complex logarithm. In general, this is not possible, but sometimes under restrictions. If $z_1 = r_1\,\mathrm{e}^{\mathrm{i}\varphi_1}$ and $z_2 = r_2\,\mathrm{e}^{\mathrm{i}\varphi_2}$, then we have with the principal value logarithm

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}\left(r_1 r_2\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}\right) = \ln(r_1 r_2) + \mathrm{i}\,\mathrm{Arg}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} = \ln r_1 + \ln r_2 + \mathrm{i}\,\mathrm{Arg}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}.$$

If $-\pi < \varphi_1 + \varphi_2 \leqslant \pi$ then $\mathrm{Arg}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} = \varphi_1 + \varphi_2$. Then indeed $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2$.
If $\varphi_1 + \varphi_2 \geqslant \pi$, then $\mathrm{Arg}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} = \varphi_1 + \varphi_2 - 2\pi$, and $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 - 2\pi\mathrm{i}$.
If $\varphi_1 + \varphi_2 < -\pi$, then $\mathrm{Arg}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} = \varphi_1 + \varphi_2 + 2\pi$, and $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\,z_1 + \mathrm{Log}\,z_2 + 2\pi\mathrm{i}$.
For example

$$0 = \mathrm{Log}\,1 = \mathrm{Log}((-1)(-1)) \neq \mathrm{Log}(-1) + \mathrm{Log}(-1) = \pi\mathrm{i} + \pi\mathrm{i} = 2\pi\mathrm{i}.$$

Particularly useful identities are: $\mathrm{Log}(\lambda z) = \ln\lambda + \mathrm{Log}\,z$ for $\lambda > 0$, and $\mathrm{Log}(z^{-1}) = -\mathrm{Log}\,z$.

Finally, from Theorem 2.9.1 it is confirmed that for any log-definition $(\log z)' = 1/z$.

The logarithm of a function

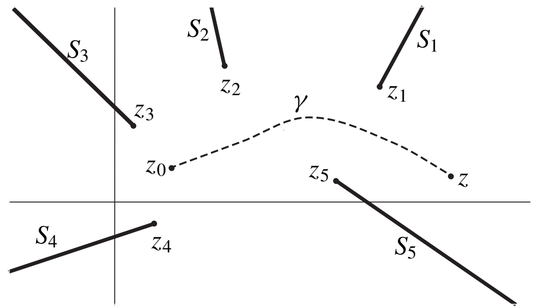
Let f be a complex function, holomorphic on U with possibly a finite number of isolated poles. Define a logarithm $\log(\cdot)$ with branch and branch cut S. Then $\log f(z)$ is defined for all $z \in U$ with $f(z) \notin S$. In other words, in order to define $\log f(z)$ as a holomorphic function we need to exclude the inverse image of S from f's definition set U. It is clear that this includes for example the zeros of f, but otherwise the geometrical structure of the definition set of $\log f(z)$ will in general be complicated.

Often it is convenient to select branch cuts directly. Take for example a rational function $R(z) = P(z)/Q(z)$ consisting of the ratio of polynomials P and Q, and with n isolated zeros and poles $z_1 \ldots z_n$. Choose a suitable point z_0 (not a zero or pole) and define $\log[R(z_0)]$ equal to one of the possible branches. Draw from each z_k a branch cut S_k to infinity, such that they do not intersect each other. Choose for $z \notin S_1 \cup \cdots \cup S_n$ a curve $\gamma(z)$ from z_0 to z which does not intersect any S_k.

Observe that $(\log R)' = P'/P - Q'/Q$ has only simple poles in $z = z_k$. Then

$$\log R(z) = \int_{\gamma(z)} \frac{P'(\zeta)}{P(\zeta)} - \frac{Q'(\zeta)}{Q(\zeta)}\, d\zeta + \log R(z_0)$$

is a logarithm of R, defined from its value in z_0 in a way that is independent of γ, and holomorphic on its simply connected domain $\mathbb{C} \setminus \{S_1 \cup \cdots \cup S_n\}$ (see Theorem 2.9.1).



EXAMPLE 2.9.4. $\log(z^2 + a^2)$ *has two branch cuts, one emanating from* $\mathrm{i}a$ *and one from* $-\mathrm{i}a$. *In the limit* $a \to 0$, *they may remain distinct, or merge into one. For example,* $\mathrm{Log}(z^2)$ *has two branch cuts, one along* $[0, \mathrm{i}\infty)$ *and one along* $[0, -\mathrm{i}\infty)$. *Another definition, via the integral of* $(z^2)'/z^2$ *and a branch cut along the negative real axis, leads to* $\log(z^2) = \int_1^z \frac{2}{\zeta}\, d\zeta = 2\,\mathrm{Log}(z)$.

Non-integer powers

By means of the logarithm it is now possible to define non-integer powers. For $z \neq 0$, $\alpha \in \mathbb{C}$, and a definition of the logarithm we define

$$z^\alpha = \mathrm{e}^{\alpha \log z}\,.$$

If $\alpha \in \mathbb{Z}$, this definition coincides with the regular definition $z^\alpha = (z \cdot z \cdot z \cdots z)^{\pm 1}$, since $\mathrm{e}^{\alpha 2\pi \mathrm{i} n} = 1$. If $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ is irrational or complex, this function has (like the logarithm) infinitely many branches. If $\alpha = p/q \in \mathbb{Q} \setminus \mathbb{Z}$ is rational ($p \in \mathbb{Z}$, $q \in \mathbb{N}$, p/q irreducible and non-integer), it has exactly q branches. In this case we call it a fractional power.

Consider, as an example, the complex power z^α defined by means of the principal value logarithm. Then

$$z^\alpha = \mathrm{e}^{\alpha\,\mathrm{Log}\, z} = \mathrm{e}^{\alpha(\ln|z| + \mathrm{i}\,\mathrm{Arg}\, z)} = |z|^\alpha\ \mathrm{e}^{\mathrm{i}\alpha\,\mathrm{Arg}\, z}, \qquad \mathrm{Arg}\, z \in (-\pi, \pi),$$

and the very complex number $\mathrm{i}^\mathrm{i} = \mathrm{e}^{\mathrm{i}\,\mathrm{Log}\,\mathrm{i}} = \mathrm{e}^{\mathrm{i}\,\pi\mathrm{i}/2} = \mathrm{e}^{-\pi/2}$ is not complex at all. Note that $a^{b\mathrm{i}}$, for $a > 0$ and $b \in \mathbb{R}$, is equal to $\mathrm{e}^{b\mathrm{i}\,\mathrm{Log}\, a}$, with $|a^{b\mathrm{i}}| = 1$ (for the principal branch; not for the other).

Naturally, the function z^α inherits the branch cut of the defining logarithm. For $x > 0$ we have along the upper side of the branch cut

$$(-x)^\alpha = (-x + \mathrm{i}0)^\alpha = \lim_{\varepsilon \downarrow 0} (-x + \mathrm{i}\varepsilon)^\alpha = x^\alpha\ \mathrm{e}^{\pi\alpha\mathrm{i}}\,.$$

Along the other side of the branch cut

$$(-x - \mathrm{i}0)^\alpha = \lim_{\varepsilon \downarrow 0} (-x - \mathrm{i}\varepsilon)^\alpha = x^\alpha\ \mathrm{e}^{-\pi\alpha\mathrm{i}}\,.$$

For $\alpha \neq 0$ we use also the notation

$$\sqrt[\alpha]{z} = z^{1/\alpha} \quad \text{and} \quad \sqrt{z} = \sqrt[2]{z} = z^{1/2}.$$

For example $\sqrt{\mathrm{i}} = \mathrm{e}^{\pi\mathrm{i}/4} = \frac{1}{2}\sqrt{2}\,(1+\mathrm{i})$ and $\sqrt{-\mathrm{i}} = \mathrm{e}^{-\pi\mathrm{i}/4} = \frac{1}{2}\sqrt{2}(1-\mathrm{i})$.

The well-known rule $(xy)^\alpha = x^\alpha y^\alpha$ can be generalised for complex numbers, but, again, under restrictions. It is clear that these properties depend on the chosen definition of logarithm.

EXAMPLE 2.9.5. *Be aware of the different branches, and don't slip unknowingly from one into another. What is wrong with the following calculation?*

$$\mathrm{e}^z = \mathrm{e}^{2\pi\mathrm{i}(z/2\pi\mathrm{i})} = 1^{z/2\pi\mathrm{i}} = \mathrm{e}^{(z/2\pi\mathrm{i})\log 1} = \mathrm{e}^0 = 1.$$

EXAMPLE 2.9.6. *Define, for $\alpha \in \mathbb{C}$, the following two versions of the power function z^α. One that has a branch cut along $[0, -\mathrm{i}\infty)$ and satisfies $1^\alpha = 1$, and the other with branch cut along $[0, \mathrm{i}\infty)$ and $1^\alpha = 1$. We indicate the first, analytic in the upper half plane, by $(z)_+^\alpha$ and the second, analytic in the lower half plane, by $(z)_-^\alpha$. Possible realisations are*

$$(z)_+^\alpha = \mathrm{e}^{\alpha(\mathrm{Log}(-\mathrm{i}z)+\frac{1}{2}\pi\mathrm{i})}, \qquad (z)_-^\alpha = \mathrm{e}^{\alpha(\mathrm{Log}(\mathrm{i}z)-\frac{1}{2}\pi\mathrm{i})}\,.$$

Then the function $(z)_+^\alpha\,(z)_-^{-\alpha}$ is identically equal to 1 along the whole right half plane $\mathrm{Re}(z) > 0$, and identically equal to $\mathrm{e}^{2\pi\alpha\mathrm{i}}$ along the whole left half plane $\mathrm{Re}(z) < 0$.

The complex square root $\sqrt{z}$

Although the complex square root is completely defined by the above definition $z^{1/2} = \mathrm{e}^{\frac{1}{2}\log z}$, it is still instructive to study it directly, because this function itself is important.

The function

$$f : \mathbb{C} \to \mathbb{C}, \quad \text{with} \quad f(z) = \sqrt{z}$$

is called *the complex square root* of z, and is defined as a solution of the equation $y^2 = z$.

This function is in a neighbourhood of every $z \neq 0$ holomorphic. This follows directly from the derivative of the defining equation $2f(z)f'(z) = 1$. On the other hand, it is not possible yet to conclude that f is holomorphic for all $z \neq 0$. $f(z)$ is not defined: we still have a choice between two solutions, and we will see that it is possible to slip from one into the other.

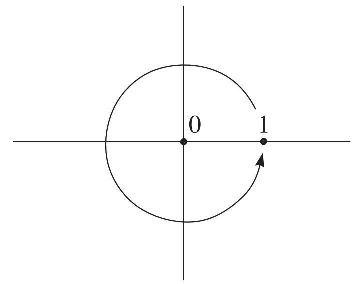
We consider in detail the behaviour near $z = 0$ by introducing $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$ such that

$$f(z) = \sqrt{r}\,\mathrm{e}^{\frac{1}{2}\varphi\mathrm{i}} \quad \text{or} \quad f(z) = -\sqrt{r}\,\mathrm{e}^{\frac{1}{2}\varphi\mathrm{i}}\,.$$

So the square root has two **branches**. (See the previous subsection: in general a q-th root has q branches.) The point $z = 0$, where $f(z) = 0$, is the **branch point**. In order to define f we have to start with choosing a branch by fixing f in one point ($\neq$ the branch point). Suppose we take $f(1) = 1$ while $\varphi = 0$ at $z = 1$, then any holomorphic f is at least in and near $z = 1$ defined by

$$f(z) = \sqrt{r}\,\mathrm{e}^{\frac{1}{2}\varphi\mathrm{i}}\,.$$

Suppose we plan to maintain this definition everywhere, with continuous φ. Then, if we trace $f(z)$ while z is moved from $z = 1$ around the branch point 0 back to 1, *i.e.* from $\varphi = 0$ to $\varphi = 2\pi$, we find $f(1) = \sqrt{1}\,\mathrm{e}^{\pi \mathrm{i}} = -1$. If we want f to be continuous everywhere we have a problem. We do not return to 1, the value we started from, but to -1, the value corresponding to the *other branch*. Only if we encircled the branch point another time, to arrive at $\varphi = 4\pi$, we would be back in $f(1) = 1$.



If f represents the same continuous function, we have a contradiction. If we encircle the branch point, we move from one branch of the square root to the other. Apparently it is not possible to define f such that it is at the same time single-valued *and* continuous everywhere. Somewhere along the encircling we have to jump from one branch to the other in order to return to our starting value. We do this by defining a **branch cut**, a simple contour from the branch point 0 to infinity. Wherever we pass this contour, the value of f changes discontinuously from one branch to the other. If S denotes the branch cut, the resulting square root function is holomorphic in the open domain $\mathbb{C}\backslash S$. If we want to define the square root also for $z \in S$ (the function is not holomorphic on S), it is a matter of choice which branch is taken.

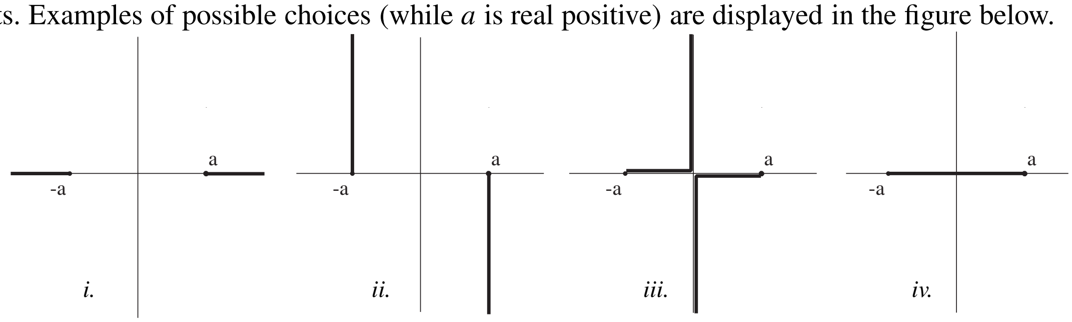
A square root is completely defined by choosing a branch cut and fixing the branch. A branch may be fixed by stating its value in a point not on the branch cut. The simplest, and most common choice of branch cut is along the negative real axis, and a branch[5] corresponding to $\sqrt{1} = 1$. This is equivalent to $\sqrt{z} = \sqrt{r}\,\mathrm{e}^{\frac{1}{2}\varphi \mathrm{i}}$ with $-\pi < \varphi \leqslant \pi$, the principal value of the argument $\mathrm{Arg}(z)$. The square root is indeed called the principal value square root. It is the common definition in computer software.

EXAMPLE 2.9.7. *We can express any square root of z that has a branch cut along the semi-infinite straight line $\{z \mid z = t\,\mathrm{e}^{\mathrm{i}\delta}, t > 0\}$, in terms of the principal value square root $\sqrt{\cdot}$ by multiplying the argument by $-\,\mathrm{e}^{-\mathrm{i}\delta}$ and correcting the result by $\pm\mathrm{i}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\delta}$ (the sign depends on the branch chosen):*

$$\pm\mathrm{i}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\delta}\sqrt{-\,\mathrm{e}^{-\mathrm{i}\delta}\,z}.$$

The complex square root $\sqrt{a^2 - z^2}$

As an example of a square root of a polynomial we consider $w(z) = \sqrt{a^2 - z^2}$. As we saw in section 2.9, this square root has two branch points, $z = a$ and $z = -a$, and therefore two branch cuts. Examples of possible choices (while a is real positive) are displayed in the figure below.



[5]To "define" the imaginary unit by $\mathrm{i} = \sqrt{-1}$ is faulty reasoning. At best it defines the used branch of $\sqrt{}$.

Note that in one case the cut runs from $-a$ to a. In reality the two branch cuts are $(-\infty, -a]$ and $(-\infty, a]$ (or equivalently $[-a, \infty)$ and $[a, \infty)$), and so they are, for a part, located on top of each other. Since the square root has only two branches, the parts that are on top of each other cancel each other ($+ \to - \to +$), so these parts of the cuts can be re-included in the domain of w.

The required branch is most easily selected by fixing w in the origin (but any other point might do as well), so by choosing $w(0) = \pm a$. In the case where a cut runs through the origin we have to modify this slightly by taking a limit, for example $w(+0 + \mathrm{i}0) = \pm a$.

EXAMPLE 2.9.8. *Verify that we can express the above square roots $w(z)$ in terms of the principal value square root $\sqrt{\cdot}$ as follows.*

i. $w(z) = \sqrt{a^2 - z^2}$ *with* $w(0) = a$.

ii. $w(z) = -\mathrm{i}\sqrt{\mathrm{i}(a - z)}\,\sqrt{\mathrm{i}(a + z)}$ *with* $w(0) = a$.

iii. $w(z) = -\mathrm{i}\sqrt{z^2 - a^2}$ *with* $w(+0 + \mathrm{i}0) = a$.

iv. $w(z) = -\mathrm{i}\sqrt{z - a}\sqrt{z + a}$ *with* $w(+0 + \mathrm{i}0) = a$.

The inverse trigonometric and hyperbolic functions

Since the complex trigonometric functions $\sin z$, $\cos z$ and $\tan z$ and the hyperbolic functions $\sinh z$, $\cosh z$ and $\tanh z$ can be expressed in complex exponentials, their respective inverses $\arcsin z$, $\arccos z$, $\arctan z$, $\operatorname{arsinh} z$, $\operatorname{arcosh} z$ and $\operatorname{artanh} z$ can be expressed as complex logarithms with square roots. As a result their definitions involve branches and branch cuts.

For example, the inverse w of $\sin z$ may be found from

$$\sin w = \frac{1}{2\mathrm{i}}\big(\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w}\big) = z \quad\Rightarrow\quad w = -\mathrm{i}\log\big(\mathrm{i}z \pm \sqrt{1 - z^2}\,\big)$$

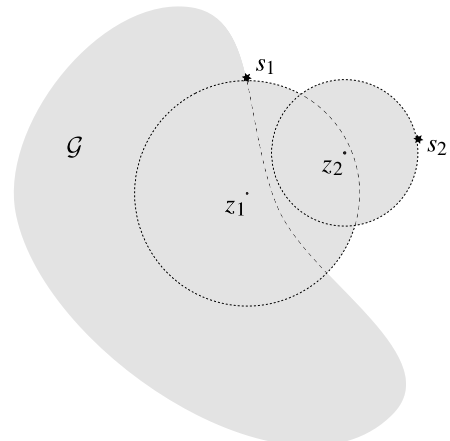
where the branch and branch cuts of the logarithm and square root have to be chosen.

The standardised forms, in terms of the principal value logarithm and square root, are as follows

$$\begin{aligned}
\arcsin z &= -\mathrm{i}\log\big(\mathrm{i}z + \sqrt{1 - z^2}\,\big),\\
\arccos z &= \mathrm{i}\log\big(z - \mathrm{i}\sqrt{1 - z^2}\,\big) = \tfrac{1}{2}\pi + \mathrm{i}\log\big(\mathrm{i}z + \sqrt{1 - z^2}\,\big),\\
\arctan z &= \tfrac{1}{2}\mathrm{i}\log(1 - \mathrm{i}z) - \tfrac{1}{2}\mathrm{i}\log(1 + \mathrm{i}z),\\
\operatorname{arsinh} z &= \log\big(z + \sqrt{z^2 + 1}\,\big),\\
\operatorname{arcosh} z &= \log\big(z + \sqrt{z + 1}\sqrt{z - 1}\,\big),\\
\operatorname{artanh} z &= \tfrac{1}{2}\log(1 + z) - \tfrac{1}{2}\log(1 - z).
\end{aligned}$$

The branch cuts are inherited from the respective logarithm and square root functions.

2.10 Continuation, uniqueness, and the modulus maximum



A function $f(z)$, which is defined and analytic on a domain $\mathcal{G} \subsetneq \mathbb{C}$, can be extended outside $\mathcal{G}$ if the boundary $\partial\mathcal{G}$ is *not essential*, *i.e.* is not covered with too many singularities of f. When we choose a suitable point near the boundary (say, z_1), we can expand

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_1)^n$$

in a Taylor series with a radius of convergence equal to the distance to the nearest singularity (say, s_1). If we start close enough to the boundary and far enough from the nearest singularity, the disc of convergence will be partly outside $\mathcal{G}$, and hence extends the domain of definition of f. We can repeat this process (see figure). This process is called **analytic continuation** of f. Note that if any of the singularities of f are branch points, the result of the continuation may not be unique as it depends on the way the branch point is encircled.

THEOREM 2.10.1. *If a functions $f(z)$, defined and analytic on an (open and connected) domain $\mathcal{D}$, is equal to zero along a curve $\gamma \subset \mathcal{D}$, then $f(z) = 0$ everywhere.*

PROOF: *Suppose $z_0 \in \gamma$, then $f(z_0) = 0$. If f is not identically zero, there is an $n \in \mathbb{N}$ and an analytic $h(z)$ with $h(z_0) \neq 0$, such that $f(z) = (z - z_0)^n h(z)$. Since h is continuous, there is a disc $|z - z_0| < \delta$ where $h(z) \neq 0$. However, inside this disc there are other points of γ where f, and hence h, is zero. So there is at least a neighbourhood of z_0 where f is identically zero. By analytic continuation f must be zero on domain $\mathcal{D}$.*

CONSEQUENCE 2.10.2 (**Uniqueness**). *If $f(z) = g(z)$ along a curve $\gamma \subset \mathcal{D}$, then $f(z) = g(z)$ everywhere.*

THEOREM 2.10.3 (**Maximum Modulus Principle**). *Suppose $f(z)$ is analytic inside and on a Jordan curve C. Then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.*

PROOF: *Suppose that $|f(z)|$ has a maximum M at z_0 inside C, and that $|f(z)| \leqslant |f(z_0)| = M$ in a disc $|z - z_0| < R$ inside C. Let $r < R$ and represent $f(z_0)$ by Cauchy's formula to obtain*

$$M = |f(z_0)| = \left| \frac{1}{2\pi \mathrm{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} d\zeta \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \mathrm{e}^{\mathrm{i}\vartheta}) d\vartheta \right| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \mathrm{e}^{\mathrm{i}\vartheta})| d\vartheta.$$

Since $M = (2\pi)^{-1} \int_0^{2\pi} M \, d\vartheta$, we have

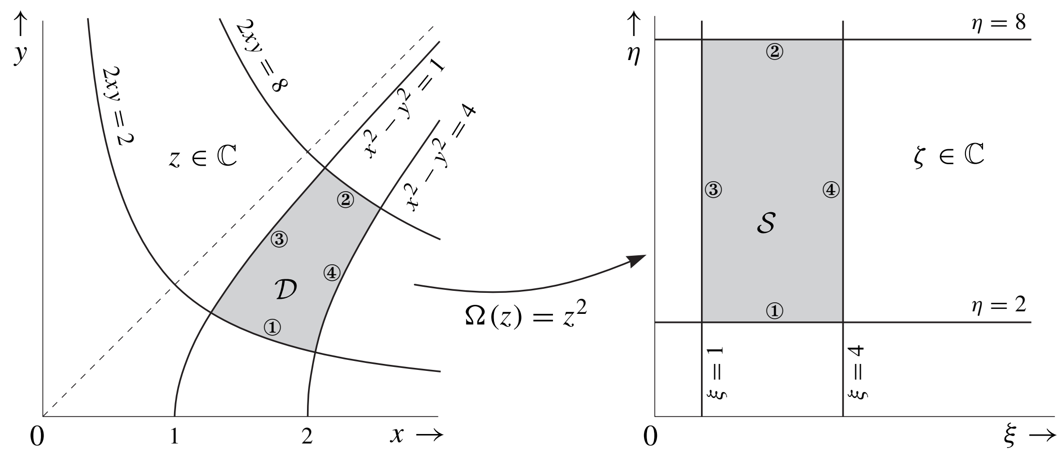
$$\int_0^{2\pi} \left[|f(z_0 + \mathrm{e}^{\mathrm{i}\vartheta})| - M \right] d\vartheta \geqslant 0.$$

Since $|f(z_0 + \mathrm{e}^{\mathrm{i}\vartheta})| - M \leqslant 0$, we must have that $|f(z_0 + \mathrm{e}^{\mathrm{i}\vartheta})| = M$ for almost all ϑ and hence (since $|f|$ is continuous) for all ϑ. Since r is arbitrary, $|f| = M$ on the disc $|z - z_0| < R$. With Cauchy-Riemann's relations it follows that f is constant on the disc, and by continuation everywhere inside C.

2.11 Applications

2.11.1 Solving 2D Potential Problems by Conformal Mapping

The function $\Omega(z) = z^2$ maps the domain $\mathcal{D} = \{z \in \mathbb{C} \mid 1 < x^2 - y^2 < 4,\ 2 < 2xy < 8\}$ to the rectangle $\mathcal{S} = \{\zeta \in \mathbb{C} \mid 1 < \xi < 4,\ 2 < \eta < 8\}$ such that every image point has only one original and hence the inverse $\Omega^{-1}(\zeta) = \sqrt{\zeta}$ exists (with the principal value square root) and is analytic. In particular, $\operatorname{Im}\Omega$ is constant along the boundaries $\mathcal{D}_1$ and $\mathcal{D}_2$ and hence the normal derivative of $\operatorname{Re}\Omega$ vanishes there. $\operatorname{Re}\Omega$ is constant along the boundaries $\mathcal{D}_3$ and $\mathcal{D}_4$.



Associate $\mathcal{D} \subset \mathbb{R}^2$ with $\mathcal{D} \subset \mathbb{C}$, and consider on $\mathcal{D}$ the potential problem

$$\nabla^2 u = 0, \quad \text{with } u|_{\mathcal{D}_1} = 0,\ u|_{\mathcal{D}_2} = A,\ \boldsymbol{n}\cdot\nabla u|_{\mathcal{D}_3\cup\mathcal{D}_4} = 0.$$

This is a rather difficult problem, because of the complicated shape of $\mathcal{D}$. It can be solved easily, however, by utilising the mapping Ω and the simple shape of image $\mathcal{S} = \Omega(\mathcal{D})$. Since u is harmonic, there is a function v such that there is an analytic

$$f(z) = u(x, y) + \mathrm{i}v(x, y).$$

Introduce the analytic function $F(\zeta) = f(\Omega^{-1}(\zeta))$, with $f(z) = F(\Omega(z))$ and $F = U + \mathrm{i}V$ for harmonic U and V. Along the boundaries $\mathcal{D}_1$ and $\mathcal{D}_2$ is $\eta = 2$ and $\eta = 8$. So for $\mathcal{S}_1$ and $\mathcal{S}_2$ we have the boundary conditions $U(\xi, 2) = 0$ and $U(\xi, 8) = A$. Along the boundaries $\mathcal{D}_3$ and $\mathcal{D}_4$ we have $\boldsymbol{n} = \nabla\xi/|\nabla\xi|$ and so

$$\boldsymbol{n}\cdot\nabla u = \big(\xi_x U_x + \xi_y U_y\big)/|\nabla\xi| = \big(U_\xi|\nabla\xi|^2 + U_\eta(\nabla\xi\cdot\nabla\eta)\big)/|\nabla\xi| = U_\xi|\nabla\xi| = 0$$

and hence $U_\xi = 0$ along $\mathcal{S}_3$ and $\mathcal{S}_4$. We conclude that we can solve first the problem $\nabla^2 U = 0$ on the simpler domain $\mathcal{S}$. This yields F, from which f follows. It is easily verified that $U = \frac{1}{6}A(\eta - 2)$, and hence $F(\zeta) = -A(\frac{1}{6}i\zeta + \frac{1}{3})$. It follows that

$$f(z) = -A(\tfrac{1}{6}\mathrm{i}z^2 + \tfrac{1}{3}), \qquad u = \tfrac{1}{3}A(xy - 1).$$

The above can be generalised as follows. For simplicity we will restrict ourselves here to Neumann and Dirichlet problems, but this is not necessary.

Problem:

Given a "difficult" domain $\mathcal{D} \subset \mathbb{R}^2$ with boundary $\partial\mathcal{D}$. A potential u satisfies on $\mathcal{D}$ the Laplace equation $\nabla^2 u = 0$. By $z = x + \mathrm{i}y$ we identify $\mathcal{D}$ in $\mathbb{R}^2$ with a domain of the same name in $\mathbb{C}$. Since u is harmonic, there is an associated analytic function $f(z)$ in $\mathcal{D}$, such that $u = \mathrm{Re}(f)$. So the problem is to find f.

Suppose there is a "simple" domain $\mathcal{S}$, connected with $\mathcal{D}$ by a conformal mapping

$$\zeta = \xi + \mathrm{i}\eta = \Omega(z) = p(x, y) + \mathrm{i}q(x, y),$$

where $\zeta \in \mathcal{S}$ if and only if $z \in \mathcal{D}$, Ω is one-to-one so the inverse Ω^{-1} exists, and $\Omega(\partial\mathcal{D}) = \partial\mathcal{S}$.

We consider the following practically important potential problems:

Dirichlet.	Find a solution u such that $u = h$ on $\partial\mathcal{D}$.
Homogeneous Neumann.	Find a solution u such that $\frac{\partial}{\partial n}u = 0$ on $\partial\mathcal{D}$.

Solution:

Dirichlet.

Suppose, the simpler geometry of $\mathcal{S}$ allows us to find a U, harmonic on $\mathcal{S}$, that satisfies $U = H$ on $\partial\mathcal{S}$, where $h(x, y) = H(p, q)$ for $(x, y) \in \mathcal{D}$. Then $u(x, y) = U(p(x, y), q(x, y))$.

Proof

If U is harmonic, there is an $F(\zeta) = U(\xi, \eta) + \mathrm{i}V(\xi, \eta)$, analytic in $\mathcal{S}$. Then is $f(z) = F(\Omega(z))$ and so u is harmonic. The boundary condition follows from $u(x, y) = H(p, q) = h(x, y)$ on $\mathcal{D}$.

Neumann.

Suppose, the simpler geometry of $\mathcal{S}$ allows us to find a U, harmonic on $\mathcal{S}$, that satisfies $\frac{\partial}{\partial n}U = 0$ on $\partial\mathcal{S}$. Then $u(x, y) = U(p(x, y), q(x, y))$.

Proof

If U is harmonic, there is an $F(\zeta) = U(\xi, \eta) + \mathrm{i}V(\xi, \eta)$, analytic in $\mathcal{S}$. Then is $f(z) = F(\Omega(z))$ and so u is harmonic. The boundary condition is verified as follows. Suppose, the boundary $\partial\mathcal{S}$ satisfies the equation $B(\xi, \eta) = 0$. Then $\nabla_{\boldsymbol{\xi}} B$ is normal to $\partial\mathcal{S}$, $\partial\mathcal{D}$ is described by $B(p, q) = 0$, and the corresponding $\nabla_{\boldsymbol{x}} B(p, q)$ is normal to $\partial\mathcal{D}$. We know that on $\partial\mathcal{S}$

$$\frac{\partial U}{\partial n} \sim U_\xi B_\xi + U_\eta B_\eta = 0.$$

By applying the chain rule and the Cauchy-Riemann relations it follows that on $\partial\mathcal{D}$

$$\begin{aligned}\frac{\partial u}{\partial n} \sim u_x B_x + u_y B_y &= (U_\xi p_x + U_\eta q_x)(B_\xi p_x + B_\eta q_x) + (U_\xi p_y + U_\eta q_y)(B_\xi p_y + B_\eta q_y) \\ &= (p_x^2 + p_y^2)(U_\xi B_\xi + U_\eta B_\eta) = 0\end{aligned}$$

Remarks.

- A similar approach is possible with a source $\mu(x, y)$. However, since $\nabla_{\boldsymbol{x}}^2 u = |\nabla_{\boldsymbol{x}} p|^2 \nabla_{\boldsymbol{\xi}}^2 U$, we have to allow for the geometric distortion factor $|\nabla_{\boldsymbol{x}} p|^2$, and solve for U with source $\mu/|\nabla_{\boldsymbol{x}} p|^2$. If $\mu(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ this source turns out to be just $\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ (provided $|\nabla_{\boldsymbol{x}} p| \neq 0$).

- Since $\nabla_\xi U \cdot \nabla_\xi V = 0$, a solution U, with $\frac{\partial}{\partial n} U = 0$ along the border, is equivalent with a solution $V =$ constant.
- For some "simple" domains, the general solution of the Dirichlet problem is explicitly known. For example, inside the unit circle $r \leqslant 1$, with $\xi = r\cos\vartheta$, $\eta = r\sin\vartheta$ and $H(\xi, \eta) = g(\vartheta)$ along $r = 1$, we have *Poisson's formula for a circle*

$$U(r, \vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) g(\varphi)}{1 - 2r\cos(\vartheta - \varphi) + r^2} \, d\varphi.$$

For a bounded solution in the upper half plane $\eta > 0$, with $H(\xi, \eta) = G(\xi)$ along $\eta = 0$, we have *Poisson's formula for the half plane*

$$U(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta G(\sigma)}{\eta^2 + (\xi - \sigma)^2} \, d\sigma.$$

2.11.2 Flow out of a duct

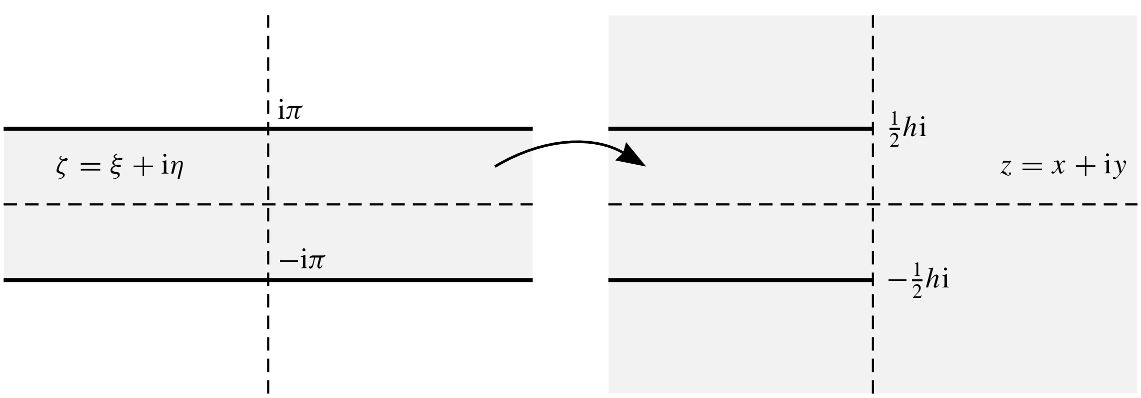
Suppose we have the 2D incompressible flow with (far upstream) a uniform velocity U inside the hard-walled duct $-\frac{1}{2}h < y < \frac{1}{2}h$, $x < 0$. The flow velocity has a potential $\varphi = \varphi(x, y)$, which satisfies $\nabla^2 \varphi = 0$ and $\frac{\partial}{\partial y}\varphi = 0$ (Neumann) on the duct wall. Far upstream inside the duct, the potential behaves like $\varphi \simeq Ux$. We write φ as the real part of the complex potential Φ.

Questions

1. Consider the complex ζ-plane with $\zeta = \xi + \mathrm{i}\eta$ and the complex z-plane with $z = x + \mathrm{i}y$, connected by the (inverse) conformal mapping $z = w(\zeta)$, given by

$$w = \frac{h}{2\pi}\big(\mathrm{e}^{\zeta} + \zeta + 1\big).$$

Verify, by tracing the images of the walls $\eta = \pm\pi$ and the lines $\xi = 0$ and $\eta = 0$, that w maps the strip $-\infty < \xi < \infty$, $-\pi < \eta < \pi$ to the inner and outer region of the semi-infinite duct, *i.e.* with $z \notin (-\infty + \frac{1}{2}h\mathrm{i}, \frac{1}{2}h\mathrm{i}] \cup (-\infty - \frac{1}{2}h\mathrm{i}, -\frac{1}{2}h\mathrm{i}]$, given in the following picture:



2. What is the simplest (non-trivial) flow in the doubly infinite duct in ζ-plane that satisfies the Neumann boundary condition?

3. Construct, by using the previous result and using the above conformal mapping w, an (implicit) description of potential Φ. What is the behaviour of Φ for $|z| \to \infty$?
HINT. The mapping w is in fact the inverse of the mapping we would like to use. Therefore, it is convenient to introduce (*formally*) the inverse of w (called γ) with $\zeta = \gamma(z)$.

4. Another approach would involve the mapping

$$z = w(\zeta) = \frac{h}{2\pi}\left(1 - \pi \mathrm{i} - \zeta^2 + 2\log\zeta\right)$$

from the upper complex half ζ-plane to the same duct geometry in z as above. What would you have to change?

Answers

1. The walls map to the walls:

$$w(\pm\pi\mathrm{i} + \lambda) = \pm\tfrac{1}{2}h\mathrm{i} + \tfrac{h}{2\pi}(1 + \lambda - \mathrm{e}^{\lambda})$$

while $1 + \lambda - \mathrm{e}^{\lambda} \leqslant 0$, monotonically increasing for $\lambda \leqslant 0$, decreasing for $\lambda \geqslant 0$, and $= 0$ for $\lambda = 0$.

2. The simplest flow is a uniform flow of velocity a and potential $\Phi(\zeta) = a\zeta + b$.

3. Note that far upstream inside the duct, the complex potential behaves like $\Phi \simeq Uz$. Define γ the inverse of w, so $\zeta = w^{-1}(z) = \gamma(z)$. Inside the duct, for $\xi \to -\infty$, then $z = w(\zeta) \simeq \frac{h}{2\pi}(\zeta + 1)$, or $\zeta = \gamma(z) \simeq \frac{2\pi}{h}z - 1$. So if we take the simplest function possible

$$\varphi(z) = \Phi(\zeta) = a\zeta + b = a\gamma(z) + b \simeq a(\tfrac{2\pi}{h}z - 1) + b = Uz$$

we have a solution if $a = \frac{Uh}{2\pi}$ and $b = a$. Hence the required potential is given by $\varphi(x, y) = \mathrm{Re}\,\Phi(\gamma(z))$.

4. For the alternative approach, we note that "far upstream inside the duct" ($-\frac{1}{2}h < y < \frac{1}{2}h$ and $x \to -\infty$) corresponds with $\zeta \to 0$. So we have $w \simeq (2\log\zeta - \pi\mathrm{i})h/2\pi$ and so $\zeta = \gamma(z) \simeq \mathrm{i}\,\mathrm{e}^{\pi z/h}$. We have to apply a source in $\zeta = 0$, which amounts to

$$\Phi(\zeta) = a\log\zeta + b = a\log\gamma(z) + b \simeq \tfrac{1}{2}\pi a\mathrm{i} + a\frac{\pi}{h}z + b = Uz$$

if $a = Uh/\pi$ and $b = -\frac{1}{2}Uh\mathrm{i}$.

Note.

The solution is the same for $U < 0$, *i.e.* when the flow is sucked into the duct. In fact, this solution is a whole lot more physical, because for positive U a real flow will leave the duct as a jet and the condition of an inviscid potential flow, negotiating the sharp edges without separation while leaving the duct, is very unlikely to find in reality (maybe superfluid Helium). Of course, also with in-flow there may be separation, now along the inside of the duct, but this is probably only a local effect if the actual Reynolds number is not too high.

2.11.3 Blasius Theorem

If the flow is irrotational, *i.e.* the vorticity vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \boldsymbol{0}$, a scalar velocity potential φ may be introduced with

$$\boldsymbol{v} = \nabla\varphi.$$

For example, in inviscid homentropic flow, any vorticity is convected with the flow, and if the flow starts irrotational it stays that way. In incompressible flow this potential is independent of pressure (except indirectly via boundary conditions) and satisfies Laplace's equation

$$\nabla^2\varphi = 0.$$

The pressure is related to the velocity by means of the equation of conservation of momentum, which in this case can be integrated to Bernoulli's equation (a form of conservation of mechanical energy)

$$p + \tfrac{1}{2}\rho|\boldsymbol{v}|^2 = p_0$$

where density ρ and stagnation pressure p_0 are constants.

In two dimensions an important class of solutions may be generated by using the property of analytic functions $\Omega(z)$ in the complex variable $z = x + \mathrm{i}y$, that both their real and imaginary parts satisfy Laplace's equation. If we introduce the complex potential $\Omega(z) = \varphi(x, y) + \mathrm{i}\psi(x, y)$, then the velocity $\boldsymbol{v} = (u, v)$ is given by

$$u - \mathrm{i}v = \Omega'(z).$$

Note that solutions may be constructed by superposition of elementary solutions (the problem is only nonlinear in pressure). For example, a uniform flow Uz and a dipole source flow R^2Uz^{-1} yield together the flow past a cylinder of radius R. As the flow is inviscid, this solution is not unique and any multiple of a line vortex flow $-\mathrm{i}\Gamma \log(z)/2\pi$ may be added to get

$$\Omega(z) = Uz + \frac{R^2U}{z} - \mathrm{i}\frac{\Gamma}{2\pi}\log(z).$$

By itself this solution is not very useful practically, because no high-Reynolds number flow will pass a cylinder without separation and creating a turbulent wake. It may, however, be a starting point for a larger family of solutions $\Omega(\zeta(z))$ to be obtained by conformal mappings $\zeta \mapsto z$. For example, the **Joukowski transformation**

$$z = \left(\zeta + \frac{\lambda^2}{\zeta}\right)\mathrm{e}^{-\mathrm{i}\alpha}$$

maps the circle $|\zeta - \zeta_c| = R$ in ζ-domain to an airfoil in z-domain if $\lambda = \xi_c + \sqrt{R^2 - \eta_c^2}$ where $\zeta_c = \xi_c + \mathrm{i}\eta_c$. Take for example figure 2.1 with $\xi_c = -0.03, \eta_c = 0.03, R = 1, \alpha = 0.05$. The corresponding flow around the airfoil is given by

$$\Omega(z) = U\zeta\,\mathrm{e}^{-\mathrm{i}\alpha} + \frac{R^2U\,\mathrm{e}^{\mathrm{i}\alpha}}{\zeta - \zeta_c} - \mathrm{i}\frac{\Gamma}{2\pi}\log(\zeta - \zeta_c)$$

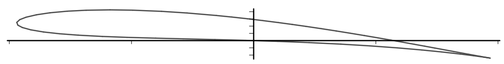


Figure 2.1: A Joukowski airfoil.

where obviously (for a suitable definition of the square root)

$$\zeta = \tfrac{1}{2}z\,\mathrm{e}^{\mathrm{i}\alpha} + \sqrt{\tfrac{1}{4}z^2\,\mathrm{e}^{2\mathrm{i}\alpha} - \lambda^2}.$$

The undetermined circulation Γ is found by requiring the flow to be non-singular at the trailing edge $\zeta = \lambda$ or $z = 2\lambda\,\mathrm{e}^{-\mathrm{i}\alpha}$ (the so-called Kutta condition) and we obtain $\Gamma = -4\pi RU\,\sin(\alpha+\beta)$ where $\beta = \arcsin(\eta_c/R)$. This condition is a remainder of the effect of viscosity near the trailing edge. Note that when we dropped viscosity in our modelling the no-slip boundary condition cannot be maintained as no solution would exist. However, dropping the no-slip condition altogether is too much and would produce a non-unique solution. It can be shown that for small angles of incidence the inviscid limit yields a condition between slip and no-slip: the no-slip condition can be dropped almost everywhere, except near the trailing edge where it degenerates to the Kutta condition of non-singular velocity.

Questions

1. Consider a 2D object with the surface described by a Jordan curve $\mathcal{C}$, in an incompressible potential flow with complex potential $\Omega(z)$. Then the aerodynamic force $\boldsymbol{F}$ applied to $\mathcal{C}$ (with outward normal vector $\boldsymbol{n}$) is the integrated normal component of the pressure

$$\boldsymbol{F} = -\int_{\mathcal{C}} p\boldsymbol{n}\,ds$$

Show that if $\boldsymbol{F} = (\boldsymbol{F}_x, \boldsymbol{F}_y)$ is described by the complex valued $F = \boldsymbol{F}_x - \mathrm{i}\boldsymbol{F}_y$, it is given by the following positively oriented complex contour integral

$$F = \tfrac{1}{2}\mathrm{i}\rho\int_{\mathcal{C}} \Omega'(z)^2\,dz.$$

This result is known as the **Blasius Theorem**.

2. Apply this to find the lift and the lift-induced drag of the above Joukowski profile. This result (the lift is independent of the geometry) is known as the **Kutta-Joukowski Lift Theorem**.

Answers

1. If $\mathcal{C}$ is parameterized by $\gamma(t)$, with $t \in [0, 2\pi)$, then γ' is tangent to $\mathcal{C}$ and so

$$\gamma'(t) = |\gamma'(t)|(\cos\tau + \mathrm{i}\sin\tau) = |\gamma'(t)|\,\mathrm{e}^{\mathrm{i}\tau}$$

where τ is the angle of the tangent vector. Since the velocity at the surface is tangent to the surface, we have

$$|\Omega'(z)| = \Omega'(z)\,\mathrm{e}^{\mathrm{i}\tau}$$

Furthermore, the angle of the normal vector is $\nu = \tau - \frac{1}{2}\pi$ and so

$$e^{i\nu} = -i\, e^{i\tau}\,.$$

Since $\oint p_0 \boldsymbol{n}\, ds = 0$, we have

$$\boldsymbol{F} = -\int_{\mathcal{C}} (p_0 - \tfrac{1}{2}\rho|\boldsymbol{v}|^2)\boldsymbol{n}\, ds = \int_{\mathcal{C}} \tfrac{1}{2}\rho|\boldsymbol{v}|^2\boldsymbol{n}\, ds = \tfrac{1}{2}\rho \int_{\mathcal{C}} |\boldsymbol{v}|^2\boldsymbol{n}\, ds$$

and so

$$\boldsymbol{F}_x = \tfrac{1}{2}\rho \int_{\mathcal{C}} |\boldsymbol{v}|^2 \cos\nu\, ds = \tfrac{1}{2}\rho \int_0^{2\pi} |\boldsymbol{v}|^2 \cos\nu |\gamma'(t)|\, dt,$$
$$\boldsymbol{F}_y = \tfrac{1}{2}\rho \int_{\mathcal{C}} |\boldsymbol{v}|^2 \sin\nu\, ds = \tfrac{1}{2}\rho \int_0^{2\pi} |\boldsymbol{v}|^2 \sin\nu |\gamma'(t)|\, dt.$$

Hence

$$\begin{aligned}
F = \boldsymbol{F}_x - i\boldsymbol{F}_y = \tfrac{1}{2}\rho \int_0^{2\pi} |\Omega'(z)|^2 |\gamma'(t)|\, e^{-i\nu}\, dt = \\
\tfrac{1}{2}\rho \int_0^{2\pi} |\Omega'(z)|^2 |\gamma'(t)|\, e^{-i\tau}\, i\, dt = \tfrac{1}{2}i\rho \int_0^{2\pi} \Omega'(z)^2\, e^{2i\tau}\, |\gamma'(t)|\, e^{-i\tau}\, dt = \\
\tfrac{1}{2}i\rho \int_0^{2\pi} \Omega'(z)^2 |\gamma'(t)|\, e^{i\tau}\, dt = \tfrac{1}{2}i\rho \int_0^{2\pi} \Omega'(z)^2 \gamma'(t)\, dt = \tfrac{1}{2}i\rho \int_{\mathcal{C}} \Omega'(z)^2\, dz
\end{aligned}$$

2. Transformation of the integral to the ζ-plane is possible but leads to complicated formulae. It is easier to deform the integration contour $\mathcal{C}$ to a large circle $\widetilde{\mathcal{C}}$ where

$$\zeta \simeq z\, e^{i\alpha} + \dots$$

such that

$$\Omega(z) \simeq Uz - i\frac{\Gamma}{2\pi}\log(z) + \dots$$

and apply residue integration in z. Since $\Omega'(z) \simeq U - i\frac{\Gamma}{2\pi z} + \dots$ for large z, it follows that

$$F = \tfrac{1}{2}i\rho \int_{\widetilde{\mathcal{C}}} \Big(U - i\frac{\Gamma}{2\pi z} + \dots\Big)^2 dz = \tfrac{1}{2}i\rho \int_{\widetilde{\mathcal{C}}} \Big(U^2 - 2i\frac{U\Gamma}{2\pi z} + \dots\Big)\, dz = i\rho U\Gamma$$

so $\boldsymbol{F} = \rho U \Gamma \boldsymbol{e}_y$, which depends only on circulation Γ, and not on details of the airfoil, other than – via the Kutta condition – the location of the trailing edge. Note: the expected angle of incidence dependence is hidden in the α dependence of the main flow.

Chapter 3

Residue calculus

An integral of an analytic function along a Jordan curve can be determined by the residues in the singular points inside the curve. In this chapter we will use this result to determine integrals in a number of different situations. In the first instance the integration contour is often not a Jordan curve but, for example, an interval. First we will discuss some techniques to determine residues.

3.1 Calculation of residues

In this chapter we will have to determine the residue of a function in a point. Some examples of such a calculation were given in §2.4. In this section we will describe other methods.

Use of the Laurent series

In formula (2.7) we have seen that in a pole a the residue of a function is given by the coefficient of $(z-a)^{-1}$ in the Laurent expansion, provided valid in a reduced neighbourhood of a. This is also true in an essential singularity. If we substitute $n=-1$ in (2.14), we find

$$c_{-1} = \frac{1}{2\pi \mathrm{i}} \int_K f(\zeta)\, d\zeta = \mathrm{Res}_{z=a} f(z),$$

where K is a Jordan curve in G, encircling a in positive direction, and located within the region of convergence of the Laurent series.

REMARK 3.1.1. *If the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is not valid in a reduced neighbourhood, but in $0 < r < |z-a| < R$, then the integral along $|z-a| = \rho \in (r, R)$ is still*

$$\frac{1}{2\pi \mathrm{i}} \int_{|z-a|=\rho} f(z)\, dz = c_{-1},$$

from (2.14) for $n=-1$. This coefficient c_{-1} is in general **not** *related to any residue of f in $z=a$. If f is analytic inside $|z-a|<r$ except for isolated singularities, c_{-1} is the sum of their residues.*

Pole of first order

If $f(z)$ has a pole of first order in a, it follows from the Laurent expansion that we can write the function as follows:

$$f(z) = \frac{A}{z-a} + h(z),$$

where $h(z)$ is analytic. For the residue A we have $A = \lim_{z\to a}(z-a)f(z)$. We conclude that in a simple pole a

$$\mathrm{Res}_{z=a}\, f(z) = \lim_{z\to a}(z-a)f(z). \tag{3.1}$$

Conversely, we can conclude from the existence of this limit that pole a is of first order.

EXAMPLE 3.1.2.

$$\mathrm{Res}_{z=1}\,\frac{z^2+1}{z^2-1} = \lim_{z\to 1}\left\{(z-1)\,\frac{z^2+1}{(z-1)(z+1)}\right\} = \lim_{z\to 1}\frac{z^2+1}{z+1} = 1.$$

Type l'Hôpital

Let the functions f and g be analytic in point a. Let a be a zero of g of order n, and (if $n \geqslant 2$) a zero of f of order $n-1$. In that case a is a pole of first order of the function $f(z)/g(z)$. By Taylor expansion of f and g (equivalent to what is also known as l'Hôpital's rule) we find

$$\lim_{z\to a}(z-a)\frac{f(z)}{g(z)} = \lim_{z\to a}(z-a)\frac{f(a)+\cdots+\frac{1}{(n-1)!}(z-a)^{n-1}f^{(n-1)}(a)+\cdots}{g(a)+\cdots+\frac{1}{n!}(z-a)^n g^{(n)}(a)+\cdots} =$$

$$\lim_{z\to a}(z-a)\frac{\frac{1}{(n-1)!}(z-a)^{n-1}f^{(n-1)}(a)+\cdots}{\frac{1}{n!}(z-a)^n g^{(n)}(a)+\cdots} = n\frac{f^{(n-1)}(a)}{g^{(n)}(a)}.$$

So we have under these conditions

$$\mathrm{Res}_{z=a}\,\frac{f(z)}{g(z)} = n\frac{f^{(n-1)}(a)}{g^{(n)}(a)}. \tag{3.2}$$

EXAMPLE 3.1.3.

For $k \in \mathbb{Z}$ is $z = (k+\frac{1}{2})\pi$ a simple zero of $\cos z$, so $n = 1$, and

$$\mathrm{Res}_{z=(k+\frac{1}{2})\pi}\,\tan z = \mathrm{Res}_{z=(k+\frac{1}{2})\pi}\,\frac{\sin z}{\cos z} = \frac{\sin(k+\frac{1}{2})\pi}{-\sin(k+\frac{1}{2})\pi} = -1.$$

For $k \in \mathbb{Z}$ is $z = 2k\pi$ a 2nd order zero of $1-\cos z$, so $n = 2$, and

$$\mathrm{Res}_{z=2k\pi}\,\frac{\sin z}{1-\cos z} = 2\frac{\cos(2k\pi)}{\cos(2k\pi)} = 2.$$

Pole of finite higher order

If $g(z)$ is analytic in a and $g(a) \neq 0$, then the function[1] $f(z) = g(z)/(z-a)^m$ has a pole of order m in a. We obtain the Laurent expansion of $f(z)$ around a from the Taylor expansion around a of the function g.

$$f(z) = \frac{1}{(z-a)^m} \sum_{n=0}^{\infty} c_n (z-a)^n = \sum_{n=0}^{\infty} c_n (z-a)^{n-m}.$$

The coefficient of $(z-a)^{-1}$ is $c_{m-1} = g^{(m-1)}(a)/(m-1)!$. So we find

$$\mathrm{Res}_{z=a} \frac{g(z)}{(z-a)^m} = \frac{g^{(m-1)}(a)}{(m-1)!}. \tag{3.3}$$

EXAMPLE 3.1.4. *The function*

$$f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{(z+\mathrm{i})^3(z-\mathrm{i})^3}$$

has a pole of order 3 *in* $z = \mathrm{i}$, *with*

$$\mathrm{Res}_{z=\mathrm{i}}\, f(z) = \frac{1}{2!}\left[\frac{d^2}{dz^2}\frac{1}{(z+\mathrm{i})^3}\right]_{z=\mathrm{i}} = \frac{-3\mathrm{i}}{16}.$$

REMARK 3.1.5. *If $f(z)$ is a function with a pole in $z = a$ of order m, then the function given by $g(z) = (z-a)^m f(z)$ has a removable singularity in $z = a$, while $g^{(m-1)}(a) = \lim_{z\to a}\{g(z)\}^{(m-1)}$. Therefore, we can write the above formula als follows:*

$$\mathrm{Res}_{z=a}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)\right].$$

Note: *unless the factor $(z-a)^m$ can be cancelled out explicitly, the formula is not as practical as its form suggests. Since $f(z)$ is singular in a, the limit is essential and usually not straightforward.*

EXAMPLE 3.1.6. *More directly, we find for the second order pole in $z = 0$ of the rational function*

$$\frac{a_0 + a_1 z + \dots}{b_0 z^2 + b_1 z^3 + \dots} = \frac{a_0 + a_1 z + \dots}{b_0 z^2 (1 + \frac{b_1}{b_0} z + \dots)} = \frac{(a_0 + a_1 z + \dots)(1 - \frac{b_1}{b_0} z + \dots)}{b_0 z^2} = \frac{a_0}{b_0 z^2} + \frac{a_1 b_0 - a_0 b_1}{b_0^2 z} + \dots$$

the residue $\dfrac{a_1 b_0 - a_0 b_1}{b_0^2}$.

[1] A function, analytic outside a pole in a of order m, can be written as $\frac{g(z)}{(z-a)^m}$ with g entire, or as $\frac{c_m}{(z-a)^m} + \dots + \frac{c_1}{z-a} + h(z)$ with h entire and constants c_k (*cf.* Theorem 2.7.5).

Essential singularity

Residues in essential singularities may be found on ad-hoc basis. We give some examples.

If $f(z) = g(1/z)$ where $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is entire (hence with a Taylor series that converges everywhere), then we have in $0 < |z| < \infty$

$$f(z) = \sum_{n=0}^{\infty} c_n z^{-n} = \sum_{n=-\infty}^{0} c_{-n} z^n, \qquad \mathrm{Res}_{z=0}(f) = c_1.$$

If $f(z) = g(1/z)h(z)$ where $g(z) = \sum_{n=0}^{\infty} c_n z^n$ is entire and $h(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in 0 with radius of convergence R, then f is analytic in $0 < |z| < R$ and has a unique Laurent series, which may be found by multiplying the series of $g(1/z)$ and $h(z)$ and collecting the terms of f as follows

$$f(z) = \left(\sum_{n=0}^{\infty} c_n z^{-n}\right)\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{\infty}\left(z^n \sum_{k=-\infty}^{\infty} c_k a_{k+n}\right)$$

where we assume $a_k, c_k = 0$ if $k < 0$. Hence

$$\mathrm{Res}_{z=0}(f) = \sum_{k=1}^{\infty} c_k a_{k-1}.$$

EXAMPLE 3.1.7. *The function $f(z) = \mathrm{e}^{1/z}/(1-z)$ has an essential singular point in $z = 0$. We find the Laurent series by multiplying the series*

$$\mathrm{e}^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots \qquad \textit{and} \qquad \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

We obtain the residue as the coefficient of z^{-1}:

$$\mathrm{Res}_{z=0} \frac{\mathrm{e}^{1/z}}{1-z} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e - 1.$$

3.2 Integrals over a finite interval

We want to find for a given function $F(u, v)$ the integral $\int_0^{2\pi} F(\cos\vartheta, \sin\vartheta)\, d\vartheta$. Often it appears advantageous to rewrite this real integral into a complex integral along the unit circle as follows.

We assume $z = \mathrm{e}^{\mathrm{i}\vartheta}$. Then z runs along the unit circle $C : |z| = 1$, and we have

$$\cos\vartheta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\vartheta = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right), \quad dz = \mathrm{i}\,\mathrm{e}^{\mathrm{i}\vartheta}\, d\vartheta, \quad \text{and so} \quad d\vartheta = \frac{dz}{\mathrm{i}z}.$$

The integral changes into

$$\int_C F\left(\tfrac{1}{2}(z + z^{-1}), \tfrac{1}{2\mathrm{i}}(z - z^{-1})\right) \frac{dz}{\mathrm{i}z}.$$

If the integrand is an analytic function, we can try to find the integral by the residues inside the unit circle. This works especially well if $F(u, v)$ is rational.

EXAMPLE 3.2.1. *Let* $0 < a < 1$. *We consider the integral*

$$I(a) = \int_0^{2\pi} \frac{d\vartheta}{1 + a\cos\vartheta}.$$

With the above substitution we find

$$I(a) = \frac{2}{\mathrm{i}a}\int_0^{2\pi} \frac{dz}{z^2 + 2z/a + 1} = \frac{2}{\mathrm{i}a}\int_0^{2\pi} \frac{dz}{(z - z_1)(z - z_2)},$$

where z_1 *and* z_2 *are the roots of the quadratic equation* $z^2 + 2z/a + 1 = 0$, *so*

$$z_1 = \frac{-1 + \sqrt{1 - a^2}}{a}, \quad z_2 = \frac{-1 - \sqrt{1 - a^2}}{a}.$$

So the integrand has two poles, in $z = z_1$ *and* $z = z_2$. *It is easily seen that* $z_2 < -1 < z_1 < 0$ *(note that* $z_1 z_2 = 1$*), so* z_1 *is inside and* z_2 *is outside* C. *It follows that*

$$I(a) = 2\pi\mathrm{i}\,\frac{2}{\mathrm{i}a}\,\mathrm{Res}_{z=z_1}\,\frac{1}{(z - z_1)(z - z_2)} = \frac{4\pi}{a}\,\frac{1}{z_1 - z_2} = \frac{2\pi}{\sqrt{1 - a^2}}.$$

The method of this section can be used to determine Fourier series. We give an example (*cf.* . example 4.1.14).

EXAMPLE 3.2.2. *Let* $-1 < a < 1$. *The function*

$$f(t) = \frac{1}{1 - 2a\cos t + a^2}$$

is 2π*-periodic. We want to find its Fourier series representation (see equation 4.1)*

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}nt}, \quad \text{with} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{e}^{-\mathrm{i}nt}\,dt \quad (n \in \mathbb{Z}).$$

Since this function is even, we have $c_n = c_{-n}$ *and it is enough to consider*

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{e}^{\mathrm{i}nt}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}nt}}{1 - 2a\cos t + a^2}dt \quad (n \in \mathbb{N}).$$

We use the above transformation $\mathrm{e}^{\mathrm{i}t} = z$, $\cos t = \frac{1}{2}(z + 1/z)$, $dt = dz/(iz)$, $\mathrm{e}^{\mathrm{i}nt} = z^n$, *and find*

$$c_n = \frac{1}{2\pi}\int_C \frac{z^n}{1 - a(z + 1/z) + a^2}\,\frac{dz}{\mathrm{i}z} = \frac{1}{2\pi\mathrm{i}}\int_C \frac{z^n}{(1 + a^2)z - (1 + z^2)a}\,dz.$$

The denominator can be factorised as

$$(1 + a^2)z - (1 + z^2)a = (1 - az)(z - a),$$

so we find

$$c_n = \frac{1}{2\pi\mathrm{i}}\int_C \frac{z^n}{(1 - az)(z - a)}\,dz.$$

Since $|a| < 1$ *and* $n \in \mathbb{N}$, *only the (simple) pole in* $z = a$ *is inside the unit circle. We find*

$$c_n = c_{-n} = \frac{1}{2\pi\mathrm{i}}\left(2\pi\mathrm{i}\,\mathrm{Res}_{z=a}\,\frac{z^n}{(1 - az)(z - a)}\right) = \frac{a^n}{1 - a^2}.$$

EXAMPLE 3.2.3. *For* $a \in \mathbb{R}$ *we define*

$$I(a) = \int_0^{2\pi} \cos(2a \sin\vartheta)\, d\vartheta.$$

As it is easier to work with the exponential than with the cosine, we write $I(a) = \operatorname{Re} T(a)$, *where*

$$T(a) = \int_0^{2\pi} \mathrm{e}^{2\mathrm{i}a\sin\vartheta}\; d\vartheta.$$

With the above substitution we obtain

$$T(a) = \int_C \exp\left\{2\mathrm{i}a\,\frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)\right\}\frac{dz}{\mathrm{i}z} = -\mathrm{i}\int_C z^{-1}\,\mathrm{e}^{az}\,\mathrm{e}^{-a/z}\; dz.$$

The integrand has an essential singularity in the origin. Otherwise, there are no singular points. We calculate the residue via the Laurent series:

$$h(z) = z^{-1}\,\mathrm{e}^{az}\,\mathrm{e}^{-a/z} = z^{-1}\left(1 + az + \frac{(az)^2}{2!} + \cdots\right)\left(1 - \frac{a}{z} + \frac{1}{2!}\,\frac{a^2}{z^2} - \cdots\right).$$

We obtain here the coefficient of z^{-1} *by taking in the product of the series the constant term:*

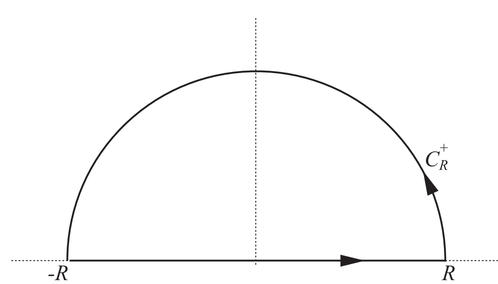
$$\operatorname{Res}_{z=0} h(z) = 1 - a^2 + \frac{a^4}{(2!)^2} - \frac{a^6}{(3!)^2} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(n!)^2} =:\; J_0(a).$$

The function J_0 *is the Bessel function of order* 0 *(see section 6.1). We conclude that*

$$I(a) = \operatorname{Re} T(a) = \operatorname{Re}\left[2\pi\mathrm{i}(-\mathrm{i})\operatorname{Res}_{z=0} h(z)\right] = 2\pi\, J_0(a).$$

3.3 Integrals over the real axis

If a function $f(x)$ is defined on $\mathbb{R}$, we can try to determine the integral $\int_{-\infty}^{\infty} f(x)\,dx$ as the limit for $R \to \infty$ of $\int_{-R}^{R} f(x)\,dx$. We assume that the function is defined and analytic in the upper half plane $\operatorname{Im} z \geqslant 0$ except for a finite number of singular points $z_1, \ldots, z_n$, not on $\mathbb{R}$. We introduce the following notation:



$$C_R^+ = \{z \,\big|\, |z| = R, \operatorname{Im} z \geqslant 0\}, \quad K_R^+ = [-R, R] + C_R^+, \quad M_R^+(f) = \max_{z\in C_R^+} |f(z)|.$$

So C_R^+ is a semi-circle in the upper half plane with centre 0 and radius R. We assume that this arc is traversed from right to left. The curve K_R^+ is the Jordan curve consisting of the interval $[-R, R]$ in $\mathbb{R}$, traversed from left to right, and the circular arc C_R^+. Now take R_0 big enough to have all singular points of $f(z)$ in the upper half plane located inside $K_{R_0}^+$. This is possible because we assumed that $f(z)$ has at most a finite number of singular points in the upper half plane. For $R > R_0$ is the value of $\int_{K_R^+} f(z)\,dz$ independent of R, namely equal to $2\pi\mathrm{i}$ times the sum of the residues of $f(z)$ in the upper half plane. If

$$\int_{C_R^+} f(z)\,dz \;\to\; 0 \quad (R\to\infty)$$

then we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty}\left\{\int_{K_R^+} f(z)\,dz - \int_{C_R^+} f(z)\,dz\right\} = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}_{z=z_k} f(z).$$

We arrive at the following result:

THEOREM 3.3.1. *If $f(z)$ is analytic in the upper half plane* $\mathrm{Im}\, z \geqslant 0$ *except for the points* $z_1, z_2, \ldots, z_n$, *which satisfy* $\mathrm{Im}\, z_k > 0$ *for* $k = 1, \ldots, n$, *and if*

$$\int_{C_R^+} f(z)\,dz \;\to\; 0 \quad (R\to\infty), \tag{3.4}$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi\mathrm{i}\sum_{k=1}^{n}\mathrm{Res}_{z=z_k} f(z). \tag{3.5}$$

The most difficult step in the application of this theorem is to prove property (3.4). In order to find a condition for this we have

LEMMA 3.3.2.

1. *If $RM_R^+(f) \;\to\; 0$ for $R \;\to\; \infty$, then (3.4) is satisfied.*
2. *If $f(z)$ is a rational function of the form $f(z) = p(z)/q(z)$, with* $\mathrm{degree}(q)-\mathrm{degree}(p) \geqslant 2$, *then we have $RM_R^+(f) \;\to\; 0$ for $R\to\infty$.*

PROOF:

1. *According to the ML-lemma we have*

$$\left|\int_{C_R^+} f(z)\,dz\right| \leqslant \pi R M_R^+(f) \;\to\; 0 \quad (R\to\infty).$$

2. *Let $p(z) = p_m z^m + \cdots + p_0$ and $q(z) = q_n z^n + \cdots + q_0$ with $m \leqslant n-2$ and $q_n \neq 0$. Then for R sufficiently large we have for $z \in C_R^+$ according to the triangle inequality*

$$\left|\frac{p(z)}{q(z)}\right| \leqslant \frac{|p_m|R^m + \cdots + |p_0|}{|q_n|R^n - |q_{n-1}|R^{n-1} - \cdots - |q_0|}.$$

From this it follows that

$$RM_R^+(f) \leqslant R^{m+1-n}\frac{|p_m| + |p_{m-1}|R^{-1}\cdots + |p_0|R^{-m}}{|q_n| - |q_{n-1}|R^{-1} - \cdots - |q_0|R^{-n}} \;\to\; 0 \quad (R\to\infty).$$

EXAMPLE 3.3.3. *We determine the integral*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

The function $f(z) = 1/(1+z^2)$ is analytic in the upper half plane except for the point $z = \mathrm{i}$, where the function has a singularity. The integrand is a rational function with the degree of its denominator being two more than the degree of its numerator. To find the residue we apply rule (3.2):

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}\,\frac{1}{1+z^2} = 2\pi\mathrm{i}\left[\frac{1}{2z}\right]_{z=\mathrm{i}} = \pi,$$

a well-known result.

EXAMPLE 3.3.4. *In a similar way we find*

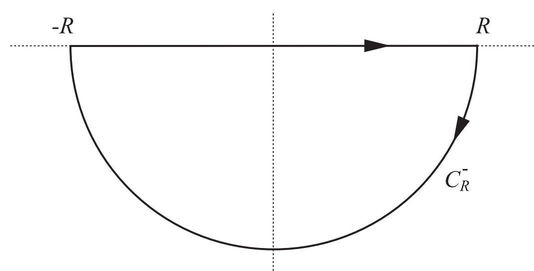
$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3}\,dx = 2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}\,\frac{1}{(1+z^2)^3} = 2\pi\mathrm{i}\,\frac{-3\mathrm{i}}{16} = \frac{3\pi}{8}.$$

Here we used the residue already found in (3.1.4).

Instead of a semi-circle in the upper half plane we could also consider a semi-circle in the lower half plane. For the most, the details are the same. We introduce

$$C_R^- = \{z \,\big|\, |z| = R, \operatorname{Im} z \leqslant 0\}, \quad K_R^- = [-R, R] + C_R^-, \quad M_R^-(f) = \max_{z\in C_R^-} |f(z)|.$$

We have to realise that Jordan curve K_R^- is now oriented clockwise, and so is followed in negative direction. In formula (3.5) we therefore have an extra minus sign. Of course, we have to estimate the function in the lower half plane. In the above examples we would have obtained the same results, and there seems to be no good reason to use this variant. This is not always true, however, and we will see that sometimes we need to work in the lower half plane.



3.4 Integrals of the type “Fourier transform”

We define[2] (see section 4.2, equation 4.2) the **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{C}$ by

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\mathrm{i}\omega x}\, dx. \tag{3.6}$$

Under suitable conditions, $f(x)$ can be reconstructed from $F(\omega)$ by the **inverse Fourier transform**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{\mathrm{i}\omega x}\, d\omega. \tag{3.7}$$

This is in particular true if f is **absolutely integrable** , *i.e.* if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. Related to the Fourier transform are the Fourier cosine transform and the Fourier sine transform:

$$F_{\cos}(\omega) = \int_{-\infty}^{\infty} f(x) \cos(\omega x)\, dx, \quad F_{\sin}(\omega) = \int_{-\infty}^{\infty} f(x) \sin(\omega x)\, dx.$$

If the contour can be closed, we can apply residue calculus to determine integrals of this type. Assume $f(x)$ is analytic in $\mathbb{C}$ except for a finite number of points, not in $\mathbb{R}$. We can apply Theorem 3.3.1, provided we carefully check condition (3.4). Assume[3] that f satisfies $RM_R^{\pm}(f) \to 0$ for $R \to \infty$, then we may apply Lemma 3.3.2 if the contour is closed such that factor $\mathrm{e}^{-\mathrm{i}\omega z}$ remains bounded. For $z = x + \mathrm{i}y$ we have $|\, \mathrm{e}^{\mathrm{i}\omega z}\, | = \mathrm{e}^{-\omega y} \leqslant 1$ if $\omega y \geqslant 0$, and $|\, \mathrm{e}^{-\mathrm{i}\omega z}\, | \leqslant 1$ if $\omega y \leqslant 0$.

EXAMPLE 3.4.1. *We want to find*

$$J(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\omega x}}{1 + x^2}\, dx$$

for all $\omega \in \mathbb{R}$. We define $g(z) = 1/(1+z^2)$ and $f(z) = \mathrm{e}^{\mathrm{i}\omega z}/(1+z^2) = \mathrm{e}^{\mathrm{i}\omega z}\, g(z)$. These functions are analytic everywhere except in the poles $z = \pm\,\mathrm{i}$.

1. *If $\omega \geqslant 0$, we close the integration contour along $[-R, R]$ with C_R^+. Then $|\, \mathrm{e}^{\mathrm{i}\omega z}\, | \leqslant 1$ for $\operatorname{Im} z \geqslant 0$, such that $RM_R^+(f) \leqslant RM_R^+(g) \to 0$ for $R \to \infty$, because of Lemma 3.3.2. So we find:*

$$J(\omega) = 2\pi\mathrm{i}\, \mathrm{Res}_{z=\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}\omega z}}{1 + z^2} = 2\pi\mathrm{i} \left[\frac{\mathrm{e}^{\mathrm{i}\omega z}}{2z} \right]_{z=\mathrm{i}} = 2\pi\mathrm{i}\, \frac{\mathrm{e}^{-\omega}}{2\mathrm{i}} = \pi\, \mathrm{e}^{-\omega},$$

   where we used rule (3.2) to determine the residue.

2. *If $\omega \leqslant 0$, we close the integration contour along $[-R, R]$ with C_R^- in the lower half plane. Then $|\, \mathrm{e}^{\mathrm{i}\omega z}\, | \leqslant 1$ for $\operatorname{Im} z \leqslant 0$, so we have $RM_R^-(f) \leqslant RM_R^-(g) \to 0$ for $R \to \infty$, because of Lemma 3.3.2. We find:*

$$J(\omega) = -2\pi\mathrm{i}\, \mathrm{Res}_{z=-\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}\omega z}}{1 + z^2} = -2\pi\mathrm{i} \left[\frac{\mathrm{e}^{\mathrm{i}\omega z}}{2z} \right]_{z=-\mathrm{i}} = -2\pi\mathrm{i}\, \frac{\mathrm{e}^{\omega}}{-2\mathrm{i}} = \pi\, \mathrm{e}^{\omega},$$

[2] In physics, ω usually denotes frequency and x distance, so ωx will rarely occur. It is common to use ωt or kx.
[3] In Lemma 3.4.3 we will see that it is sufficient that $M_R^{\pm}(f) \to 0$, if $\omega \neq 0$.

We can summarise the results as follows:

$$J(\omega) = \pi\, \mathrm{e}^{-|\omega|}\,.$$

If we take the real and imaginary parts of this result, we find

$$\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{1+x^2}\,dx = \pi\,\mathrm{e}^{-|\omega|}, \qquad \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{1+x^2}\,dx = 0.$$

Of course, we could have known this last result more directly (how?).

REMARK 3.4.2. *If we had started with $\int_{-\infty}^{\infty} \frac{\cos(\omega x)}{1+x^2}\,dx$, rather than the complex integral, we could have been tempted to determine in a similar way the residue of function $h(z) = \cos(\omega z)/(1+z^2)$ in $z = \mathrm{i}$. This, however, produces the answer $2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}}\,h(z) = \frac{1}{2}\pi(\mathrm{e}^{\omega}+\mathrm{e}^{-\omega})$, which is apparently wrong! The reason for this is that the function $\cos(\omega z)$ is not bounded in the upper half plane (it is bounded, of course, on the real axis, but that is not sufficient). See the notes after Property 1.6.9.*

Therefore, if we are to determine an integral of the type $J(\omega) = \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$, we have to replace the cosine by an exponential. We can do this in two ways:

If $f(x)$ is real and $\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,d(x)$ exists, we can write the integral as

$$J(\omega) = \mathrm{Re}\left[\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}\omega x}\,dx\right].$$

If $f(x)$ is not real this is not correct. In that case we can write:

$$J(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}\omega x}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\omega x}\,dx.$$

It is possible, however, that this reformulation yields non-converging integrals, making this method to fail. Later we will present an example, and show how this difficulty can be circumvented.

In order to estimate $I_R = \int_{C_R^+} f(z)\,dz$ we used the inequality $|I_R| \leqslant \pi R M_R^+(f)$, based on the ML-lemma. With Fourier integrals we used, for example, the inequality $|\,\mathrm{e}^{\mathrm{i}\omega z}\,| = \mathrm{e}^{-\omega y} \leqslant 1$. If $\omega > 0$, then quantity $\mathrm{e}^{-\omega y}$ is on the greater part of C_R^+ very small, and the used estimate is very crude. Therefore, we may not be able to conclude in this way that integral I_R tends to zero, while this is yet the case. A better result is provided by the Lemma of Jordan.

LEMMA 3.4.3. *(**Jordan's Lemma**) Let $\omega > 0$.*

1. *If* $\lim_{R\to\infty} M_R^+(f) = 0$, *then* $\lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{e}^{\mathrm{i}\omega z}\,dz = 0.$
2. *If* $\lim_{R\to\infty} M_R^-(f) = 0$, *then* $\lim_{R\to\infty} \int_{C_R^-} f(z)\,\mathrm{e}^{-\mathrm{i}\omega z}\,dz = 0.$

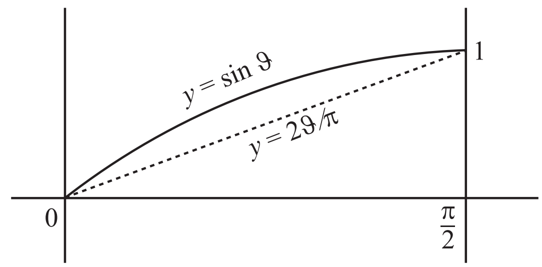
PROOF: *We start with case 1. Parametrise the integral by $z = R\,\mathrm{e}^{\mathrm{i}\vartheta}$ with $0 \leqslant \vartheta \leqslant \pi$. Then*

$$I(R)=\int_{C_R^+} f(z)\,\mathrm{e}^{\mathrm{i}\omega z}\,dz=\int_0^\pi f(R\,\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{e}^{\mathrm{i}\omega R\,\mathrm{e}^{\mathrm{i}\vartheta}}\,\mathrm{i}R\,\mathrm{e}^{\mathrm{i}\vartheta}\,d\vartheta$$
$$=iR\int_0^\pi f(R\,\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{e}^{\mathrm{i}\omega R\cos\vartheta-\omega R\sin\vartheta}\,\mathrm{e}^{\mathrm{i}\vartheta}\,d\vartheta.$$

From this it follows

$$|I(R)|\leqslant R\int_0^\pi \left|f(R\,\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{e}^{\mathrm{i}\omega R\cos\vartheta-\omega R\sin\vartheta}\,\mathrm{e}^{\mathrm{i}\vartheta}\right|d\vartheta$$
$$\leqslant RM_R^+(f)\int_0^\pi \mathrm{e}^{-\omega R\sin\vartheta}\,d\vartheta=2RM_R^+(f)\int_0^{\pi/2}\mathrm{e}^{-\omega R\sin\vartheta}\,d\vartheta.$$

Along the interval $[0,\frac{1}{2}\pi]$ *we have* $\sin\vartheta\geqslant 2\vartheta/\pi$, *as is easily seen from the graph of* $\sin\vartheta$. *We find*



$$\int_0^{\pi/2}\mathrm{e}^{-\omega R\sin\vartheta}\,d\vartheta\leqslant\int_0^{\pi/2}\mathrm{e}^{-2\omega R\vartheta/\pi}\,d\vartheta=$$
$$\frac{\pi}{2\omega R}\left(1-\mathrm{e}^{-\omega R}\right)\leqslant\frac{\pi}{2\omega R}.$$

Then it follows that

$$|I(R)|\leqslant 2RM_R^+(f)\,\frac{\pi}{2\omega R}=\frac{\pi}{\omega}M_R^+(f)\;\to\;0\quad(R\to\infty).$$

The proof of the second case is the same.

In a similar way as in Lemma 3.3.2 we can understand that condition $M_R^+(f)\to 0$ is satisfied if $f(z)$ is a rational function of the form $f(z)=p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials with $\mathrm{degree}(q)-\mathrm{degree}(p)\geqslant 1$. We see here too that the condition is now weaker. The fact that the integral itself exists (*i.e.* the part along the real axis), is not obvious anymore, and indeed not true if $\mathrm{degree}(q)-\mathrm{degree}(p)=1$ and $\omega=0$.

EXAMPLE 3.4.4. *Determine for* $a>0$

$$I(a)=\int_0^\infty\frac{x\sin x}{x^2+a^2}\,dx.$$

We note that

$$I(a)\stackrel{1}{=}\frac{1}{2}\int_{-\infty}^\infty\frac{x\sin x}{x^2+a^2}\,dx\stackrel{2}{=}\frac{1}{2\mathrm{i}}\int_{-\infty}^\infty\frac{x\,\mathrm{e}^{\mathrm{i}x}}{x^2+a^2}\,dx.$$

($\stackrel{1}{=}$ *is true because the integrand is even,* $\stackrel{2}{=}$ *is true because* $x\cos x/(x^2+a^2)$ *is odd.) The function* $f(z)=z/(z^2+a^2)$ *is a rational function of which the degree of the numerator is smaller than the degree of denominator. So we can apply Jordan's Lemma 3.4.3 (Lemma 3.3.2 doesn't work here). The function* $f(z)$ *has in the upper half plane only a pole in* $z=\mathrm{i}a$. *So we find*

$$I(a)=\frac{1}{2\mathrm{i}}\left(2\pi\mathrm{i}\,\mathrm{Res}_{z=\mathrm{i}a}\frac{z\,\mathrm{e}^{\mathrm{i}z}}{z^2+a^2}\right)=\left(\pi\left[\frac{z\,\mathrm{e}^{\mathrm{i}z}}{2z}\right]_{z=\mathrm{i}a}\right)=\left(\pi\frac{ia\,\mathrm{e}^{-a}}{2\mathrm{i}a}\right)=\frac{1}{2}\pi\,\mathrm{e}^{-a}.$$

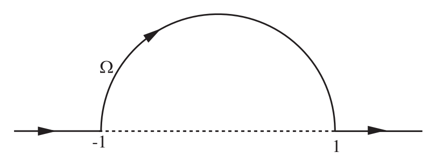
Sometimes it is convenient to deform or indent the integration contour.

EXAMPLE 3.4.5. *Determine*

$$J = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

The obvious choice would be to write $\sin x$ *again as* $\operatorname{Im} \mathrm{e}^{\mathrm{i}x}$. *This, however, is not possible because this yields a pole in* $z = 0$. *Although the function* $(\sin z)/z$ *has a removable singularity in* $z = 0$, *this is not true of the function* $\mathrm{e}^{\mathrm{i}z}/z$, *making the integral* $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x}/x\, dx$ *undefined.*

We will remove this difficulty by deforming the integration contour first, in such a way that it avoids the origin. We define the path $\Omega = (-\infty, 1] - C_1^+ + [1, \infty)$. *Note that* C_1^+ *is now traversed from left to right, so we write a minus sign. We can think of integration contour* Ω *as the real axis with interval* $[-1, 1]$ *being replaced by* $-C_1^+$.



Since $\sin z/z$ *is an entire function, this deformation has no effect on the integral. After this deformation we cannot use the identity* $\sin z = \operatorname{Im} \mathrm{e}^{\mathrm{i}z}$, *because the integrand is not real any more. We can write, however,* $\sin z = (\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z})/(2\mathrm{i})$. *So we find*

$$J = \int_\Omega \frac{\sin z}{z}\, dz = \frac{J_+ - J_-}{2\mathrm{i}}, \quad \text{where} \quad J_+ = \int_\Omega \frac{\mathrm{e}^{\mathrm{i}z}}{z}\, dz, \quad J_- = \int_\Omega \frac{\mathrm{e}^{-\mathrm{i}z}}{z}\, dz.$$

(Note the order: to split first *and deform* later *is a logical error.) Now we can apply Jordan's Lemma 3.4.3. For the calculation of* J_+ *we close the contour in the upper half plane by an arc of the form* C_R^+ *with* $R > 1$. *The integrand has no singularities above* Ω, *so the contour integral is zero. We conclude that* $J_+ = 0$. *For the calculation of* J_- *we close via the lower half plane with with* C_R^-. *Then there is a pole inside the integration contour, namely in* $z = 0$. *We thus find*

$$J_- = -2\pi\mathrm{i}\operatorname{Res}_{z=0} \frac{\mathrm{e}^{-\mathrm{i}z}}{z} = -2\pi\mathrm{i} \cdot 1 = -2\pi\mathrm{i}, \quad \text{so} \quad J = \pi.$$

3.5 Integrals of the type "Laplace transform"

Let $f(t)$ be a piecewise-continuous function defined for $t \geqslant 0$. Let $\alpha \in \mathbb{R}$ and assume that $f(t)$ is α-**exponentially bounded**, *i.e.* there is a number M such that $|f(t)| \leqslant M\,\mathrm{e}^{\alpha t}$ for all $t \geqslant 0$. Then the integral

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt$$

converges for $\operatorname{Re} s > \alpha$. We call $F(s)$ the **Laplace transform** of $f(t)$ (see section 4.3, equations 4.3). One can prove that the function $F(s)$ is analytic for $\operatorname{Re} s > \alpha$. We can reconstruct the function $f(t)$ from the corresponding $F(s)$ by means of the **inverse Laplace transform** given by

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_L F(s)\,\mathrm{e}^{st}\, ds,$$

where $L = \{s = \sigma + \mathrm{i}\omega \,\big|\, \sigma > \alpha, \omega \in \mathbb{R}\}$ is a vertical line in the s-plane, traversed in upward direction, and located on the right of the line $\operatorname{Re} s = \alpha$. For a given $F(s)$ (and we don't know f

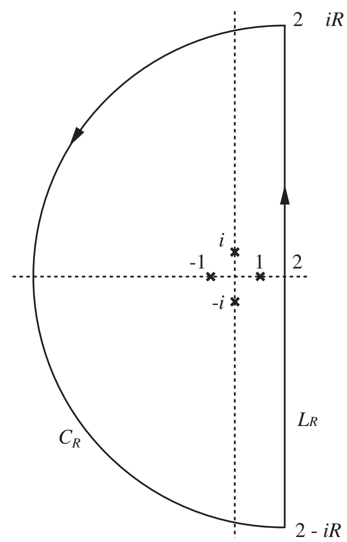
nor α yet) we have to move the integration contour L far enough to the right, such that $F(s)$ is analytic in and on the right-hand side of L. If $t \leqslant 0$ and $F \to 0$ for $|s| \to \infty$ in and on the right of L, we can close the contour along the right half plane, yielding $f(t) = 0$. This reflects the idea that the semi-infinite Laplace integral of above was really a doubly infinite Fourier integral of a function $f(t)$ that vanishes for $t \leqslant 0$. Furthermore, if $t > 0$, we can sometimes determine the inverse transform by means of residue calculus.

EXAMPLE 3.5.1. *Determine the inverse transform $f(t)$ from*

$$F(s) = 1/(s^4 - 1).$$

The function $F(s) = 1/(s^4 - 1)$ has poles in the points $s = \pm 1, \pm \mathrm{i}$. We choose the location of the integration path L such that at and on the right of L there are no singularities. We achieve this by choosing $L = \{s \in \mathbb{C} \,\big|\, \operatorname{Re} s = 2\}$. We continue in a similar way as with the previous examples. We form a Jordan curve K_R by $K_R = L_R + C_R$, where $L_R = \{s \in L \,\big|\, |\operatorname{Im} s| \leqslant R\}$, a straight line from $2 - \mathrm{i}R$ to $2 + \mathrm{i}R$, and $C_R = \{s \in \mathbb{C} \,\big|\, |s - 2| = R, \operatorname{Re} s \leqslant 2\}$, a semicircle from $2 + \mathrm{i}R$ to $2 - \mathrm{i}R$. For sufficiently large R we have

$$\int_{K_R} \mathrm{e}^{st} F(s)\, ds = J = 2\pi \mathrm{i} \sum_{s=\pm 1, s=\pm \mathrm{i}} \operatorname{Res}_s(\mathrm{e}^{st} F(s)) \quad (3.8)$$



A Bromwich contour

If we can prove that $\int_{C_R} \mathrm{e}^{st} F(s)\, ds \to 0$ for $R \to \infty$, then we find

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_L \mathrm{e}^{st} F(s)\, ds = \frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{L_R} \mathrm{e}^{st} F(s)\, ds = \frac{J}{2\pi \mathrm{i}}.$$

Along C_r we have $|\mathrm{e}^{st}| = \mathrm{e}^{(\operatorname{Re} s)t} \leqslant \mathrm{e}^{2t}$ and $|s| \geqslant R - 2$ (the triangle inequality) and so $|F(s)| \leqslant 1/((R-2)^4 - 1)$.

From this it follows that $|\int_{C_R} \mathrm{e}^{st} F(s)\, ds| \leqslant \pi R\, \mathrm{e}^{2t}\ 1/((R-2)^4 - 1) \to 0$ for $R \to \infty$. We find

$$f(t) = \sum_{\substack{s=\pm 1\\ s=\pm \mathrm{i}}} \operatorname{Res}\big(\mathrm{e}^{st} F(s)\big) = \sum \left[\frac{\mathrm{e}^{st}}{4s^3}\right]_{\substack{s=\pm 1\\ s=\pm \mathrm{i}}} = \frac{\mathrm{e}^t}{4} - \frac{\mathrm{e}^{-t}}{4} + \frac{\mathrm{e}^{\mathrm{i}t}}{4\mathrm{i}^3} + \frac{\mathrm{e}^{-\mathrm{i}t}}{4(-\mathrm{i})^3} = \tfrac{1}{2}\sinh t - \tfrac{1}{2}\sin t.$$

EXAMPLE 3.5.2. *Determine the Laplace transform*[4] *of* $\dfrac{H(t)}{\sqrt{t}}$.
With transformation $t = y^2$ we obtain

$$\int_0^\infty \frac{\mathrm{e}^{-st}}{\sqrt{t}}\, dt = 2\int_0^\infty \mathrm{e}^{-sy^2}\, dy.$$

Let $\sqrt{s}$ denote the principal value square root of s. If $\operatorname{Re}(s) > 0$, then $\left|\arg(\sqrt{s})\right| < \frac{1}{4}\pi$, and hence $\operatorname{Re}(\sqrt{s}\,y)^2 > 0$. So we can rotate the integration contour in the complex y-plane over an angle $-\arg(\sqrt{s})$ without contribution of the arc at infinity. We obtain with $\tau = \sqrt{s}\,y$

$$2\int_0^\infty \mathrm{e}^{-sy^2}\, dy = \frac{2}{\sqrt{s}} \int_0^\infty \mathrm{e}^{-\tau^2}\, d\tau = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

[4] $H(t)$ denotes the Heaviside function, which is strictly speaking superfluous here. See also exercise 4.2.10.

3.6 Integrals with logarithms and non-integer powers

We can use residue calculus to determine integrals with logarithms or non-integer powers.

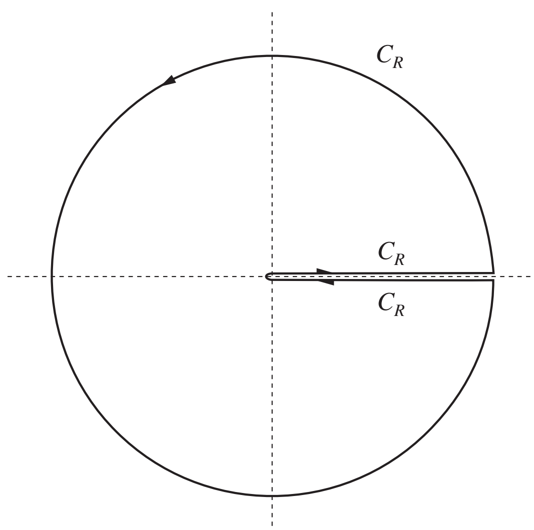
Integrals of the type "Mellin transform"

The **Mellin transform** of a function $f(x)$, defined for $x \geqslant 0$, is given by

$$F(\mu) = \int_0^\infty f(x) x^{\mu-1}\, dx.$$

Here we assume that $\mu > 0$. If we assume that $f(x)$ is continuous and there exist positive numbers M and α such that $|f(x)| \leqslant M x^{-\alpha}$ for all $x > 0$, then it is easily seen that the integral is absolutely convergent for $\mu < \alpha$. (Note that we have to examine convergence near $x = 0$ and near $x = \infty$.)

We will use residue theory to determine the Mellin transform of a rational function $f(x)$ and μ not an integer. We assume that condition $\mu < \alpha$ is satisfied. Note that we can take α equal to (the degree of the denominator) $-$ (the degree of the numerator). We denote the poles of $f(z)$ by $z_1, \ldots, z_m$. We will use the auxiliary function $g(z) = (-z)^{\mu-1} f(z)$. This function is analytic in $\mathbb{C}$ except for the poles of $f(z)$ and along the positive real axis, which is the branch cut of the function $(-z)^{\mu-1}$. It is to get the cut there, that we use this choice of function $g(z)$ instead of the more obvious $z^{\mu-1} f(z)$.



We determine the integral $\int_{K_R} g(z)\, dz$ with K_R the contour[5] $K_R = C_R^+ + C_R - C_R^-$. Here is $C_R^\pm = \{x \pm \mathrm{i}0 \mid 0 \leqslant x \leqslant R\}$, and C_R is the circle with centre 0 and radius R.

First, we note that $\int_{C_R} \to 0$ for $R \to \infty$, because along C_R we have $|g(z)| \leqslant M R^{\mu-1-\alpha}$.

[5]This contour K_R is to be considered as the limit of $K_R(\varepsilon) = C_R^+(\varepsilon) + C_R(\varepsilon) - C_R^-(\varepsilon) - C_0(\varepsilon)$ for $\varepsilon \downarrow 0$, where $C_R^\pm(\varepsilon) = \{x \pm \mathrm{i}\varepsilon | 0 \leqslant x \leqslant R\}$, $C_R(\varepsilon) = \{R\,\mathrm{e}^{\mathrm{i}\varphi} | \varepsilon_R \leqslant \varphi \leqslant 2\pi\mathrm{i} - \varepsilon_R\}$, $C_0(\varepsilon) = \{\varepsilon\,\mathrm{e}^{\mathrm{i}\varphi} | \frac{\pi}{2} \leqslant \varphi \leqslant \frac{3\pi}{2}\}$, $\varepsilon_R = \arcsin(\varepsilon/R)$.

Furthermore, $\int_{K_R} g(z)\,dz$ is equal to $2\pi\mathrm{i}$ times the sum of the residues in the points $z_1, \dots, z_m$ within K_R. We consider the integrals over $C_R^{\pm}$:

For $x > 0$ we have

$$g(x+\mathrm{i}0) = (-x-\mathrm{i}0)^{\mu-1} f(x) = x^{\mu-1}\ \mathrm{e}^{-\pi\mathrm{i}(\mu-1)}\ f(x)$$
$$g(x-\mathrm{i}0) = (-x+\mathrm{i}0)^{\mu-1} f(x) = x^{\mu-1}\ \mathrm{e}^{\pi\mathrm{i}(\mu-1)}\ f(x).$$

Therefore

$$\int_{C_R^+} g(z)\,dz = \mathrm{e}^{-\pi\mathrm{i}(\mu-1)} \int_0^R x^{\mu-1} f(x)\,dx, \qquad \int_{C_R^-} g(z)\,dz = \mathrm{e}^{\pi\mathrm{i}(\mu-1)} \int_0^R x^{\mu-1} f(x)\,dx.$$

We arrive at the following result:

$$\int_{K_R} g(z)\,dz = \int_{C_R^+} g(z)\,dz - \int_{C_R^-} g(z)\,dz + \int_{C_R} g(z)\,dz \to [\mathrm{e}^{-\pi\mathrm{i}(\mu-1)} - \mathrm{e}^{\pi\mathrm{i}(\mu-1)}] \int_0^\infty x^{\mu-1} f(x)\,dx$$

for $R \to \infty$. Otherwise, $\int_{K_R} g(z)\,dz$ is for large values of R a constant equal to $2\pi\mathrm{i}\sum_{k=1}^m \mathrm{Res}_{z=z_k}\, g(z)$.

Hence, we have the result

$$\int_0^\infty x^{\mu-1} f(x) dx = \frac{\pi}{\sin(\pi\mu)} \sum_{k=1}^m \mathrm{Res}_{z=z_k}[(-z)^{\mu-1} f(z)],$$

provided μ is not an integer. The minus sign at the start of the right hand side disappears because we replaced $\sin(\pi(\mu-1))$ by $-\sin(\pi\mu)$.

EXAMPLE 3.6.1. *If for example $f(x) = 1/(x^2+1)$, then we have poles in $z_1 = \mathrm{i}$, $z_2 = -\mathrm{i}$. We must have $\mu < 2$. We find*

$$\int_0^\infty \frac{x^{\mu-1}}{1+x^2}\,dx = \frac{\pi}{\sin(\pi\mu)}\left(\mathrm{Res}_{z=\mathrm{i}} \frac{(-z)^{\mu-1}}{z^2+1} + \mathrm{Res}_{z=-\mathrm{i}} \frac{(-z)^{\mu-1}}{z^2+1}\right) = \frac{\pi}{\sin(\pi\mu)}\left(\frac{(-\mathrm{i})^{\mu-1}}{2\mathrm{i}} + \frac{\mathrm{i}^{\mu-1}}{-2\mathrm{i}}\right)$$
$$= \frac{\pi}{\sin(\pi\mu)}\left(\frac{\mathrm{e}^{-\pi\mathrm{i}(\mu-1)/2}}{2\mathrm{i}} + \frac{\mathrm{e}^{\pi\mathrm{i}(\mu-1)/2}}{-2\mathrm{i}}\right) = \frac{\pi\cos(\frac{1}{2}\pi\mu)}{\sin(\pi\mu)} = \frac{\frac{1}{2}\pi}{\sin(\frac{1}{2}\pi\mu)}.$$

NOTE. *Although this derivation is, in the first instance, not valid for $\mu = 1$, the final result remains true for $\mu = 1$.*

Integrals of the type $\int_0^\infty f(x)\,dx$

If we try to use a similar method to determine $\int_0^\infty f(x)\ln x\,dx$ by means of a contour integral of the function $g(z) = f(z)\log(-z)$, then this appears to fail. We do have an interesting result, however. Let us make the same assumptions about $f(x)$ as in 3.6 and use the same contour K_R.

Then we have for $x > 0$

$$g(x + \mathrm{i}0) = f(x)\log(-x - \mathrm{i}0) = f(x)(\ln x - \pi\mathrm{i})$$
$$g(x - \mathrm{i}0) = f(x)\log(-x + \mathrm{i}0) = f(x)(\ln x + \pi\mathrm{i}).$$

It follows that

$$\int_{C_R^+} g(z)\,dz = \int_0^R f(x)\ln x\,dx - \pi\mathrm{i}\int_0^R f(x)\,dx$$
$$\int_{C_R^-} g(z)\,dz = \int_0^R f(x)\ln x\,dx + \pi\mathrm{i}\int_0^R f(x)\,dx$$

We arrive at the following result:

$$\int_{K_R} g(z)\,dz = \int_{C_R^+} g(z)\,dz - \int_{C_R^-} g(z)\,dz + \int_{C_R} g(z)\,dz \to 2\pi\mathrm{i}\int_0^\infty f(x)\,dx$$

for $R \to \infty$. Again, $\int_{K_R} g(z)\,dz$ is constant for large values of R, equal to $2\pi\mathrm{i}\sum_{k=1}^m \mathrm{Res}_{z=z_k}\, g(z)$.

So we find

$$\int_0^\infty f(x)dx = -\sum_{k=1}^m \mathrm{Res}_{z=z_k}\, f(z)\log(-z).$$

We see that we do not find a formula for $\int_0^\infty f(x)\ln x\,dx$, but we do find one for $\int_0^\infty f(x)\,dx$. This is a new result. Until now we could only determine doubly infinite integrals, *i.e.* integrals of the form $\int_{-\infty}^\infty f(x)\,dx$, but not single sided infinite integrals (except for even integrands). We give an example of the obtained result.

EXAMPLE 3.6.2. *Assume that $f(x) = 1/(x^3+1)$, and we want to determine the integral*

$$\int_0^\infty \frac{dx}{x^3+1}.$$

We introduce the function $g(z) = \log(-z)/(z^3+1)$. The poles of this function are represented by the equation $z^3 + 1 = 0$, so by $(-z)^3 = 1$. We find $-z_1 = 1$, $-z_2 = \mathrm{e}^{2\pi\mathrm{i}/3}$, $-z_3 = \mathrm{e}^{-2\pi\mathrm{i}/3}$.

It follows that

$$\int_0^\infty \frac{dx}{1+x^3} = -\sum_{k=1}^3 \mathrm{Res}_{z=z_k} \frac{\log(-z)}{z^3+1} = -\sum_{k=1}^3 \frac{\log(-z_k)}{3z_k^2}.$$

The (principal value) logarithms can be determined easily

$$\log(-z_1) = \ln 1 = 0, \quad \log(-z_2) = \log(\mathrm{e}^{2\pi\mathrm{i}/3}) = \tfrac{2}{3}\pi\mathrm{i}, \quad \log(-z_3) = -\tfrac{2}{3}\pi\mathrm{i}.$$

hence

$$\int_0^\infty \frac{dx}{1+x^3} = -\left[\frac{\frac{2}{3}\pi\mathrm{i}}{3\,\mathrm{e}^{4\pi\mathrm{i}/3}} + \frac{-\frac{2}{3}\pi\mathrm{i}}{3\,\mathrm{e}^{-4\pi\mathrm{i}/3}}\right] = \tfrac{2}{9}\pi\mathrm{i}\,2\mathrm{i}\,\frac{\mathrm{e}^{4\pi\mathrm{i}/3} - \mathrm{e}^{-4\pi\mathrm{i}/3}}{2\mathrm{i}} = -\tfrac{4}{9}\pi\ \sin(\tfrac{4}{3}\pi) = \frac{2\pi}{3\sqrt{3}}.$$

The question remains how to determine integrals of the form $\int_0^\infty f(x)\ln x\,dx$. This is possible by integration of $g(z) = f(z)(\log(-z))^2$ along K_R. We omit the details.

Chapter 4

Fourier Analysis

4.1 Fourier series

DEFINITION 4.1.1. *A function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ defined by the (convergent) series*

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{\mathrm{i}nx} \stackrel{\text{def}}{=} \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \, \mathrm{e}^{\mathrm{i}nx}$$

is called a Fourier series, with Fourier coefficients c_n. Note that upper and lower limits $\pm N$ run symmetrically to infinity.

$\tilde{f}(x)$ is periodic with period 2π. Likewise, the function $\tilde{f}(2\pi x/L)$ is periodic with period L. An alternative representation is

$$\tilde{f}(x) = c_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad a_n = c_n + c_{-n}, \quad b_n = \mathrm{i}(c_n - c_{-n}).$$

$\tilde{f}$ is real, if and only if c_0 is real and $c_n = \overline{c}_{-n}$, in which case a_n and b_n are real. If the series converges uniformly, for example if it converges absolutely (see (1.3.15))

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty,$$

then $\tilde{f}$ is continuous (see (1.3.13)). If

$$\tilde{h}(x) = \sum_{n=-\infty}^{\infty} \mathrm{i}nc_n \, \mathrm{e}^{\mathrm{i}nx}$$

is uniformly convergent, then is $\tilde{h}$ continuous, can be integrated and $\tilde{f}'(x) = \tilde{h}(x)$.

Provided summation and integration can be interchanged, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(x)\,\mathrm{e}^{-\mathrm{i}mx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}(n-m)x}\,dx = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-m)x}\,dx = c_m,$$

We say that $\{c_n\}$ are the Fourier coefficients of $\tilde{f}$. This brings us to the important question whether for a given f, the 2π-periodic Fourier series $\tilde{f}$ constructed from the Fourier coefficients of f

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n\,\mathrm{e}^{\mathrm{i}nx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\mathrm{e}^{-\mathrm{i}nx}\,dx \tag{4.1}$$

(suitably adapted for L-periodic series) can be identified with f, and of course whether $\tilde{f}$ makes sense at all. If this is the case, we say that the Fourier series of f *converges* to $[f]_{0,2\pi}$, where $[f]_{a,b}$ denotes the function, which is defined as f on $[a,b)$ and then continued periodically.

In fact, no simple test is known that is both necessary and sufficient to relate a periodic function with its Fourier coefficients. There is, however, a vast amount of partial results, from which we present here a selection.

THEOREM 4.1.2. *If for all points $x \in (0,2\pi)$ the left and right derivatives of f, i.e. the limits $\lim_{h\downarrow 0}(f(x)-f(x-h))/h$ and $\lim_{h\downarrow 0}(f(x+h)-f(x))/h$, exist, then the Fourier series (4.1) of f at x converges to $f(x)$.*

THEOREM 4.1.3. *If f is continuous on $[0,2\pi]$ while $f(0)=f(2\pi)$, then the Fourier series (4.1) of f converges uniformly to f and is continuous.*

DEFINITION 4.1.4 (**Piecewise continuous**). *The function f is piecewise continuous on $[0,L]$ if there are a finite number of open subintervals $(0,x_1),\ldots,(x_{N-1},L)$ on which f is continuous, while the limits $f(0+), f(x_1\pm),\ldots,f(L-)$ exist.*

In other words, the discontinuities are only finite in number and no worse than simple jumps.

DEFINITION 4.1.5 (**Piecewise smooth**). *The function f is piecewise smooth on $[0,L]$ if f and its derivative f' are both piecewise continuous.*

For such a function f we have the following theorem.

THEOREM 4.1.6 (**Existence of Fourier series**). *If a function f is piecewise smooth*[1] *on the interval $[0,2\pi]$, while $f(x)=\frac{1}{2}[f(x+)+f(x-)]$, then the Fourier series of f converges for every x to the 2π-periodic continuation of f.*

For a given Fourier series $\sum c_n\,\mathrm{e}^{\mathrm{i}nx}$ we have the following theorem.

THEOREM 4.1.7 (**Continuity of Fourier series**). *If a Fourier series is absolutely convergent, i.e. $\sum |c_n| < \infty$, then it converges absolutely and uniformly to a continuous periodic function f, such that c_n are just the Fourier coefficients of f.*

[1] Alternative versions of these **Dirichlet Conditions** are common, for example: ... f is piecewise continuous with a finite number of extrema ...

COROLLARY 4.1.8. *If f and f' are piecewise smooth, the Fourier coefficients c_n of f behave asymptotically for $n \to \infty$ like $c_n = O(n^{-1})$.*

PROOF: *Integration by parts yields*

$$c_n = \frac{1}{2\pi \mathrm{i} n} \sum_{x=x_d} \Big[f(x)\, \mathrm{e}^{-\mathrm{i}nx} \Big]_{x_d-}^{x_d+} + \frac{1}{\mathrm{i}n} c'_n$$

where the summation runs over all points x_d of discontinuity of f (possibly including the end points), and c'_n is the n-th Fourier coefficient of f'. As $c'_n \to 0$, the result follows.

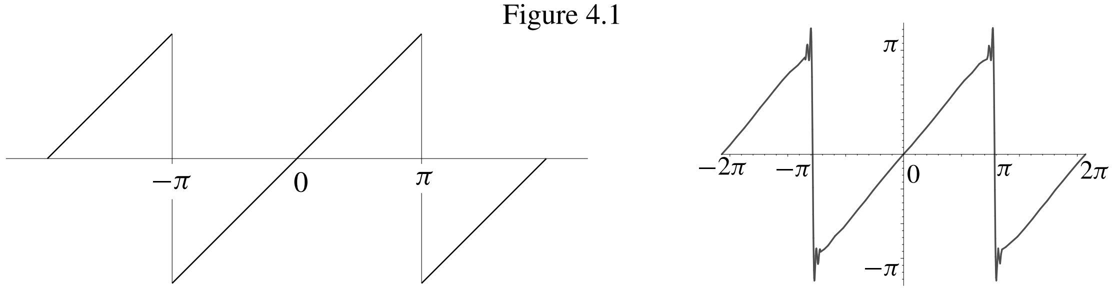
REMARK 4.1.9. *Analogously, this may be generalised to higher orders for suitably smooth f.*

EXAMPLE 4.1.10. *Consider the function*

$$f(x) = x \;\text{ for }\; x \in (-\pi, \pi), \quad \text{and } \; f(-\pi) = 0$$

extended periodically beyond $[-\pi, \pi)$ to obtain the saw-tooth function *of Fig. 4.1a.*

Figure 4.1



(a) Sawtooth function. (b) Gibbs' phenomenon (50 terms).

Since $c_0 = 0$, $c_n = (-1)^n \mathrm{i}/n$, we deduce, on account of Theorem 4.1.6, that the resulting series

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} c_n\, \mathrm{e}^{\mathrm{i}nx} = -2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n}$$

converges to $f(x)$ for any $x \in \mathbb{R}$.

However, in view of f being discontinuous, uniform convergence cannot be expected (Theorem 4.1.7) and indeed not the case ($c_n \sim 1/n$). At the discontinuities of f the series converges to 0, the average between the left- and right limits, as is shown graphically by Fig. 4.1b. We see an interesting phenomenon here: there is an overshoot to the left and an "undershoot" to the right. This is known as Gibbs phenomenon. *See exercise (4.1.14).*

The function in example 4.1.10 was clearly odd and could be expressed as a sine-series. This is generally true. If f is odd (*i.e.* $f(-x) = -f(x)$) we have a *Fourier sine series* and if f is even (*i.e.* $f(-x) = f(x)$) we have a *Fourier cosine series*.

EXAMPLE 4.1.11. *The following sine and cosine series define periodic functions with period* 1.

$$\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} = \left[\tfrac{1}{2} - x\right]_{0,1}, \qquad \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n} = -\log|2\sin \pi x|,$$

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{\pi^2 n^2} = \left[x^2 - x + \tfrac{1}{6}\right]_{0,1}, \qquad \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2 - \frac{1}{4}} = 2 - \pi|\sin \pi x|.$$

The following sine series define odd functions with period 2.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(\pi n x)}{\pi n} = \left[-\tfrac{1}{2}x\right]_{-1,1}, \qquad \sum_{n=1}^{\infty} (-1)^n \frac{\sin(\pi n x)}{\pi^3 n^3} = \left[\tfrac{1}{12}x(x^2-1)\right]_{-1,1},$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(\pi n x)}{\pi^5 n^5} = \left[-\tfrac{1}{240}x(x^2-1)(x^2-\tfrac{7}{3})\right]_{-1,1}.$$

Again we see only uniform convergence for continuous functions.
Of special interest is the behaviour of the non-uniformly convergent Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n x)}{\sqrt{n}} = \frac{\operatorname{sign}(x)}{\sqrt{|2x|}} + \dots \quad (x \to 0)$$

The series is pointwise equal to zero in $x = 0$*, but represents a function that diverges there. Another interesting example is the block-wave function, defined along* $[-1, 1)$ *by* $\operatorname{sign} x$*, or*

$$4 \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x}{(2n+1)\pi} = [\operatorname{sign} x]_{-1,1}.$$

If f is square-integrable on $[0, 2\pi]$, *i.e.* $\int_0^{2\pi} |f(x)|^2\,dx < \infty$, then we have the following

THEOREM 4.1.12 (**Parseval's Identity for Fourier series**). *Let* f *and* g *be square-integrable on* $[0, 2\pi]$ *with Fourier coefficients* c_n *and* d_n*. Then*

$$\int_0^{2\pi} f(x)\overline{g(x)}\,dx = 2\pi \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

In particular, if $g = f$ *then*

$$\int_0^{2\pi} |f(x)|^2\,dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

EXAMPLE 4.1.13. *With Parseval's Identity applied to the Fourier series found in example 4.1.10 we obtain* $\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^3 = 4\pi \sum_{n=1}^{\infty} 1/n^2$*. See the exercises for more examples.*

The Laurent series as a Fourier series

Every Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$, defined for $|z| = R$, can be identified at $|z| = R$ with a Fourier series of the form

$$F(\vartheta) = f(R\,\mathrm{e}^{\mathrm{i}\vartheta}) = \sum_{n=-\infty}^{\infty} c_n R^n\,\mathrm{e}^{\mathrm{i}n\vartheta} = \sum_{n=-\infty}^{\infty} \alpha_n\,\mathrm{e}^{\mathrm{i}n\vartheta}, \qquad \vartheta \in [0, 2\pi), \quad \alpha_n = c_n R^n.$$

EXAMPLE 4.1.14. *Consider for $a > 1$ the function*

$$f(z) = \frac{(a^2-1)z}{(a-z)(az-1)} = \frac{a}{a-z} + \frac{1}{az-1} = \sum_{n=-\infty}^{\infty} a^{-|n|} z^n$$

on the circle $|z| = 1$. Then we have

$$F(\vartheta) = \frac{a^2-1}{a^2 - 2a\cos\vartheta + 1} = \sum_{n=-\infty}^{\infty} a^{-|n|}\,\mathrm{e}^{\mathrm{i}n\vartheta}\,.$$

REMARK 4.1.15. *If $|z| = R$ is inside the annular domain of convergence of the Laurent series, the Fourier series is infinitely differentiable (C^∞) and $\{c_n\}$ converge faster than any power of n^{-1}.*

4.2 Fourier transforms

Consider the L-periodic Fourier series of an absolute and square-integrable[2] function $f : \mathbb{R} \to \mathbb{C}$, restricted to $[-\frac{1}{2}L, \frac{1}{2}L]$. There exists an integral analogue to the Fourier series, if we let the interval of periodicity tend to infinity. First we have with

$$c_n = \frac{1}{L}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(y)\,\mathrm{e}^{-\mathrm{i}yn\Delta\alpha}\,dy, \quad \Delta\alpha = \frac{2\pi}{L},$$

the representation

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \Delta\alpha \left(\frac{1}{2\pi}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(y)\,\mathrm{e}^{-\mathrm{i}yn\Delta\alpha}\,dy \right) \mathrm{e}^{\mathrm{i}xn\Delta\alpha}\,.$$

Then we recognize a Riemann sum along $-2\pi N/L \leqslant \alpha \leqslant 2\pi N/L$ of the function

$$g(\alpha; L) = \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}\alpha x}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} f(y)\,\mathrm{e}^{-\mathrm{i}\alpha y}\,dy,$$

with piecewise constant approximations of $g(\alpha; L)$ at the points $\alpha = n\Delta\alpha$, $n = -N \cdots N$, multiplied by the interval width $\Delta\alpha$. Hence by a (subtle) limit argument for $N, L \to \infty$ we find

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(y)\,\mathrm{e}^{-\mathrm{i}\alpha y}\,dy \right] \mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha,$$

[2] Since we use in this document only Riemann integration, we do not write $f \in L^1$ or L^2

which leads to $\hat{f}$, the *Fourier transform* of f, which is together with its inversion given by

$$\hat{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}\alpha x}\,dx, \tag{4.2a}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha. \tag{4.2b}$$

(Note that other, equivalent, definitions occur, which may cause confusion.) The Fourier transform (or also called: Fourier integral) plays an important rôle in the analysis of problems where we have a continuous spectrum of wave numbers or frequencies. One can show that f and $\hat{f}$ cannot vanish simultaneously outside a finite domain. Furthermore, a sufficient condition for existence of $\hat{f}$ is that $|f|$ is integrable ($\hat{f}$ is even continuous), but the resulting $\hat{f}$ does not have to be absolutely integrable (see example 4.2.1, with indeed a non-continuous f). However, if f is also square-integrable, the situation is more symmetric. Then $\hat{f}$ exists and is also square-integrable.

EXAMPLE 4.2.1. *(i) The "top-hat" function, given by $f(x) = H(x+1) - H(x-1)$, or*

$$f(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 1, \\ 0, & |x| > 1, \end{cases}$$

*has a Fourier transform, closely related to the **sinc-function**[3]:*

$$\hat{f}(\alpha) = \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}\alpha x}\,dx = \frac{\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha}}{\mathrm{i}\alpha} = 2\frac{\sin\alpha}{\alpha} = 2\,\mathrm{sinc}(\alpha).$$

(ii) The decaying exponential with decay rate $p > 0$, vanishing for $x < 0$, and defined by

$$f(x) = \begin{cases} \mathrm{e}^{-px}, & x > 0, \\ 0\quad, & x < 0, \end{cases}$$

has a Fourier transform consisting of a single pole in the upper complex α-plane

$$\hat{f}(\alpha) = \int_{0}^{\infty} \mathrm{e}^{-px}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,dx = \left[-\frac{\mathrm{e}^{-(p+\mathrm{i}\alpha)x}}{p+\mathrm{i}\alpha}\right]_0^{\infty} = \frac{1}{p+\mathrm{i}\alpha} = \frac{-\mathrm{i}}{\alpha - \mathrm{i}p}.$$

(iii) Another important example is $f(x) = p^{-1/2}\,\mathrm{e}^{-\frac{1}{2}(x/p)^2}$, where $p > 0$, with the similar

$$\hat{f}(\alpha) = \int_{-\infty}^{\infty} p^{-1/2}\,\mathrm{e}^{-\frac{1}{2}(x/p)^2}\,\mathrm{e}^{-\mathrm{i}\alpha x}\,dx = (2\pi p)^{1/2}\,\mathrm{e}^{-\frac{1}{2}(p\alpha)^2}.$$

Note that where f depends on p, $\hat{f}$ depends on its reciprocal. See also section 4.4.

If $\alpha^n \hat{f}(\alpha)$ is Fourier transformable, where $n \in \mathbb{N}$ and $\hat{f}(\alpha)$ is the Fourier transform of $f(x)$, then

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathrm{i}\alpha)^n \hat{f}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha.$$

[3] Apart from the definition $\mathrm{sinc}(x) = \sin(x)/x$, there is the normalised definition $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$.

Convolution and Parseval's Identity for Fourier Transforms

DEFINITION 4.2.2. *The convolution product $f*g$ of square-integrable f and g, is defined as*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

THEOREM 4.2.3 (**Convolution Theorem and Parseval's Identity for Fourier Transforms**). *The Fourier transformed convolution product of absolute and square-integrable f and g is the product of their Fourier transforms:*

$$\widehat{f*g} = \hat{f}\hat{g}.$$

In other words, the inverse Fourier transform of $\hat{f}\hat{g}$ is equal to the convolution product:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

If we consider this for $x = 0$ and take $g(y) = \overline{f(-y)}$ with $\hat{g}(\alpha) = \overline{\hat{f}(\alpha)}$, we obtain the analogue of Parseval's Identity (Theorem 4.1.12) for integrals[4]

$$\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\alpha)\right|^2 d\alpha.$$

PROOF: *As f and g are square-integrable, we may change the order of integration to get*

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\hat{g}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\int_{-\infty}^{\infty} g(y)\,\mathrm{e}^{-\mathrm{i}\alpha y}\,dy\;\mathrm{e}^{\mathrm{i}\alpha x}\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} \hat{f}(\alpha)\,\mathrm{e}^{\mathrm{i}\alpha(x-y)}\,d\alpha\,dy = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.\end{aligned}$$

EXAMPLE 4.2.4. *Consider example 4.2.1 (ii) again. We obtain indeed*

$$\int_{-\infty}^{\infty} \left|f(x)\right|^2 dx = \int_{0}^{\infty} \mathrm{e}^{-2px}\,dx = \frac{1}{2p},$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\alpha)\right|^2 d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|p+\mathrm{i}\alpha|^2}\,d\alpha = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{p^2+\alpha^2}\,d\alpha = \frac{1}{2p}.$$

See the exercises for more examples.

[4] sometimes referred to as the *energy theorem*.

4.3 Laplace transforms

Let $f(t)$ be a piecewise-continuous function defined for $t \geqslant 0$. Let $\alpha \in \mathbb{R}$ and assume that $f(t)$ is α-**exponentially bounded**, *i.e.* there is a number M such that $|f(t)| \leqslant M\,e^{\alpha t}$ for all $t \geqslant 0$. Then the integral

$$F(s) = \int_0^\infty e^{-st}\, f(t)\, dt \tag{4.3a}$$

converges for $\operatorname{Re} s > \alpha$. Indeed, $|\,e^{-st}\, f(t)| \leqslant M\, e^{(\alpha - \operatorname{Re} s)t}$.

We call $F(s)$ the **Laplace transform** of $f(t)$. One can prove that the function $F(s)$ is analytic for $\operatorname{Re} s > \alpha$. We can reconstruct the function $f(t)$ from the corresponding $F(s)$ by means of the **inverse Laplace transform**. We derive this formula from the inverse Fourier transform (4.2). We have, with the notation $s = \sigma + i\omega$, $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}$,

$$F(s) = \int_0^\infty f(t)\, e^{-st}\ dt = \int_0^\infty f(t)\, e^{-\sigma t}\, e^{-i\omega t}\ dt = \int_{-\infty}^\infty g(t)\, e^{-i\omega t}\ dt,$$

where $g(t)$ is the following causal function (g is **causal** if there is a t_0 with $g(t) \equiv 0$ for $t < t_0$)

$$g(t) = H(t) f(t)\, e^{-\sigma t}, \qquad H(t) = \begin{cases} 1 & (t \geqslant 0), \\ 0 & (t < 0). \end{cases}$$

Function $H(t)$ is known as **Heaviside's unit step function**. For fixed $\sigma > \alpha$, $F(\sigma + i\omega)$ is the Fourier transform of $g(t)$. It is also clear that $g(t)$ is absolutely integrable. The inverse Fourier transform gives us now:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty F(\sigma + i\omega)\, e^{i\omega t}\ d\omega.$$

For $t \geqslant 0$ it follows that

$$f(t) = e^{\sigma t}\, g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty F(\sigma + i\omega)\, e^{(\sigma + i\omega)t}\ d\omega,$$

which can be considered as the parametrised complex integral

$$f(t) = \frac{1}{2\pi i} \int_L F(s)\, e^{st}\ ds, \tag{4.3b}$$

where $L = \{s = \sigma + i\omega \,\big|\, \sigma > \alpha, \omega \in \mathbb{R}\}$ (also known as a Bromwich contour; see example 3.5.1) is a vertical line in the s-plane, traversed in upward direction, and located on the right of the line $\operatorname{Re} s = \alpha$. For a given $F(s)$ we have to move the integration contour L far enough to the right, such that $F(s)$ is analytic at and on the right-hand side of L. The integral is then independent of the path, as follows from the above calculation. Sometimes it is possible to determine the inverse transform by means of residue calculus.

Solving certain differential equations

A popular application of the Laplace Transform is to construct the solution of an ordinary linear differential equations with initial conditions. The method is straightforward for constant coefficients. Consider for example for $t \geqslant 0$

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1,$$

while f is continuous and exponentially bounded with Laplace transform $F(s)$. Assuming that y is Laplace transformable (this may be checked afterwards) we introduce

$$\int_0^\infty y(t)\,\mathrm{e}^{-st}\,dt = Y(s),\ \int_0^\infty y'(t)\,\mathrm{e}^{-st}\,dt = sY(s) - y_0,\ \int_0^\infty y''(t)\,\mathrm{e}^{-st}\,dt = s^2Y(s) - y_1 - sy_0.$$

After Laplace transformation of the left hand and right hand sides of the equation, we find

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = F(s),$$

leading, for suitable Bromwich contour L, to

$$y(t) = \frac{1}{2\pi\mathrm{i}} \int_L \frac{F(s) + (as + b)y_0 + ay_1}{as^2 + bs + c}\,\mathrm{e}^{st}\,ds.$$

Response $y(t)$ is clearly brought about by driving force $f(t)$ and initial conditions y_0 and y_1. It depends greatly on the poles of $F(s)$ and the zeros of polynomial $as^2 + bs + c$.

Variations on this theme are systems of ordinary differential equations, or a possibly helpfull reduction of linear partial differential equations, for example

$$\frac{\partial u}{\partial t} = \nabla^2 u,\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \quad \Longrightarrow \quad sU = \nabla^2 U + u_0,\ U(\boldsymbol{x}, s) = \int_0^\infty u(\boldsymbol{x}, t)\,\mathrm{e}^{-st}\,dt.$$

Sometimes we can trade a non-constant coefficient of polynomial type in t for derivatives in s:

$$\int_0^\infty (-t)^n y(t)\,\mathrm{e}^{-st}\,dt = \frac{d^n}{ds^n} Y(s).$$

EXAMPLE 4.3.1. *Consider the harmonic oscillator with resonance frequency ω_0, excited from rest by a harmonic force $\mathrm{e}^{\mathrm{i}\omega t}$ of frequency ω.*

$$y'' + \omega_0^2 y = \mathrm{e}^{\mathrm{i}\omega t}, \quad y(0) = 0, \quad y'(0) = 0,$$

From the exposition above we find

$$y(t) = \frac{1}{2\pi\mathrm{i}} \int_L \frac{\mathrm{e}^{st}}{(s - \mathrm{i}\omega)(s^2 + \omega_0^2)}\,ds.$$

If $\omega \neq \omega_0$ we have three simple poles along the imaginary s-axis leading to the solution

$$y(t) = H(t)\left[\frac{\cos(\omega_0 t)}{\omega_0(\omega - \omega_0)} - \frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega^2 - \omega_0^2}\right].$$

If $\omega = \omega_0$ we have one simple pole in $s = -\mathrm{i}\omega_0$ and one pole of second order in $s = \mathrm{i}\omega_0$, leading to the algebraically growing resonant solution

$$y(t) = H(t)\left[\frac{\mathrm{i}\sin(\omega_0 t)}{2\omega_0^2} - \frac{\mathrm{i}t\,\mathrm{e}^{\mathrm{i}\omega_0 t}}{2\omega_0}\right].$$

4.4 Poisson Summation Formula and some applications

Poisson Summation Formula is an excellent tool to accelerate certain slowly converging series, while it also may clarify and regularise difficult asymptotic behaviour.

Let f be a Fourier transformable function on $\mathbb{R}$, with Fourier transform $\hat{f}$ defined in (4.2). Noting that the function[5]

$$\sum_{n=-\infty}^{\infty} f(nL+x)$$

is periodic in x with positive period L, it has a Fourier series and we can write

$$\begin{aligned}\sum_{n=-\infty}^{\infty} f(nL+x) &= \sum_{k=-\infty}^{\infty}\left[\frac{1}{L}\int_0^L \sum_{n=-\infty}^{\infty} f(nL+y)\,\mathrm{e}^{-\mathrm{i}\alpha_k y}\,dy\right]\mathrm{e}^{\mathrm{i}\alpha_k x} \\ &= \frac{1}{L}\sum_{k=-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\eta)\,\mathrm{e}^{-\mathrm{i}\alpha_k\eta}\,d\eta\right]\mathrm{e}^{\mathrm{i}\alpha_k x} = \frac{1}{L}\sum_{k=-\infty}^{\infty}\hat{f}(\alpha_k)\,\mathrm{e}^{\mathrm{i}\alpha_k x},\end{aligned}$$

with $\alpha_k = \dfrac{2\pi k}{L}$. In other words we have the **Generalised Poisson Summation Formula**

$$\sum_{n=-\infty}^{\infty} f(nL+x) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\hat{f}\big(\tfrac{2\pi n}{L}\big)\,\mathrm{e}^{\mathrm{i}\frac{2\pi n}{L}x}, \tag{4.4}$$

which simplifies for $x=0$ to the **Poisson Summation Formula**

$$\sum_{n=-\infty}^{\infty} f(nL) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\hat{f}\big(\tfrac{2\pi}{L}n\big), \tag{4.5}$$

where the average of the left and right limit is to be taken at any discontinuities of f or $\hat{f}$.

EXAMPLE 4.4.1. *With example 4.2.1 we have for the "top-hat" function*

$$f(x) = H(x+1) - H(x-1) \quad \textit{with} \quad \hat{f}(\alpha) = 2\frac{\sin\alpha}{\alpha}$$

and $L=4$

$$\sum_{n=-\infty}^{\infty} f(4n) = f(0) = 1 = \frac{1}{4}\left[2 + 4\sum_{n=1}^{\infty}\frac{\sin\frac{1}{2}\pi n}{\frac{1}{2}\pi n}\right] = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

from which follows Leibniz's formula for π, *stating that* $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{1}{4}\pi$.

The following examples illustrate the convergence enhancing potential of Poisson's Formula.

[5] This derivation is due to N.G. de Bruijn.

EXAMPLE 4.4.2. *Let* $\mathrm{Re}(p) > 0$ *and*

$$f(x) = H(x)\,\mathrm{e}^{-px} \quad \textit{with} \quad \hat{f}(\alpha) = \frac{1}{p + \mathrm{i}\alpha}.$$

If $L = 2\pi$ *then*

$$\frac{1}{2\pi p} + \frac{p}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2 + p^2} = \frac{1}{2} + \sum_{n=1}^{\infty}\mathrm{e}^{-2\pi pn} = \frac{1}{2}\coth(\pi p)$$

The left-hand side converges algebraically slowly, in contrast to the fast, exponential convergence of the right-hand side. As a bonus, we have in this case even an explicit expression if we recognize the geometric series with common ratio $\mathrm{e}^{-2\pi p}$.

If we write $p = 1/\varepsilon$, *we obtain for small* ε

$$\sum_{n=0}^{\infty}\frac{1}{1 + n^2\varepsilon^2} = \frac{\pi}{2\varepsilon} + \frac{1}{2} + EST \qquad (\varepsilon \to 0).$$

(EST = exponentially small terms.)

EXAMPLE 4.4.3. *Another important example is*

$$f(x) = \frac{\mathrm{e}^{-\frac{x^2}{2p^2}}}{\sqrt{p}} \quad \textit{with} \quad \hat{f}(\alpha) = \sqrt{2\pi p}\,\mathrm{e}^{-\frac{1}{2}\alpha^2 p^2},$$

where $p > 0$. *With* $L = \sqrt{2\pi}$ *we obtain the famous*

$$\sum_{n=0}^{\infty}\mathrm{e}^{-\pi n^2 p^2} = \frac{1}{p}\sum_{n=0}^{\infty}\mathrm{e}^{-\pi\frac{n^2}{p^2}}$$

If $p = \varepsilon/\sqrt{\pi}$ *we obtain for small* ε

$$\sum_{n=0}^{\infty}\mathrm{e}^{-n^2\varepsilon^2} = \frac{\sqrt{\pi}}{\varepsilon} + EST \qquad (\varepsilon \to 0).$$

EXAMPLE 4.4.4. *For*

$$f(x) = \frac{\mathrm{e}^{-x^2} - 1}{x^2}$$

we find

$$\hat{f}(\alpha) = \pi|\alpha| - \pi\alpha\,\mathrm{erf}(\alpha) - 2\sqrt{\pi}\,\mathrm{e}^{-\frac{1}{4}\alpha^2}$$

such that with $L = \varepsilon$

$$\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-n^2\varepsilon^2} - 1}{n^2\varepsilon^2} = \frac{1}{\varepsilon}\sum_{n=-\infty}^{\infty}\left[\frac{2\pi^2}{\varepsilon}|n| - \frac{2\pi^2}{\varepsilon}n\,\mathrm{erf}\Big(\frac{2\pi n}{\varepsilon}\Big) - 2\sqrt{\pi}\,\mathrm{e}^{-\frac{\pi^2 n^2}{\varepsilon^2}}\right]$$

From this we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-n^2\varepsilon^2}}{n^2} &= \tfrac{1}{6}\pi^2 - \sqrt{\pi}\,\varepsilon + \tfrac{1}{2}\varepsilon^2 + 2\pi^2\sum_{n=1}^{\infty}n\,\mathrm{erfc}\Big(\frac{2\pi n}{\varepsilon}\Big) - 2\varepsilon\sqrt{\pi}\sum_{n=1}^{\infty}\mathrm{e}^{-\frac{\pi^2 n^2}{\varepsilon^2}}\\ &= \tfrac{1}{6}\pi^2 - \sqrt{\pi}\,\varepsilon + \tfrac{1}{2}\varepsilon^2 + EST \qquad (\varepsilon \to 0)\end{aligned}$$

4.5 Discrete Fourier Transform

The standard numerical implementation of the calculation of a Fourier transform or Fourier coefficient is the *Fast Fourier Transform* (FFT) algorithm[6]. This algorithm calculates for a given complex array $\{x_k\}$, $k = 0, \ldots, N-1$ very efficiently (especially if N is a power of 2) the *Discrete Fourier Transform* (DFT)

$$X_n = \sum_{k=0}^{N-1} x_k \, \mathrm{e}^{-2\pi \mathrm{i} kn/N}, \qquad n = 0, \ldots, N-1. \tag{4.6}$$

Its (exact[7]) inverse form is

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \, \mathrm{e}^{2\pi \mathrm{i} kn/N}, \qquad n = 0, \ldots, N-1.$$

A Fourier coefficient (4.1) is approximately calculated by discretising the integral

$$c_n = \frac{1}{T} \int_0^T \varphi(t) \, \mathrm{e}^{-2\pi \mathrm{i} nt/T} \, dt \simeq \frac{1}{N} \sum_{k=0}^{N-1} \varphi\Big(\frac{kT}{N}\Big) \, \mathrm{e}^{-2\pi \mathrm{i} kn/N}$$

and relating $x_k = \varphi(kT/N)$ and $c_n = X_n/N$.

To identify a Fourier integral (4.2) to a DFT, we restrict the infinite integral to a large enough finite interval $[-\frac{1}{2}T, \frac{1}{2}T]$,

$$\hat{\varphi}(\omega) = \int_{-\infty}^{\infty} \varphi(t) \, \mathrm{e}^{-\mathrm{i}\omega t} \, dt \simeq \int_{-\frac{1}{2}T}^{\frac{1}{2}T} \varphi(t) \, \mathrm{e}^{-\mathrm{i}\omega t} \, dt = \int_0^{\frac{1}{2}T} \varphi(t) \, \mathrm{e}^{-\mathrm{i}\omega t} \, dt + \int_{\frac{1}{2}T}^{T} \varphi(t-T) \, \mathrm{e}^{-\mathrm{i}\omega t} \, dt.$$

If we consider only the values $\omega = \frac{2\pi}{T} n$, for $n = -\frac{1}{2}N, \ldots, \frac{1}{2}N - 1$ for the discretised integrals

$$\hat{\varphi}\Big(\frac{2\pi}{T} n\Big) \simeq \frac{T}{N} \left[\sum_{k=0}^{\frac{1}{2}N-1} \varphi\Big(\frac{k}{N} T\Big) \, \mathrm{e}^{-2\pi \mathrm{i} kn/N} + \sum_{k=\frac{1}{2}N}^{N-1} \varphi\Big(\frac{k-N}{N} T\Big) \, \mathrm{e}^{-2\pi \mathrm{i} kn/N} \right],$$

we obtain the required result by identifying

$$x_k = \begin{cases} \varphi\Big(\dfrac{k}{N} T\Big) & \text{if} \quad 0 \leqslant k \leqslant \frac{1}{2}N - 1, \\ \varphi\Big(\dfrac{k-N}{N} T\Big) & \text{if} \quad \frac{1}{2}N \leqslant k \leqslant N - 1, \end{cases}$$

$$\hat{\varphi}\Big(\frac{2\pi n}{T}\Big) = \frac{T}{N} \begin{cases} X_{n+N} & \text{if} \quad -\frac{1}{2}N \leqslant n \leqslant -1, \\ X_n & \text{if} \qquad 0 \leqslant n \leqslant \frac{1}{2}N - 1. \end{cases}$$

[6] (re-)invented in 1965 by James William Cooley and John Wilder Tukey.

[7] as may be readily seen by using the identity $\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} k(m-n)/N} = 0$ if $m \neq n$ and $= N$ if $m = n$.

4.6 Applications

4.6.1 Excitation of a mass-spring-damper system

Consider the response (in this case the displacement) $y(t)$ of a mass-spring-damper system with mass m, damping coefficient R and spring constant K, to the excitation of an external force $f(t)$

$$my''(t) + Ry'(t) + Ky(t) = f(t).$$

For physical reasons, the parameters m, R and K are positive.

Questions

1. Solve the problem for harmonic excitation $f(t) = \mathrm{Re}[A\,\mathrm{e}^{\mathrm{i}\omega t}]$, and initial conditions $y(0) = y'(0) = 0$, by Laplace transformation.

2. If we are only interested in the steady-state situation, a more straightforward approach is to assume that the response follows the harmonic time dependence of the force. Show that the solution y is given by $y = \mathrm{Re}(u)$ with
$$u(t) = \frac{A\,\mathrm{e}^{\mathrm{i}\omega t}}{-\omega^2 m + \mathrm{i}\omega R + K}.$$

3. The found solution u is a particular solution, *i.e.* not the most general solution. In particular, it does not satisfy any initial conditions. Indeed, there is also an "undriven" or "homogeneous" solution, *i.e.* a solution of the problem with $F \equiv 0$. This solution has to be added if we want to satisfy the initial conditions.

   What happens with this homogeneous solution for $t \to \infty$? We will further ignore this homogeneous solution.

4. Since the problem is linear, solutions for more general excitations can be constructed by Fourier synthesis, *i.e.* summation over harmonic solutions. For example, if we take solutions of any ω and amplitudes $A = \frac{1}{2\pi}\hat{f}(\omega)$, the general excitation, given by the Fourier integral
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\,d\omega$$
gives the solution
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}}{-\omega^2 m + \mathrm{i}\omega R + K}\,d\omega.$$
Show that, for a real f, its Fourier transform $\hat{f}$ has to satisfy the condition $\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$. Show that y is then also real.

5. Consider here the constant amplitudes $\hat{f}(\omega) = A_0$. Determine $y(0)$. What about $y'(0)$?

6. For the general case of $t \neq 0$ we have to make a distinction between $t < 0$ and $t > 0$. Find the respective expressions for $y(t)$.

Answers

1. If we write $y(t) = \operatorname{Re} u(t)$, then

$$u(t) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{A}{(s - \mathrm{i}\omega)(ms^2 + Rs + K)} \,\mathrm{e}^{st}\, dt$$
$$= \frac{A}{m}\left[\frac{\mathrm{e}^{\mathrm{i}\omega t}}{(\mathrm{i}\omega - s_1)(\mathrm{i}\omega - s_2)} + \frac{\mathrm{e}^{s_1 t}}{(s_1 - \mathrm{i}\omega)(s_1 - s_2)} + \frac{\mathrm{e}^{s_2 t}}{(s_2 - \mathrm{i}\omega)(s_2 - s_1)}\right]$$

where L is located at the right side of the poles in $s = s_1$, $s = s_2$ and $s = \mathrm{i}\omega$, while

$$s_{1,2} = -\frac{R}{2m} \pm \mathrm{i}\sqrt{\frac{K}{m} - \frac{R^2}{4m^2}}.$$

2. It follows by direct substitution.

3. The homogeneous solutions are proportional to $\mathrm{e}^{\mathrm{i}\omega_1 t}$ and $\mathrm{e}^{\mathrm{i}\omega_2 t}$, with $\omega_{1,2} = \mathrm{i}\frac{R}{2m} \pm \sqrt{\frac{K}{m} - \frac{R^2}{4m^2}}$. Since $\operatorname{Im}\omega_{1,2} > 0$, even if $4mK < R^2$, they tend exponentially to zero.

4. If f is real, it is equal to its complex conjugate $\overline{f}$, and so

$$2\pi f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)\,\mathrm{e}^{\mathrm{i}\omega t}\, d\omega = \int_{-\infty}^{\infty} \overline{\hat{f}(\omega)}\,\mathrm{e}^{-\mathrm{i}\omega t}\, d\omega = \int_{-\infty}^{\infty} \overline{\hat{f}(-\omega)}\,\mathrm{e}^{\mathrm{i}\omega t}\, d\omega,$$

which is only possible for *all* t if $\hat{f}(\omega) = \overline{\hat{f}(-\omega)}$. Since

$$\frac{\hat{f}(\omega)}{-\omega^2 m + \mathrm{i}\omega R + K} = \overline{\frac{\hat{f}(-\omega)}{-(-\omega)^2 m + \mathrm{i}(-\omega)R + K}},$$

it follows in a similar way that $y = \overline{y}$, and hence is real.

5. We have simple poles in $\omega = \omega_1$ and $\omega = \omega_2$ in the upper half plane. We can close the contour either up- or downward. Downward is easiest, because no poles are captured and it is directly clear that the integral is zero:

$$y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0}{-m(\omega - \omega_1)(\omega - \omega_2)}\, d\omega = 0.$$

For $y'(0)$ the integral does not converge. In fact, $y'(t \uparrow 0) = 0$ while $y'(t \downarrow 0) = A_0/m$.

6. Simple poles in $\omega = \omega_1$ and $\omega = \omega_2$ in the upper half plane. We can close the contour only in the direction where $\mathrm{e}^{\mathrm{i}\omega t} \to 0$. That is upward for $t > 0$, and downward for $t < 0$. We obtain altogether

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0\,\mathrm{e}^{\mathrm{i}\omega t}}{-m(\omega - \omega_1)(\omega - \omega_2)}\, d\omega = \frac{\mathrm{i}A_0 H(t)}{-m(\omega_1 - \omega_2)}\left(\mathrm{e}^{\mathrm{i}\omega_1 t} - \mathrm{e}^{\mathrm{i}\omega_2 t}\right)$$
$$= \frac{2A_0 H(t)}{\sqrt{4mK - R^2}} \exp\left(-\frac{R}{2m}t\right) \sin\left(\sqrt{\frac{K}{m} - \frac{R^2}{4m^2}}\, t\right)$$

where $H(t)$ is Heaviside's unit step function.

4.6.2 Kramers-Kronig Relation

Causal. A real function $g(t)$ is **causal** if there is a t_0 with $g(t) = 0$ for $t < t_0$.

Principal Value. A **Principal Value** integral is a redefined integral in order to attach a meaning to the integral of certain singular functions. If $h(x)$ is singular in $x = a$ but otherwise smooth and integrable, we define, provided the limit exists,

$$\mathcal{P}\int_{-\infty}^{\infty} h(x)\,dx \stackrel{\text{def}}{=} \lim_{\varepsilon\downarrow 0}\left[\int_{-\infty}^{a-\varepsilon} + \int_{a+\varepsilon}^{\infty}\right] h(x)\,dx.$$

The limit exists typically if the singularity is a simple pole.

Questions

1. Show that if $F(z)$ is analytic in the lower[8] complex half plane $\mathbb{C}^- = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \leqslant 0\}$, while $\lim\limits_{R\to\infty} \max\limits_{-\pi\leqslant\vartheta\leqslant 0} |F(R\,\mathrm{e}^{\mathrm{i}\vartheta})| = 0$, we can write

$$F(z) = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{F(\zeta)}{\zeta - z}\,d\zeta \quad \text{for any} \quad z \in \mathbb{C}^-,\ \mathrm{Im}(z) \neq 0$$

2. Show that for $x \in \mathbb{R}$ we can write

$$F(x) = -\frac{1}{\pi\mathrm{i}}\mathcal{P}\int_{-\infty}^{\infty} \frac{F(\xi)}{\xi - x}\,d\xi.$$

The real (U) and imaginary (V) parts of F are related by the so-called Hilbert transforms

$$U(x) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{V(\xi)}{\xi - x}\,d\xi, \qquad V(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{U(\xi)}{\xi - x}\,d\xi.$$

3. Show that for a causal and (absolutely) integrable function $g(t)$ that satisfies $|g(t)| \leqslant M\,\mathrm{e}^{-\varepsilon t}$ for some M and $\varepsilon > 0$ [9], its Fourier transform

$$\hat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,dt$$

is analytic in $\omega \in \mathbb{C}^-$.

4. Show that the real (u) and imaginary (v) parts of the Fourier transform $\hat{g}(\omega)$ of a causal and integrable real function $g(t)$ are related by the Kramers-Kronig relation

$$u(\omega) = -\frac{2}{\pi}\mathcal{P}\int_0^{\infty} \frac{\omega' v(\omega')}{\omega'^2 - \omega^2}\,d\omega'$$

where ω is real.

[8] The lower half plane is necessary for causality. For the upper half plane it would be similar although not the same by virtue of minus signs.

[9] This condition can be relaxed.

Answers

1. Integral theorem of Cauchy along $z \in [-R, R]$ and $z = R\,\mathrm{e}^{\mathrm{i}\vartheta},\ \vartheta \in [-\pi, 0]$. Take the limit $R \to \infty$. Note the minus sign.

2. Take the limit of $\mathrm{Im}(z) \uparrow 0$ for the representation integral of question 1, while making a (vanishingly) small semi-circular indentation of the contour. The contribution of the integral along this indentation is just half the residue.

3. The conditions allow exchange of differentiation and integration. The integral

$$\hat{g}(\omega) = \frac{1}{2\pi} \int_{t_0}^{\infty} g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\ dt$$

exists for all $\omega \in \mathbb{C}^-$ because $|g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}| = |g(t)|\,\mathrm{e}^{\mathrm{Im}(\omega)t}$ converges for all $\omega \in \mathbb{C}^-$ including a strip along the real axis of height ε. Since this convergence is *uniformly* in any $\mathrm{Im}\,\omega \leqslant \varepsilon_0 < \varepsilon$, we can exchange integration and differentiation, provided the derivative exists. This is indeed the case for all derivatives, because $|t^n g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}| < M(n/\alpha)^n\,\mathrm{e}^{-\frac{1}{2}\alpha t}$ with $\alpha = \varepsilon - \mathrm{Im}\,\omega$. Hence, the integral is analytic in ω.

4. Use the Hilbert transforms to relate u and v. Then apply the reality of $g(t)$, implying

$$\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\ dt = \overline{\int_{-\infty}^{\infty} g(t)\,\mathrm{e}^{\mathrm{i}\omega t}\ dt} = \overline{\hat{g}(-\omega)}.$$

Use symmetry of $u(\omega') = u(-\omega')$ and/or anti-symmetry of $v(\omega') = -v(-\omega')$.
Note the minus sign. This is opposite to what is usually found in the literature (Wikipedia), but this depends on the definition of the Fourier transform.

4.6.3 Heat kernel

Consider the following canonical 1D heat diffusion problem for $u(x, t)$, $x \in \mathbb{R}$, $t \in (0, \infty)$, due to a constant point source in $x = 0$ switched on at $t = 0$

$$u_t - k u_{xx} = 0, \quad \text{with} \quad u(x, 0) = \delta(x) \quad \text{and} \quad u \to 0 \quad \text{if} \quad |x| \to \infty.$$

Question

Find an explicit expression for $u(x, t)$ by means of Laplace transformation in time.

Answer

Consider $u \equiv 0$ for $t < 0$ and introduce the Laplace transform

$$U(x,s) = \int_0^\infty u(x,t)\, \mathrm{e}^{-st}\, dt,$$

such that

$$sU - kU_{xx} = \delta(x) \quad \text{with} \quad U \to 0 \quad \text{if} \quad |x| \to \infty.$$

It is not difficult to verify that the solution is given by

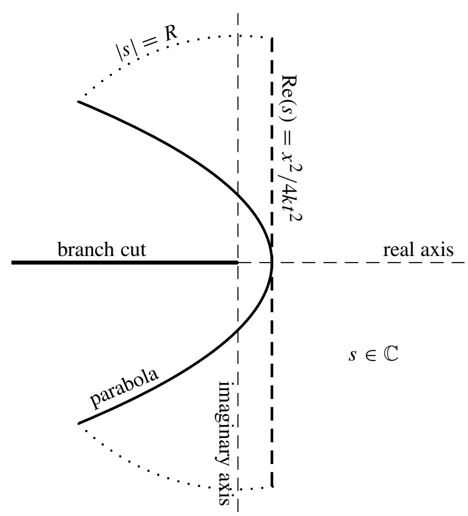
$$U(x,s) = \frac{1}{2\sqrt{sk}}\, \mathrm{e}^{-\sqrt{s/k}|x|},$$

where $\sqrt{s}$ is the principal value square root of s, such that $\mathrm{Re}(\sqrt{s}) > 0$ for all $s \notin (-\infty, 0]$. Finally, for arbitrary $c > 0$ is

$$u(x,t) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+c}^{\mathrm{i}\infty+c} \frac{1}{2\sqrt{sk}}\, \mathrm{e}^{st-\sqrt{s/k}|x|}\, ds.$$

This last integral can be evaluated by noting that for $t > 0$ there is along any arc $|s| = R$ in the 2nd or 3rd quadrant for $R \to \infty$ no contribution (Jordan's Lemma 3.4.3), so for $t > 0$ and $c = x^2/4kt^2$ we can deform the integration contour to the following parabola, parametrised by the real parameter $y \in (-\infty, \infty)$, and given by

$$s = -\frac{y^2}{t} + \frac{x^2}{4kt^2} + \mathrm{i}\frac{|x|y}{t\sqrt{kt}} = \frac{1}{t}\Big(\mathrm{i}y + \frac{|x|}{2\sqrt{kt}}\Big)^2.$$



Hence

$$\sqrt{st} = \mathrm{i}y + \frac{|x|}{2\sqrt{kt}},$$

$$st - \sqrt{s/k}|x| = -y^2 - \frac{x^2}{4kt},$$

and

$$ds = \frac{2\mathrm{i}}{t}\Big(\mathrm{i}y + \frac{|x|}{2\sqrt{kt}}\Big)dy = \frac{2\mathrm{i}\sqrt{s}}{\sqrt{t}}dy.$$

We obtain

$$u(x,t) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{sk}}\, \mathrm{e}^{-y^2-\frac{x^2}{4kt}}\, \frac{2\mathrm{i}\sqrt{s}}{\sqrt{t}}\, dy = \frac{1}{2\pi\sqrt{kt}}\, \mathrm{e}^{-\frac{x^2}{4kt}} \int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\, dy = \frac{1}{\sqrt{4\pi kt}}\, \mathrm{e}^{-\frac{x^2}{4kt}}.$$

u is known as the **heat kernel**.

4.6.4 Heat diffusion in a finite bar

A classic and subtle application of Fourier series[10] is the solution of the heat equation on a finite interval. Consider a temperature distribution $u(x,t)$, which is for $t > 0$ continuous on $[0,\pi]$, satisfying homogeneous boundary conditions of Dirichlet type and a sufficiently smooth initial distribution $f(x)$, satisfying[11]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{in } x \in (0,\pi), \text{ while } u(x,0) = f(x),\ u(0,t) = u(\pi,t) = 0.$$

Anticipating the homogeneous boundary conditions, we assume an anti-symmetric periodic continuation of u and f producing (*cf.* example 4.1.11) a sine-series representation

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx) \quad \text{such that} \quad \sum_{n=1}^{\infty}\Big(\frac{\partial c_n}{\partial t} + n^2 c_n\Big)\sin(nx) = 0 \quad \text{for all } x$$

yielding

$$c_n(t) = A_n\,\mathrm{e}^{-n^2 t}\,.$$

A_n may be determined by the initial condition $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x)$, to give

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Due to the sine-type basis functions, u satisfies apparently the boundary conditions

$$u(0,t) = \sum_{n=1}^{\infty} A_n\,\mathrm{e}^{-n^2 t}\sin(n0) = 0, \quad u(\pi,t) = \sum_{n=1}^{\infty} A_n\,\mathrm{e}^{-n^2 t}\sin(n\pi) = 0$$

However, it is important to note that this result is not enough. It only proves convergence in the endpoints, and nothing is known yet for the other points. Moreover, even if the series converge everywhere, we do not know if the sum u is continuous.

For this we have to verify, *a posteriori*, that the convergence is uniform (Thm. 4.1.7). For a piecewise smooth f and f', A_n behaves at least like $A_n = O(1/n)$ for $n \to \infty$ (Cnsq. 4.1.8), which is for $t > 0$ enough for absolute, and hence uniform, convergence of u due to the exponential convergence provided by the combination $A_n\,\mathrm{e}^{-n^2 t}$. This guarantees indeed a solution $u(x,t)$, continuous in x and with boundary conditions satisfied in a meaningful way.

The above procedure is designed for the homogeneous boundary conditions $u(0,t) = u(\pi,t) = 0$ of Dirichlet type, but it is easily adapted to more general problems. For example, inhomogeneous boundary conditions like $u(0,t) = \vartheta_0$, $u(\pi,t) = \vartheta_1$ can be turned into homogenous ones by rewriting $u(x,t) = w(x,t) + (\pi - x)\vartheta_0 + x\vartheta_1$. Homogeneous Neumann condition $u_x(0,t) = u_x(\pi,t) = 0$ can be dealt with by adopting a symmetric periodic continuation, which leads to a $\cos(nx)$-series for u and therefore a $\sin(nx)$-series in u_x. Other variants may be constructed in similar ways.

[10] We will introduce a Fourier representation directly, and skip the usual approach via the construction of an eigenvalue problem and expansion in eigenfunctions.

[11] By rescaling t and x we can always write $u_t = \kappa u_{xx}$ on $x \in [0,L]$ in this way.

Chapter 5

Asymptotic Analysis

5.1 Basic definitions

It is sometimes of interest to analyse the behaviour of a function near a particular point y_0, say $y \to y_0$, especially when this point is a singularity of some kind. We distinguish between the behaviour on the right side ($y \downarrow y_0$) and on the left side ($y \uparrow y_0$) if y_0 is finite, and $y \to \pm\infty$ if y_0 is infinite. By a simple coordinate transformation y_0 can always be assumed to be 0, approached from the right. If y_0 is finite, we can transform $\varepsilon = y - y_0$ or $\varepsilon = y_0 - y$. If it is $\pm\infty$, we can transform $\varepsilon = 1/y$ or $\varepsilon = -1/y$. In general, we consider therefore $f(\varepsilon)$ or $f(x;\varepsilon)$ for $\varepsilon \downarrow 0$.

DEFINITION 5.1.1. $f(\varepsilon) = O(\varphi(\varepsilon))$ *as* $\varepsilon \to 0$ *if there are a fixed constant* $K > 0$ *and an interval* $(0, \varepsilon_1)$ *such that*

$$|f(\varepsilon)| \leqslant K|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_1.$$

Intuitive interpretation: f can be embraced completely by $|\varphi|$ (up to a multiplicative constant) in a neighbourhood of 0.

Examples: $\sin\varepsilon = O(\varepsilon)$, $(1-\varepsilon)^{-1} = O(1)$, $\sin(1/\varepsilon) = O(1)$, $(\varepsilon + \varepsilon^2)^{-1} = O(\varepsilon^{-1})$, $\ln((1+\varepsilon)/\varepsilon) = O(\ln\varepsilon)$.

DEFINITION 5.1.2. $f(\varepsilon) = o(\varphi(\varepsilon))$ *as* $\varepsilon \to 0$ *if for every* $\delta > 0$ *there is an interval* $(0, \varepsilon_1)$ *such that*

$$|f(\varepsilon)| \leqslant \delta|\varphi(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_\delta.$$

Intuitive interpretation: f is always smaller than any multiple of $|\varphi|$ (however small) in a neighbourhood of 0.

Examples: $\sin(2\varepsilon) = o(1)$, $\cos\varepsilon = o(\varepsilon^{-1})$, $\mathrm{e}^{-a/\varepsilon} = o(\varepsilon^n)$ for any $a > 0$ and any n.

DEFINITION 5.1.3. $f(\varepsilon) = O_s(\varphi(\varepsilon))$ *as* $\varepsilon \to 0$ *if* $f(\varepsilon) = O(\varphi(\varepsilon))$ *and* $f(\varepsilon) \neq o(\varphi(\varepsilon))$.

Intuitive interpretation: f behaves exactly the same as φ (up to a multiplicative constant) in a neighbourhood of 0.

Examples: $2\sin\varepsilon = O_s(\varepsilon)$, $3\cos\varepsilon = O_s(1)$, but there is *no* n such that $\ln\varepsilon = O_s(\varepsilon^n)$.

THEOREM 5.1.4.

(i) If $f = o(\varphi)$ *then* $f = O(\varphi)$.

(ii) If $\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = 0$ *then* $f = o(\varphi)$.

(iii) If $\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = L \in [0,\infty)$ *then* $f = O(\varphi)$.

(iv) If $\lim_{\varepsilon\downarrow 0}\left|\frac{f(\varepsilon)}{\varphi(\varepsilon)}\right| = L \in (0,\infty)$ *then* $f = O_s(\varphi)$.

(v) If $f = O(\varphi)$ *and* $\varphi = O(f)$ *then* $f = O_s(\varphi)$.

REMARK 5.1.5. *The reverse is certainly not true: (i)* $\sin\varepsilon = O(\varepsilon)$ *but* $\sin\varepsilon \neq o(\varepsilon)$. *(ii) If* $f \equiv 0$ *and* $\varphi \equiv 0$ *on an interval containing* $\varepsilon = 0$, *then* $f = o(\varphi)$ *but* $\lim_{\varepsilon\downarrow 0}|f/\varphi|$ *does not exist. (iii,iv)* $\varepsilon\sin(1/\varepsilon) = O_s(\varepsilon)$, *but* $\lim_{\varepsilon\downarrow 0}|\sin(1/\varepsilon)|$ *does not exist. (v)* $\sin(1/\varepsilon) = O_s(1)$ *but* $1 \neq O(\sin(1/\varepsilon))$.

DEFINITION 5.1.6 (**Asymptotic approximation**). $\varphi(\varepsilon)$ *is an asymptotic approximation to* $f(\varepsilon)$ *as* $\varepsilon \to 0$, *denoted by* $f \sim \varphi$, *if*

$$f(\varepsilon) = \varphi(\varepsilon) + o(\varphi(\varepsilon)) \quad \text{as } \varepsilon \to 0,$$

Intuitive interpretation: If $\lim_{\varepsilon\to 0} f/\varphi = 1$ then $f \sim \varphi$. *Note:* $f \sim 0$ is only possible if $f \equiv 0$.

Examples: $\sin\varepsilon \sim \varepsilon$, $(\varepsilon+\varepsilon^2)^{-1} \sim 1/\varepsilon$, $\ln(a\varepsilon) \sim \ln\varepsilon$ for any $a > 0$.

DEFINITION 5.1.7 (**Pointwise asymptotic approximation**). $\varphi(x,\varepsilon)$ *is a pointwise asymptotic approximation to* $f(x,\varepsilon)$ *as* $\varepsilon \to 0$ *if*

$$f(x,\varepsilon) \sim \varphi(x,\varepsilon) \quad \textit{for fixed } x.$$

Intuitive interpretation: $f(x,\varepsilon)$ is approximated by $\varphi(x,\varepsilon)$ asymptotically for $\varepsilon \to 0$ and x *fixed*. Nothing is known if we allow x to vary (for example as a function of ε) within the domain.

Examples: $\sin(x+\varepsilon) \sim \sin x$ and $\sin x \neq 0$, $1/(\varepsilon + x) \sim 1/x$ and $x \neq 0$. Note that in the last example the approximation fails if we would scale $x = \varepsilon^n t$ for any $n \geqslant 1$.

DEFINITION 5.1.8 (**Uniform asymptotic approximation**). *The continuous function* $\varphi(x,\varepsilon)$ *is a uniform asymptotic approximation to the continuous function* $f(x,\varepsilon)$ *for* $x \in \mathcal{D}$ *as* $\varepsilon \to 0$, *if the way* φ *approaches* f *is the same for all* x.
More precisely: if for any positive number δ *there is an* ε_1 *(independent of* x *and* ε*) such that*

$$|f(x,\varepsilon) - \varphi(x,\varepsilon)| \leqslant \delta|\varphi(x,\varepsilon)| \quad \textit{for } x \in \mathcal{D} \quad \textit{and } 0 < \varepsilon < \varepsilon_1.$$

Intuitive interpretation: $f(x,\varepsilon)$ is approximated uniformly by $\varphi(x,\varepsilon)$, if the approximation is preserved with any scaling of x in ε within the domain of f.

Examples:

$\cos(\varepsilon)+\mathrm{e}^{-x/\varepsilon}$	~ 1	only pointwise for $x \in (0,\infty)$. Not uniform: take $x=\varepsilon t$.
$\cos(\varepsilon)+\mathrm{e}^{-x/\varepsilon}$	~ 1	pointwise and uniformly for $x \in [a,\infty)$, $a>0$.
$\cos(\varepsilon)+\mathrm{e}^{-t}$	$\sim 1+\mathrm{e}^{-t}$	uniformly for $t \in [0,\infty)$.
$\sin(\varepsilon x+\varepsilon)$	$\sim \varepsilon(x+1)$	only pointwise for $x \in (-\infty,\infty)$. Take $x=t/\varepsilon$.
$\sin(\varepsilon x+\varepsilon)$	$\sim \varepsilon(x+1)$	uniformly for $x \in [-a,a]$, $0<a<\infty$.
$2+\sin(t+\varepsilon)$	$\sim 2+\sin(t)$	uniformly for $t \in (-\infty,\infty)$.
$2+\mathrm{e}^{\varepsilon}\sin(t+\varepsilon t)$	$\sim 2+\sin(t)$	only pointwise for $t \in \mathbb{R}$. Note that $\sin(t+\varepsilon t)=\sin t+O(\varepsilon t)$.
$2+\mathrm{e}^{\varepsilon}\sin(\tau)$	$\sim 2+\sin(\tau)$	uniform for $\tau \in \mathbb{R}$. Note that we rescaled $\tau=(1+\varepsilon)t$.

REMARK 5.1.9. Uniform *implies* pointwise, *but the reverse is not necessarily true. Compare for any* $n>0$ *the function* $\varepsilon^n+\mathrm{e}^{-x/\varepsilon}\sim\varepsilon^n$ *on* $[a,\infty)$, $a>0$, *(both pointwise and uniform), and the same function on* $(0,\infty)$ *(only pointwise).*

THEOREM 5.1.10. *If* f *and* φ *are absolutely integrable, and* $f(x,\varepsilon)\sim\varphi(x,\varepsilon)$ *uniformly on a domain* $\mathcal{D}$, *while* $\int_{\mathcal{D}}|\varphi|dx=O(\int_{\mathcal{D}}\varphi dx)$, *then* $\int_{\mathcal{D}} f(x,\varepsilon)\,dx\sim\int_{\mathcal{D}}\varphi(x,\varepsilon)\,dx$.

PROOF: *Let* δ *be given, then there are* ε_1 *and* M *with*

$$\left|\int_{\mathcal{D}} f-\varphi\,dx\right|\leqslant\int_{\mathcal{D}}\left|f-\varphi\right|dx\leqslant\delta M^{-1}\int_{\mathcal{D}}|\varphi|\,dx\leqslant\delta\left|\int_{\mathcal{D}}\varphi\,dx\right|$$

along $(0,\varepsilon_1)$, *from which the result follows.*

DEFINITION 5.1.11 (**Asymptotic sequence**). *The sequence* $\{\mu_n(\varepsilon)\}$ *is called an asymptotic sequence, if* $\mu_{n+1}=o(\mu_n)$ *as* $\varepsilon\to 0$ *for each* $n=0,1,2,\cdots$. *This is denoted symbolically*

$$\mu_0\gg\mu_1\gg\mu_2\gg\cdots\gg\mu_n\gg\ldots$$

Examples. Common examples are $\mu_n=\varepsilon^n$, or more generally $\mu_n=\delta(\varepsilon)^n$ if $\delta(\varepsilon)=o(1)$. Combinations of ε and $\ln(\varepsilon)$ yield the sequence $\mu_{n,k}=\varepsilon^n\ln(\varepsilon)^k$, where $k=n,\cdots,0$ and

$$\ln\varepsilon\gg 1\gg\varepsilon\ln(\varepsilon)\gg\varepsilon\gg\varepsilon^2\ln(\varepsilon)^2\gg\varepsilon^2\ln(\varepsilon)\gg\varepsilon^2\gg\ldots$$

DEFINITION 5.1.12 (**Asymptotic expansion**). *If* $\{\mu_n(\varepsilon)\}$ *is an asymptotic sequence, then* $f(\varepsilon)$ *has an asymptotic expansion of* $N+1$ *terms with respect to this sequence, denoted by*

$$f(\varepsilon)\sim\sum_{n=0}^{N}a_n\mu_n(\varepsilon),$$

where the coefficients a_n *are independent of* ε, *if for each* $M=0,\ldots,N$

$$f(\varepsilon)-\sum_{n=0}^{M}a_n\mu_n(\varepsilon)=o(\mu_M(\varepsilon))\quad\text{as }\ \varepsilon\to 0.$$

$\mu_n(\varepsilon)$ is called a **gauge function** or an **order function**. If $\mu_n(\varepsilon)=\varepsilon^n$, we call the expansion an **asymptotic power series**.

Any Taylor series in ε around $\varepsilon = 0$, with a positive radius of convergence and truncated to finite terms, is at the same time an asymptotic expansion. However, an asymptotic power series is not necessarily a Taylor series. Many interesting series do not converge to any function for $n \to \infty$ outside $\varepsilon = 0$, but are, with a finite number of terms, nevertheless asymptotic approximations for $\varepsilon \to 0$. (See Example 5.2.4 with $\varepsilon = 1/s$.)

Examples. Asymptotic expansions, based on Taylor expansions in ε^n, of elementary functions:

$$\begin{aligned}
e^{\varepsilon} &= 1 + \varepsilon + \tfrac{1}{2}\varepsilon^2 + \dots \\
\sin(\varepsilon) &= \varepsilon - \tfrac{1}{6}\varepsilon^3 + \dots \\
\cos(\varepsilon) &= 1 - \tfrac{1}{2}\varepsilon^2 + \dots \\
(1-\varepsilon)^{-1} &= 1 + \varepsilon + \varepsilon^2 + \dots \\
\ln(1-\varepsilon) &= -\varepsilon - \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{3}\varepsilon^3 - \dots \\
\ln(1+\varepsilon) &= \varepsilon - \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{3}\varepsilon^3 - \dots \\
(1+\varepsilon)^{\alpha} &= 1 + \alpha\varepsilon + \tfrac{1}{2}\alpha(\alpha-1)\varepsilon^2 + \dots
\end{aligned}$$

Examples of combinations (which are sometimes not Taylor expansions in ε^n)

$$\begin{aligned}
\varepsilon^{\varepsilon} = e^{\varepsilon\ln\varepsilon} &= 1 + \varepsilon\ln\varepsilon + \tfrac{1}{2}\varepsilon^2(\ln\varepsilon)^2 + \dots \\
\ln(\sin\varepsilon) &= \ln\varepsilon - \tfrac{1}{6}\varepsilon^2 + \dots \\
\ln(\cos\varepsilon) &= -\tfrac{1}{2}\varepsilon^2 - \tfrac{1}{12}\varepsilon^4 + \dots \\
(1 - f(\varepsilon))^{-1} &= 1 + f(\varepsilon) + f(\varepsilon)^2 + \dots, \qquad \text{if } f(\varepsilon) = o(1).
\end{aligned}$$

THEOREM 5.1.13. *The coefficients a_n of an asymptotic expansion can be determined uniquely (for given $\mu_n(\varepsilon)$) by the following recursive procedure*

$$a_0 = \lim_{\varepsilon\to 0}\frac{f(\varepsilon)}{\mu_0(\varepsilon)}, \quad a_1 = \lim_{\varepsilon\to 0}\frac{f(\varepsilon) - a_0\mu_0(\varepsilon)}{\mu_1(\varepsilon)}, \ \dots \ a_N = \lim_{\varepsilon\to 0}\frac{f(\varepsilon) - \sum_{n=0}^{N-1} a_n\mu_n(\varepsilon)}{\mu_N(\varepsilon)},$$

provided μ_n are nonzero for ε near 0 and each of the limits exist.

THEOREM 5.1.14 (**Fundamental theorem of asymptotic expansions**). *An asymptotic expansion vanishes only if the coefficients vanish, i.e.*

$$\{a_0\mu_0(\varepsilon) + a_1\mu_1(\varepsilon) + a_2\mu_2(\varepsilon) + \dots = 0 \quad (\varepsilon \to 0)\} \ \Leftrightarrow \ \{a_0 = a_1 = a_2 = \dots = 0\}.$$

PROOF: *$\Leftarrow$ Is trivial. $\Rightarrow$ For any δ there is an interval with $|\mu_1| \leqslant \delta|\mu_0|$. It follows that $|a_0\mu_0| = |a_1\mu_1 + \dots| \leqslant |2\delta a_1\mu_0|$, which is only possible if $a_0 = 0$. The same for a_1, etc.*

DEFINITION 5.1.15 (**Poincaré expansion**). *Let $\{\mu_n(\varepsilon)\}$ be an asymptotic sequence of* **order functions**. *If $f(x,\varepsilon)$ has an asymptotic expansion with respect to this sequence, given by*

$$f(x,\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\mu_n(\varepsilon),$$

where the **shape functions** *$a_n(x)$ are* **independent** *of ε, then this expansion is called a Poincaré expansion.* Note*: a Poincaré expansion is* never *Poincaré anymore after (nontrivial) rescaling x.*

DEFINITION 5.1.16 (**Regular and singular expansion**). *If a Poincaré expansion is uniform in x on a given domain $\mathcal{D}$ this expansion is called a regular expansion. Else, the expansion is called a singular expansion.*

REMARK 5.1.17. *A typical indication for non-uniformity is a scaling, such that the asymptotic ordering of terms is violated. In other words, an $x = x(\varepsilon)$ with $a_1(x)\mu_1(\varepsilon) \not\ll a_0(x)\mu_0(\varepsilon)$, etc.*

Indeed, regular and singular depends (among other things) on the chosen variable x and the domain. For example, $\mathrm{e}^{-x/\varepsilon} + \sin(x+\varepsilon) = \sin(x) + O(\varepsilon)$ is regular on any positive interval $[a,b]$ with $a, b = O(1)$ but is singular on $(0,b]$, while $\mathrm{e}^{-t} + \sin(\varepsilon t + \varepsilon) = \mathrm{e}^{-t} + \varepsilon(t+1) + O(\varepsilon^3)$ is regular on any finite fixed interval.

THEOREM 5.1.18 (**Elementary manipulations of asymptotic expansions**). *Let $f(x,\varepsilon)$ and $g(x,\varepsilon)$ have Poincaré expansions on $\mathcal{D}$ with asymptotic sequence $\{\mu_n(\varepsilon)\}$*

$$f(x,\varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots$$
$$g(x,\varepsilon) = \mu_0(\varepsilon)b_0(x) + \mu_1(\varepsilon)b_1(x) + \cdots$$

Addition. Then the sum has the following asymptotic expansion

$$f + g = \mu_0(a_0 + b_0) + \mu_1(a_1 + b_1) + \cdots$$

Multiplication. If $\{\mu_k\mu_m\}$ can be asymptotically ordered to the asymptotic sequence $\{\gamma_n\}$, with $\gamma_0 = \mu_0^2$, $\gamma_1 = \mu_0\mu_1$, $\gamma_2 = O(\mu_0\mu_2 + \mu_1^2)$, etc., then the product has the asymptotic expansion

$$fg = (\mu_0 a_0 + \mu_1 a_1 + \cdots)(\mu_0 b_0 + \mu_1 b_1 + \cdots) = \gamma_0 a_0 b_0 + \gamma_1(a_0 b_1 + a_1 b_0) + \gamma_2(\cdots) + \cdots$$

Integration. If the approximation is uniform, f, a_0, a_1, etc. are absolute-integrable on $\mathcal{D}$, while $\int_{\mathcal{D}} a_n\,dx \neq 0$, then we can integrate term by term and obtain the asymptotic expansion

$$\int_{\mathcal{D}} f(x,\varepsilon)\,dx = \mu_0(\varepsilon)\int_{\mathcal{D}} a_0(x)\,dx + \mu_1(\varepsilon)\int_{\mathcal{D}} a_1(x)\,dx + \cdots$$

Differentiation. This is the least obvious. Consider the counter example

$$f(x,\varepsilon) = \tfrac{1}{2}x^2 + \varepsilon\cos\left(\tfrac{x}{\varepsilon}\right) = \tfrac{1}{2}x^2 + O(\varepsilon), \quad \text{but} \quad f'(x,\varepsilon) = x - \sin\left(\tfrac{x}{\varepsilon}\right) \neq x + O(\varepsilon).$$

However, if both f and f' have asymptotic expansions with asymptotic sequence $\{\mu_n(\varepsilon)\}$,

$$f(x,\varepsilon) = \mu_0(\varepsilon)a_0(x) + \mu_1(\varepsilon)a_1(x) + \cdots, \qquad f'(x,\varepsilon) = \mu_0(\varepsilon)q_0(x) + \mu_1(\varepsilon)q_1(x) + \cdots$$

then the derivative of the expansion of f is the expansion of derivative f', and satisfy

$$a_0' = q_0, \quad a_1' = q_1, \quad \text{etc.}$$

5.2 Integrals and Watson's Lemma

In the following sections we will consider methods to determine the asymptotic behaviour of functions defined by integrals. From Theorem 5.1.10 we know that a uniform asymptotic approximation can be integrated directly. For a non-uniform approximation the situation is more subtle. An example is the following integral. The result is important in its own right but in particular the proof is typical and interesting.

THEOREM 5.2.1 (**A Result for Cauchy integrals**). *Let f be locally integrable*[1] *in $\mathbb{R}$, such that there is a $p > 0$ with*

$$f(t) = O(|t|^{-p}) \quad \text{for} \quad t \to \pm\infty.$$

Then the following Cauchy-type integral in $z \in \mathbb{C}$, $z \notin \mathbb{R}$ has the asymptotic behaviour

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt = \begin{cases} O(z^{-p}) & \text{if} \quad 0 < p < 1, \\ O(z^{-1}\log z) & \text{if} \quad p = 1, \\ O(z^{-1}) & \text{if} \quad p > 1, \end{cases} \qquad \text{for} \quad |z| \to \infty, \quad |\arg(\pm z)| \geqslant \delta > 0.$$

PROOF: *We consider the right half-range integral on $(0,\infty)$ first. The left half is analogous. By definition there are numbers K and t_0 such that*

$$|f(t)| \leqslant K t^{-p} \quad \text{for} \quad t > t_0.$$

We split up the integral and apply the above.

$$\left|\int_0^{\infty} \frac{f(t)}{t-z}\,dt\right| \leqslant \int_0^{\infty} \left|\frac{f(t)}{t-z}\right| dt \leqslant \int_0^{t_0} \frac{|f(t)|}{|t-z|}\,dt + \int_{t_0}^{\infty} \frac{K}{t^p|t-z|}dt$$

Write $z = r\,e^{i\vartheta}$ and assume that $r \geqslant 2t_0$ such that along $[0, t_0]$ we have $|t-z| \geqslant r - t \geqslant \frac{1}{2}r$, then

$$\int_0^{t_0} \frac{|f(t)|}{|t-z|}\,dt \leqslant \frac{2}{r}\int_0^{t_0} |f(t)|\,dt.$$

If $0 < p < 1$, then t^{-p} is integrable at $t = 0$. We can estimate $|t-z| \geqslant \left|\sin\frac{1}{2}\vartheta\right|(t+r)$. Then

$$\int_{t_0}^{\infty} \frac{K}{t^p|t-z|}dt \leqslant \int_0^{\infty} \frac{K}{t^p|t-z|}dt \leqslant \frac{K}{r^p\left|\sin\frac{1}{2}\vartheta\right|}\int_0^{\infty} \frac{1}{\tau^p(\tau+1)}d\tau = \frac{K}{r^p\left|\sin\frac{1}{2}\vartheta\right|}\frac{\pi}{\sin(\pi p)}.$$

If $p > 1$, we find similarly

$$\int_{t_0}^{\infty} \frac{K}{t^p|t-z|}dt \leqslant \frac{K}{r\left|\sin\frac{1}{2}\vartheta\right|}\int_{t_0}^{\infty} \frac{1}{t^p}dt = \frac{K}{r\left|\sin\frac{1}{2}\vartheta\right|}\frac{t_0^{1-p}}{p-1}.$$

If $p = 1$, we find

$$\int_{t_0}^{\infty} \frac{K}{t|t-z|}dt \leqslant \frac{K}{r\left|\sin\frac{1}{2}\vartheta\right|}\int_{t_0}^{\infty}\Big(\frac{1}{t} - \frac{1}{t+r}\Big)dt = \frac{K}{r\left|\sin\frac{1}{2}\vartheta\right|}\Big(\log r + \log\big(t_0^{-1} + r^{-1}\big)\Big).$$

The results follow now immediately.

[1] $|f|$ is integrable on any finite interval.

Another important type is the following Laplace integral[2]. If $f(t)$ is N times continuously differentiable on $[0, \infty)$ and bounded, then the main contribution for $\varepsilon \to 0$ of

$$\int_0^\infty f(t)\,\mathrm{e}^{-t/\varepsilon}\,dt$$

comes from the neighbourhood of $t = 0$, because elsewhere the function is exponentially small. We can utilise this by splitting the integration interval in a convenient way as follows

$$= \int_0^{\sqrt{\varepsilon}} + \int_{\sqrt{\varepsilon}}^\infty f(t)\,\mathrm{e}^{-t/\varepsilon}\,dt = \varepsilon \int_0^{1/\sqrt{\varepsilon}} f(\varepsilon y)\,\mathrm{e}^{-y}\,dy + \int_{\sqrt{\varepsilon}}^\infty f(t)\,\mathrm{e}^{-t/\varepsilon}\,dt$$

The last integral is exponentially small because

$$\left|\int_{\sqrt{\varepsilon}}^\infty f(t)\,\mathrm{e}^{-t/\varepsilon}\,dt\right| \leqslant \int_{\sqrt{\varepsilon}}^\infty |f(t)\,\mathrm{e}^{-t/\varepsilon}|dt \leqslant \int_{\sqrt{\varepsilon}}^\infty K\,\mathrm{e}^{-t/\varepsilon}\,dt = K\varepsilon\,\mathrm{e}^{-1/\sqrt{\varepsilon}}.$$

Since $f(\varepsilon y) = f(0) + \varepsilon y f'(0) + \frac{1}{2}\varepsilon^2 y^2 f''(0) + \ldots$ (uniformly) for $\varepsilon y \in [0, \sqrt{\varepsilon})$, we have finally

$$= \varepsilon \int_0^{1/\sqrt{\varepsilon}} \big(f(0) + \varepsilon y f'(0) + \ldots\big)\,\mathrm{e}^{-y}\,dy + O(\varepsilon\,\mathrm{e}^{-1/\sqrt{\varepsilon}}) = \varepsilon f(0) + \varepsilon^2 f'(0) + \cdots + O(\varepsilon^{N+1}).$$

This result is a special case of

THEOREM 5.2.2 (**Watson's Lemma**). *Let $f(t)$ be continuous[3] on $(0, \infty)$ and exponentially bounded, i.e. there is a real constant $c > 0$ such that $f(t) = O(\mathrm{e}^{ct})$ for $t \to \infty$. There are real constants $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ such that f has an asymptotic expansion for $t \to 0$ given by*

$$f(t) \sim \sum_{n=0}^{N} a_n t^{\lambda_n - 1} \quad \text{for} \quad t \downarrow 0.$$

Then f's Laplace transform has the following asymptotic expansion for $s \to \infty$

$$F(s) = \int_0^\infty \mathrm{e}^{-st}\,f(t)\,dt \sim \sum_{n=0}^{N} a_n \frac{\Gamma(\lambda_n)}{s^{\lambda_n}} \quad \text{for} \quad s \to \infty \quad \text{and} \quad |\arg(s)| \leqslant \beta \in (0, \tfrac{1}{2}\pi)$$

PROOF: *Since f is continuous on $(0, \infty)$ and $O(\mathrm{e}^{ct})$, there is for any $L > 0$ a constant M with $|f(t)| \leqslant M\,\mathrm{e}^{ct}$ on $[L, \infty)$. Then for any s with $\mathrm{Re}(s) > c$ we have*

$$\left|\int_L^\infty f(t)\,\mathrm{e}^{-st}\,dt\right| \leqslant \int_L^\infty \left|f(t)\,\mathrm{e}^{-st}\right|\,dt \leqslant M \int_L^\infty \mathrm{e}^{-\mathrm{Re}(s)t + ct}\,dt = \frac{M\,\mathrm{e}^{cL}}{\mathrm{Re}(s) - c}\,\mathrm{e}^{-\mathrm{Re}(s)L} = O(\mathrm{e}^{-sL}),$$

which is asymptotically for $s \to \infty$ smaller than any power of s. In a similar way is

$$\int_0^\infty t^{\lambda_n - 1}\,\mathrm{e}^{-st}\,dt = s^{-\lambda_n}\Gamma(\lambda_n) = \int_0^L t^{\lambda_n - 1}\,\mathrm{e}^{-st}\,dt + O(\mathrm{e}^{-sL}).$$

[2] A Laplace integral is an integral in the form of a Laplace transform.

[3] This can be relaxed, *e.g.* to piecewise continuous.

Choose a real constant K. Then (by assumption) there is an $L > 0$ such that

$$\left| f(t) - \sum_{n=0}^{N} a_n t^{\lambda_n - 1} \right| \leqslant K t^{\lambda_N - 1} \quad \textit{for} \quad 0 < t < L.$$

Hence for s with $\mathrm{Re}(s) > 0$

$$\left| \int_0^L \left(f(t) - \sum_{n=0}^{N} a_n t^{\lambda_n - 1} \right) \mathrm{e}^{-st}\, dt \right| \leqslant K \int_0^\infty t^{\lambda_N - 1}\, \mathrm{e}^{-\mathrm{Re}(s)t}\, dt = K\Gamma(\lambda_N)\, \mathrm{Re}(s)^{-\lambda_N} \leqslant K\Gamma(\lambda_N) |s^{-\lambda_N}|.$$

We write symbolically f for $f(t)\,\mathrm{e}^{-st}$ and Σ for $\sum_{n=0}^{N} a_n t^{\lambda_n - 1}\,\mathrm{e}^{-st}$, and split

$$\int_0^\infty f\, dt = \int_0^\infty (f - \Sigma)\, dt + \int_0^\infty \Sigma\, dt = \int_0^\infty \Sigma\, dt + \int_0^L (f - \Sigma)\, dt + \int_L^\infty f\, dt - \int_L^\infty \Sigma\, dt.$$

After taking all parts together, we have, for $\mathrm{Re}(s)$ *large enough (i.e. for large s inside a cone $|\arg(s)| \leqslant \beta \in [0, \frac{1}{2}\pi)$), the claimed result*

$$F(s) = \sum_{n=0}^{N} a_n \frac{\Gamma(\lambda_n)}{s^{\lambda_n}} + o(s^{-\lambda_N}).$$

In short, Watson's Lemma tells us when integration and asymptotic expansion in t can be exchanged, to result in an asymptotic expansion in s^{-1}.

CONSEQUENCE 5.2.3. *Any finite integral $\int_0^L f(t)\,\mathrm{e}^{-st}\, dt$ of a function of the form $f(t) = t^\sigma g(t)$, with $\sigma > -1$ and $g(t)$ analytic in $t = 0$, satisfies the conditions of Watson's Lemma 5.2.2.*

EXAMPLE 5.2.4.

$$\int_0^\infty \frac{\mathrm{e}^{-st}}{t+1}\, dt \sim \sum_{n=0} (-1)^n \frac{n!}{s^{n+1}}$$

$$\int_0^\infty \mathrm{e}^{-st} \log(1 + t^2)\, dt \sim \sum_{n=1} \frac{(-1)^{n+1}}{n} \frac{(2n)!}{s^{2n+1}}$$

$$\int_0^{\frac{1}{2}\pi} \mathrm{e}^{-s\tan^2(\vartheta)}\, d\vartheta = \frac{1}{2} \int_0^\infty \frac{\mathrm{e}^{-st}}{\sqrt{t}(1+t)}\, dt \sim \frac{1}{2} \sum_{n=0} (-1)^n \frac{\Gamma(n + \frac{1}{2})}{s^{n+\frac{1}{2}}}$$

Watson's Lemma is stronger than might appear at first sight. Many integrals can be recast by a coordinate transformation into the required form. More examples can be found in the exercises.

5.3 Laplace's Method

For generalisations of Laplace integrals, *i.e.* integrals of the form

$$f(s) = \int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,dt$$

(with g and h sufficiently smooth) considered for $s \to \infty$, we may typically have the dominant contribution at an end of the interval or somewhere halfway.

Dominant contribution at left end. The dominant contribution is near $t = a$ if h is strict-monotonically increasing near a, *i.e.* $h'(a) > 0$, and h remains sufficiently bounded from below along the rest of the interval. Assume $g(a) \neq 0$. By a variant of Theorem 5.2.2 we have then

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,dt \simeq \int_a^{a+s^{-\frac{1}{2}}} g(a)\,\mathrm{e}^{-sh(a)-s(t-a)h'(a)}\,dt \simeq \frac{g(a)\,\mathrm{e}^{-sh(a)}}{s}\int_0^{s^{\frac{1}{2}}} \mathrm{e}^{-yh'(a)}\,dt \simeq \frac{g(a)\,\mathrm{e}^{-sh(a)}}{sh'(a)}.$$

Dominant contribution at right end. The dominant contribution is near $t = b$ if h is strict-monotonically decreasing near b, *i.e.* $h'(b) < 0$, and h remains sufficiently bounded from below along the rest of the interval. In a similar way as before (assume $g(b) \neq 0$) we have then

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,dt \simeq \int_{b-s^{-\frac{1}{2}}}^{b} g(b)\,\mathrm{e}^{-sh(b)-s(t-b)h'(b)}\,dt \simeq \frac{g(b)\,\mathrm{e}^{-sh(b)}}{s}\int_{-s^{\frac{1}{2}}}^{0} \mathrm{e}^{-yh'(b)}\,dy \simeq -\frac{g(b)\,\mathrm{e}^{-sh(b)}}{sh'(b)}.$$

Dominant contribution halfway (Laplace's Method). Let h have an absolute minimum in $c \in (a, b)$, such that[4] $h'(c) = 0$, $h''(c) > 0$ and $h(t) = h(c) + \frac{1}{2}(t-c)^2 h''(c) + \dots$. Let $g(c) \neq 0$ and h be sufficiently bounded from below along the rest of the interval. Then we obtain, by using $\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha t^2}\,dt = \sqrt{\pi/\alpha}$, the asymptotic approximation for $s \to \infty$

$$\int_a^b g(t)\,\mathrm{e}^{-sh(t)}\,dt \simeq \int_{c-s^{-\frac{1}{4}}}^{c+s^{-\frac{1}{4}}} g(c)\,\mathrm{e}^{-sh(c)-\frac{1}{2}s(t-c)^2h''(c)}\,dt \simeq \sqrt{\frac{2\pi}{sh''(c)}}\,g(c)\,\mathrm{e}^{-sh(c)}\,.$$

(Note: $O(s^{-\frac{1}{2}})$ here vs. $O(s^{-1})$ at the ends.) Proofs may be constructed in a similar way as with Watson's Lemma (5.2.2), by splitting the integration interval in an asymptotically small region near a, b, or c respectively for the dominant contribution, and a rest with a negligible contribution.

EXAMPLE 5.3.1. *The modified Bessel function $K_0(z)$ is for $z \to \infty$*

$$K_0(z) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z\cosh(t)}\,dt \sim \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z},$$

by noting that $h(t) = \cosh(t)$ satisfies $h(0) = 1$, $h'(0) = 0$ and $h''(0) = 1$.

EXAMPLE 5.3.2. *A famous example is **Stirling's formula**. Via the transformation $t = n\tau$ is*

$$n! = \Gamma(n+1) = \int_0^\infty t^n\,\mathrm{e}^{-t}\,dt = n^{n+1}\int_0^\infty \mathrm{e}^{-n(\tau-\log\tau)}\,d\tau \sim \sqrt{2\pi}n^{n+\frac{1}{2}}\,\mathrm{e}^{-n} \quad (n \to \infty),$$

by noting that $h(\tau) = \tau - \log(\tau)$ has $h(1) = 1$, $h'(1) = 0$ and $h''(1) = 1$.

[4]The case for vanishing higher derivatives is analogous.

5.4 Method of Stationary Phase

A generalisation of the above for Fourier-type integrals, *i.e.* integrals of the form

$$f(s) = \int_a^b g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}\; dt$$

(with g absolutely integrable and h strictly monotonic except in discrete points) considered for $s \to \infty$, is similar, although there are differences. For example, proving the vanishing of the non-contributing parts of the integral takes more work. For this we need a version of the Riemann-Lebesgue Lemma including a generalisation.

LEMMA 5.4.1 (**Riemann-Lebesgue Lemma**). *If the function $\varphi(x)$ is absolutely integrable*[5]*, then the (finite or infinite) Fourier integral*

$$\int_a^b \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx \to 0 \quad \text{as} \quad |s| \to \infty.$$

PROOF: *A sketch of the proof is as follows. We write*

$$\int_a^b \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx = -\int_a^b \varphi\left(x + \frac{\pi}{s}\right) \mathrm{e}^{\mathrm{i}sx}\, dx + \int_a^{a+\frac{\pi}{s}} \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx - \int_b^{b+\frac{\pi}{s}} \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx$$

by a simple substitution, whence

$$\begin{aligned} 2\left|\int_a^b \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx\right| &= \left|\int_a^b \left\{\varphi(x) - \varphi\left(x + \frac{\pi}{s}\right)\right\} \mathrm{e}^{\mathrm{i}sx}\, dx + \int_a^{a+\frac{\pi}{s}} - \int_b^{b+\frac{\pi}{s}} \varphi(x)\, \mathrm{e}^{\mathrm{i}sx}\, dx\right| \\ &\leqslant \int_a^b \left|\varphi(x) - \varphi\left(x + \frac{\pi}{s}\right)\right| dx + \int_a^{a+\frac{\pi}{s}} |\varphi(x)|\, dx + \int_b^{b+\frac{\pi}{s}} |\varphi(x)|\, dx \end{aligned}$$

which tend to 0 *as* $|s| \to \infty$ *by fundamental theorems of integration.*

COROLLARY 5.4.2. *If the function $\mu(x)$ is strictly monotonic, with $|\mu'(x)| > \delta > 0$, such that we can define the inverse $x = \mu^{-1}(z)$, then*

$$\int_a^b \varphi(x)\, \mathrm{e}^{\mathrm{i}s\mu(x)}\, dx = \int_{\mu(a)}^{\mu(b)} \frac{\varphi(\mu^{-1}(z))}{\mu'(\mu^{-1}(z))}\, \mathrm{e}^{\mathrm{i}sz}\, dz \to 0 \quad \text{as} \quad |s| \to \infty.$$

If we return to the original integral, we may typically have the dominant contribution at the ends of the interval or somewhere halfway.

Dominant contribution at the ends. The dominant contribution is at the ends if h is strictly monotonic, such that we can write by partial integration

$$f(s) = \int_a^b g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}\; dt = \left.\frac{g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}}{\mathrm{i}sh'(t)}\right|_a^b - \frac{1}{\mathrm{i}s} \int_a^b \frac{d}{dt}\left[\frac{g(t)}{h'(t)}\right] \mathrm{e}^{\mathrm{i}sh(t)}\; dt \simeq \left.\frac{g(t)\, \mathrm{e}^{\mathrm{i}sh(t)}}{\mathrm{i}sh'(t)}\right|_a^b$$

since by the Riemann-Lebesgue Lemma the second integral decays faster than $O(1/s)$.

[5] This condition can be relaxed to piecewise continuous and uniform convergence at the ends and discontinuities.

Dominant contribution halfway (Method of Stationary Phase). Let h have a 2-nd order stationary point in $c \in (a, b)$, such that $h'(c) = 0$, $h''(c) \neq 0$ and $h(t) = h(c) + \frac{1}{2}(t-c)^2 h''(c) + \dots$, while h is sufficiently smooth and strictly monotonic along the rest of the interval, and $g(c) \neq 0$. Then we obtain, by using $\int_{-\infty}^{\infty} \mathrm{e}^{\pm \mathrm{i}\alpha t^2}\, dt = \mathrm{e}^{\pm\frac{1}{4}\pi\mathrm{i}} \sqrt{\pi/\alpha}$, the asymptotic approximation for $s \to \infty$

$$\int_a^b g(t)\,\mathrm{e}^{\mathrm{i}sh(t)}\, dt \simeq \int_{c-s^{-\frac{1}{4}}}^{c+s^{-\frac{1}{4}}} g(c)\,\mathrm{e}^{\mathrm{i}sh(c)+\frac{1}{2}\mathrm{i}s(t-c)^2 h''(c)}\, dt \simeq g(c)\,\mathrm{e}^{\mathrm{i}sh(c)}\,\mathrm{e}^{\pm\frac{1}{4}\pi\mathrm{i}} \sqrt{\pm\frac{2\pi}{sh''(c)}}\,,$$

where the $\pm$ sign can be chosen according to what is most convenient. Similar to Laplace's Method, a contribution here is $O(s^{-\frac{1}{2}})$, while $O(s^{-1})$ at the ends. The limit $s \to -\infty$, and stationary points of higher order can be dealt with in an analogous way (make sure to distinguish even and odd orders). Proofs may be found by splitting the integration interval in an asymptotically small region near c, and a remaining part where the Riemann-Lebesgue Lemma can be applied.

EXAMPLE 5.4.3. *The n-th order Bessel function of the first kind $J_n(x)$ (section 6.1) has an asymptotic behaviour for large values of the argument x, given by*

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\pi} \mathrm{e}^{\mathrm{i}x\sin t - \mathrm{i}nt}\, dt\Big] \\ &\simeq \frac{1}{\pi}\,\mathrm{Re}\left[\mathrm{e}^{-\mathrm{i}n\frac{1}{2}\pi}\,\mathrm{e}^{\mathrm{i}x\sin(\frac{1}{2}\pi)}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\sqrt{\frac{2\pi}{x\sin(\frac{1}{2}\pi)}}\,\right] = \sqrt{\frac{2}{x\pi}}\cos\big(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\big) \end{aligned}$$

due to the stationary point at $t = \frac{1}{2}\pi$.

EXAMPLE 5.4.4. *The asymptotic behaviour of $J_n(n)$ for large n cannot be found by the standard formula. However, a slight adaptation following the same lines of reasoning yields*

$$J_n(n) = \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\pi} \mathrm{e}^{\mathrm{i}n(\sin t - t)}\, dt\Big] \simeq \frac{1}{\pi}\,\mathrm{Re}\Big[\int_0^{\infty} \mathrm{e}^{-\mathrm{i}\frac{1}{6}nt^3}\, dt\Big] = \frac{2^{\frac{1}{3}}}{n^{\frac{1}{3}}3^{\frac{2}{3}}\Gamma(\frac{2}{3})}$$

due to phase function $h(t) = \sin(t) - t = -\frac{1}{6}t^3 + \dots$, expanded around the stationary point $t = 0$.

EXAMPLE 5.4.5. *Another example that requires some preparation is the integral*

$$\int_0^{\frac{1}{2}\pi} t\sin(x\cos t)\, dt = \mathrm{Im}\Big[\int_0^{\frac{1}{2}\pi} t\,\mathrm{e}^{\mathrm{i}x\cos t}\, dt\Big]$$

The point of stationary phase is at left end $t = 0$, which would normally lead to half its contribution, but at the same time the function $g(t) = t$ vanishes there. However, by partial integration we can compensate for this and find the dominating contributions from the ends

$$= \mathrm{Im}\left[\frac{\mathrm{i}t}{x\sin t}\,\mathrm{e}^{\mathrm{i}x\cos t}\Big|_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \frac{d}{dt}\Big(\frac{\mathrm{i}t}{x\sin t}\Big)\mathrm{e}^{\mathrm{i}x\cos t}\, dt\right] = \frac{1}{x}\big(\tfrac{1}{2}\pi - \cos x\big) + o(x^{-1}).$$

5.5 Method of Steepest Descent or Saddle Point Method

The Method of Steepest Descent (Peter Debije, 1909) is essentially a recipe to optimise the integration contour of an integral (for analytic functions g and h) of the form

$$f(s) = \int_{\mathcal{C}} g(z)\, \mathrm{e}^{-sh(z)}\, dz,$$

such that the integral is, after parametrisation, amenable to Laplace's method.

As will be shown, the contribution to the integral comes primarily from the neighbourhood of the point (or points) where $h'(z) = 0$. Write $h = u + \mathrm{i}v$. Suppose h' vanishes at a point z_0 with $h(z_0) = u_0 + \mathrm{i}v_0$. (We start with the assumption that $h''(z_0) \neq 0$, but higher order generalisations are similar.) In order to make the integral not oscillatory anymore, we deform the contour $\mathcal{C}$ (at least for the part where h dominates) into a contour $\mathcal{L}$ where $v = v_0$ is constant. Such a contour is at the same time a contour of **steepest descent** for u: (i) ∇v is orthogonal to level curve $\mathcal{L}$; (ii) because of the Cauchy-Riemann relations is ∇u, the direction where u varies most, orthogonal to ∇v, and hence directed along $\mathcal{L}$.

Since $h'(z_0) = 0$, $h(z) = h(z_0) + \frac{1}{2}h''(z_0)(z - z_0)^2 + \ldots$. If we write $\frac{1}{2}h''(z_0) = \rho\, \mathrm{e}^{\mathrm{i}\alpha}$ and $z = z_0 + r\, \mathrm{e}^{\mathrm{i}\vartheta}$, it follows that $u = u_0 + \rho r^2 \cos(2\vartheta + \alpha)$, from which can be concluded that z_0 is a **saddle point** of u. Therefore, there are two steepest descent contours $v = v_0$: one of hill-type and one of valley-type, which intersect at z_0. Obviously, only the valley-type is useful, where u has a minimum at $z = z_0$. Along $\mathcal{L}$ we have thus $h = u + \mathrm{i}v_0$ where u is real. With parametrisation $z = \gamma(t)$, with $z_0 = \gamma(0)$, is then $h(\gamma(t)) = h(z_0) + \frac{1}{2}\beta t^2 + \ldots$ where $\beta = h''(z_0)\gamma'(0)^2$ is by the above construction real positive. Taking note of the direction z_0 is crossed, we obtain

$$f(s) = \int_{\mathcal{L}} g(z)\, \mathrm{e}^{-sh(z)}\, dz = \int_a^b g(\gamma(t))\, \mathrm{e}^{-sh(\gamma(t))}\, \gamma'(t)\, dt \simeq \pm\sqrt{\frac{2\pi}{sh''(z_0)}}\, g(z_0)\, \mathrm{e}^{-sh(z_0)}\,.$$

The method is best explained by an example:

EXAMPLE 5.5.1. *A classic example is Hankel's asymptotic expansion of the Bessel functions. Consider (see section 6.1) the representation of the n-th order Hankel function of the first kind*

$$H_n^{(1)}(s) = \frac{1}{\pi\mathrm{i}} \int_{\mathcal{C}} \mathrm{e}^{s \sinh z - nz}\, dz,$$

with integration contour $\mathcal{C}$ from $-\infty$ to $\infty + \pi\mathrm{i}$. We are interested in its behaviour for $s \to \infty$.

Consider the landscape of the following function

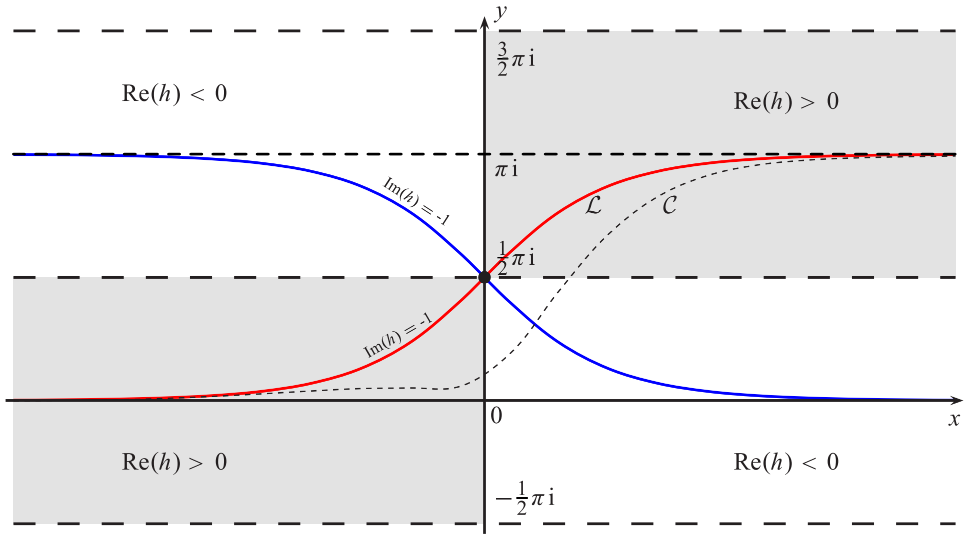
$$h(z) = -\sinh(z) = -\sinh(x)\cos(y) - \mathrm{i}\cosh(x)\sin(y).$$

$h(z)$ has a stationary point, where $h'(z) = 0$, at $z_0 = \frac{1}{2}\pi\mathrm{i}$. It is clearly a saddle point of $\mathrm{Re}(h)$, as $\mathrm{Re}(h)$ is negative in the right-lower and left-upper semi-infinite strip (see the figure below), and positive in the right-upper and left-lower strip (gray). Evidently, the integration contour $\mathcal{C}$ has to run from the left-lower to the right-upper strip, otherwise the integral would not converge.

There are two paths of steepest descent of $\mathrm{Re}(h)$ *through saddle point* z_0*. They are given by* $\mathrm{Im}(h) = -\cosh(x)\sin(y) = -1$*. After some algebra they are found to be given by*

$$y = 2\arctan(\mathrm{e}^{x}) \quad \textit{(red)}, \qquad y = 2\arctan(\mathrm{e}^{-x}) \quad \textit{(blue)}.$$

For the blue path, z_0 *is a maximum of* $\mathrm{Re}(h)$*, and this path is useless for our purposes. For the red path, however, it is a minimum. So we deform* $\mathcal{C}$ *into the red path* $\mathcal{L}$*, in order to be able to apply Laplace's method.*



With the following parametrisation γ *and its properties*

$$z = \gamma(t) = t + \mathrm{i}\,2\arctan(\mathrm{e}^{t}), \quad \gamma'(t) = 1 + \mathrm{i}\cosh(t)^{-1}, \quad \sinh\gamma(t) = -\sinh(t)\tanh(t) + \mathrm{i}$$

we obtain

$$H_n^{(1)}(s) = \frac{1}{\pi\,\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-s\gamma(t)-n\gamma(t)}\,\gamma'(t)\,dt = \frac{\mathrm{e}^{\mathrm{i}s}}{\pi\,\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-s\sinh(t)\tanh(t)-nt-\mathrm{i}2n\arctan(\mathrm{e}^{t})}(1+\mathrm{i}\cosh(t)^{-1})\,dt,$$

Since the contribution of the integrand[6] *is now concentrated near* $t = 0$ *(i.e.* z_0*), we have*

$$H_n^{(1)}(s) \simeq \frac{\mathrm{e}^{\mathrm{i}s}}{\pi\,\mathrm{i}}\int_{-\infty}^{\infty} \mathrm{e}^{-st^2-\mathrm{i}2n\arctan(1)}(1+\mathrm{i})\,dt = \sqrt{\frac{2}{\pi s}}\,\mathrm{e}^{\mathrm{i}s-\frac{1}{2}\pi n\mathrm{i}-\frac{1}{4}\pi\mathrm{i}} \qquad (s \to \infty).$$

[6] For $n \gtrsim \sqrt{s}$ and $st^2 = O(1)$, the term nt is not negligible against $-st^2$ and the approximation breaks down.

5.6 Applications

5.6.1 Group velocity

Many linear waves in x and t, for example water waves, are such that components $\sim \mathrm{e}^{-\mathrm{i}kx}$ of wave number k only exist in combination with components $\sim \mathrm{e}^{\mathrm{i}\omega t}$ of a specific frequency ω. This frequency is given by a relation $\omega = \Omega(k)$, called the dispersion relation. By superposition these waves can be written in general by the Fourier integral

$$\psi(x,t) = \int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}kx}\,dk = \int_{-\infty}^{\infty} g(k)\,\mathrm{e}^{\mathrm{i}t(\Omega(k) - k\frac{x}{t})}\,dk.$$

Applying the method of stationary phase we conclude that an observer, moving with velocity v along $x = vt$, sees for large t only the component of wave number k_v given by the stationary phase

$$\frac{d}{dk}\big(\Omega(k) - kv\big) = \Omega'(k) - v = 0.$$

In other words, these waves propagate with the group velocity, $c_g = \frac{d\omega}{dk} = \Omega'(k)$, rather than the phase velocity, $c_f = \frac{\omega}{k} = \Omega(k)/k$. With $\omega_v = \Omega(k_v)$ we have eventually

$$\psi(x,t) \simeq g(k_v)\,\mathrm{e}^{\mathrm{i}\omega_v t - \mathrm{i}k_v x}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sqrt{\frac{2\pi}{t\Omega''(k_v)}} \quad \text{for} \quad t \to \infty \quad \text{along} \quad x = vt.$$

Linear water waves on depth h with gravity g and neglecting surface tension satisfy $\omega = \Omega(k) = \sqrt{gk\tanh(kh)}$, which is for deep water $\simeq \sqrt{gk}$. The group velocity $c_g = \frac{1}{2}\sqrt{g/k}$ is then exactly half the phase velocity $c_f = \sqrt{g/k}$. This is nicely seen when throwing a stone in a pond.

5.6.2 Doppler effect of a moving sound source.

The observed pitch of a moving sound source of frequency ω_0 is higher if the source approaches the observer and lower if it recedes from it. This frequency shift, called *the Doppler effect*, occurs if the time scale of the tone ω_0^{-1} is much smaller than the time scale T of the motion, *i.e.* if $\omega_0 T \gg 1$.

Let the sound field $p(\boldsymbol{x},t)$ of a time-harmonic point source, of frequency ω_0 and moving subsonically along the path $\boldsymbol{x} = \boldsymbol{x}_S(t)$, be given by the following inhomogeneous wave equation

$$c_0^{-2}\tfrac{\partial^2}{\partial t^2}p - \nabla^2 p = 4\pi q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t}\,\delta(\boldsymbol{x} - \boldsymbol{x}_S(t)).$$

According to Liénard and Wiechert[7] the solution in free space is given by

$$p(\boldsymbol{x},t) = \frac{q_0\,\mathrm{e}^{\mathrm{i}\omega_0 t_e}}{R_e(1 - M_e\cos\vartheta_e)}$$

where $t_e = t_e(\boldsymbol{x},t)$ is the emission time. This is the time of emission of the signal that travelled (along a straight line with the sound speed c_0) from the source in $\boldsymbol{x}_S$ at time t_e to the observer in $\boldsymbol{x}$ at time t. It is a function of $\boldsymbol{x}$ and t, implicitly given by the equation

$$t = t_e + \big\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\big\|c_0^{-1}.$$

[7] The so-called **Liénard-Wiechert potential** is independently found in 1898 and 1900.

For subsonically moving sources, this equation has exactly one solution. Furthermore, the distance R_e between source and observer, the Mach number M_e of the source speed, and the angle ϑ_e between the observer direction and the source velocity, are functions of t_e and given by

$$R_e = \|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\|, \quad M_e = \frac{\|\dot{\boldsymbol{x}}_S(t_e)\|}{c_0}, \quad \cos\vartheta_e = \frac{(\boldsymbol{x} - \boldsymbol{x}_S(t_e)) \cdot \dot{\boldsymbol{x}}_S(t_e)}{\|\boldsymbol{x} - \boldsymbol{x}_S(t_e)\| \, \|\dot{\boldsymbol{x}}_S(t_e)\|}.$$

Let the typical time associated to the source motion be T, so we can write $\boldsymbol{x}_S(t) = X_0 \boldsymbol{\xi}(t/T)$, where X_0 characterises a typical position and $\boldsymbol{\xi} = O(1)$ is a dimensionless function. Assuming that time variations due to frequency ω_0 are much larger than T, we are interested in the Fourier-transformation in time of p.

For this we ignore for the moment the $\boldsymbol{x}$ dependence and consider for smoothly varying amplitude A and phase $\omega_0 T\varphi$ the slowly varying, *almost* harmonic signal p

$$p(t) = A(t/T)\,\mathrm{e}^{\mathrm{i}\omega_0 T\varphi(t/T)}, \quad \omega_0 T \gg 1.$$

If T is large compared to t, then $\mathrm{e}^{\mathrm{i}\omega_0 T\varphi(t/T)} = \mathrm{e}^{\mathrm{i}\omega_0 T\varphi(0) + \mathrm{i}\omega_0\varphi'(0)t + \dots}$ and we see that the observed frequency is about $\omega_0\varphi'(0)$. To generalise this for arbitrary t, we consider the Fourier transform

$$P(\omega) = \int_{-\infty}^{\infty} p(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,dt,$$

to see which part of the spectrum dominates and when. We assume that ω is of the order of magnitude of ω_0. Introduce the large parameter $\lambda = \omega_0 T$. Make time t dimensionless on the slow time scale, with $t = T\tau$. Scale ω on ω_0 such that $\omega = \omega_0\nu$, with $\nu = O(1)$. We obtain then

$$P(\omega) = P(\omega_0\nu) = T\int_{-\infty}^{\infty} A(\tau)\,\mathrm{e}^{\mathrm{i}\lambda\varphi(\tau) - \mathrm{i}\lambda\nu\tau}\,d\tau,$$

For large λ this becomes, by using the method of stationary phase,

$$P(\omega_0\nu) \simeq T A(\tau_s)\,\mathrm{e}^{\mathrm{i}\lambda\varphi(\tau_s) - \mathrm{i}\lambda\nu\tau_s}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}\sqrt{\frac{2\pi}{\lambda\varphi''(\tau_s)}}$$

with $\tau_s = \tau_s(\nu)$ defined by the stationary phase equation

$$\varphi'(\tau_s) = \nu.$$

In other words: to find frequency $\omega = \omega_0\nu = \omega_0\varphi'(t_s/T)$ we have to look at time $t = t_s = T\tau_s$. Therefore, $\omega_0\varphi'(t/T)$ is sometimes called the *instantaneous frequency*.

Returning to our original problem of the moving point source with $t_e = T\varphi(t/T)$, we find that the instantaneous frequency ω observed at position $\boldsymbol{x}$ and time t is then given by[8]

$$\omega = \omega_0\varphi'(t/T) = \omega_0\frac{dt_e}{dt} = \frac{\omega_0}{1 - M_e\cos\vartheta_e}.$$

This formula expresses the famous Doppler shift.

[8]Check. It requires implicit differentiation.

Chapter 6

Special functions

The following functions are not part of the exam, but they are important in applied analysis, especially in physical applications. They play a role in, and have been subject of asymptotic analysis.

6.1 Bessel Functions

The Bessel equation of non-negative integer order n

$$y'' + \frac{1}{z}y' + \Big(1 - \frac{n^2}{z^2}\Big)y = 0,$$

and therefore its solutions the Bessel Functions, appears naturally when the Laplace operator is rewritten in polar coordinates[1]. We have the following standard forms

$$\begin{aligned}
J_n(z) &= \frac{1}{\pi}\int_0^{\pi} \cos(z\sin t - nt)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}z\sin t - \mathrm{i}nt}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{2}z)^{n+2k}}{k!(n+k)!},\\
Y_n(z) &= \frac{1}{\pi}\int_0^{\pi} \sin(z\sin t - nt)\,dt - \frac{1}{\pi}\int_0^{\infty}\big[\mathrm{e}^{nt} + (-1)^n\,\mathrm{e}^{-nt}\big]\,\mathrm{e}^{-z\sinh t}\,dt = \frac{2}{\pi}\log(\tfrac{1}{2}z\,\mathrm{e}^{\gamma})J_n(z)\\
&\quad - \frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(\tfrac{1}{2}z)^{-n+2k} - \frac{1}{\pi}\sum_{k=0}^{\infty}\Big(\sum_{j=1}^{k}\frac{1}{j} + \sum_{j=1}^{n+k}\frac{1}{j}\Big)\frac{(-1)^k(\frac{1}{2}z)^{n+2k}}{k!(n+k)!},\\
H_n^{(1)}(z) &= J_n(z) + \mathrm{i}Y_n(z) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty+\pi\mathrm{i}} \mathrm{e}^{z\sinh t - nt}\,dt,\\
H_n^{(2)}(z) &= J_n(z) - \mathrm{i}Y_n(z) = -\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty-\pi\mathrm{i}} \mathrm{e}^{z\sinh t - nt}\,dt,
\end{aligned}$$

and the principal value log. J_n is called the ordinary n-th order Bessel Function of the 1st kind; Y_n is called the n-th order Bessel Function of the 2nd kind or Neumann Function; $H_n^{(1)}$, $H_n^{(2)}$ are called the n-th order Hankel Functions of the 1st and 2nd kind or Bessel Functions of the 3d kind.

[1] In 2D we have $\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}$. General solutions of $\nabla^2\phi = 0$ are found by separation of variables $\phi = f(r)g(\theta)$, leading to $\phi = \sum f_n(r)g_n(\theta)$ where $r^2 f_n'' + r f_n' + (r^2 - n^2)f_n = 0$ and $g_n'' + n^2 g_n = 0$, for $n \in \mathbb{Z}$.

6.2 Gamma Function

For $\mathrm{Re}(z) > 0$ is $\Gamma(z)$, known as the Gamma Function, and the related $z!$ defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}\,\mathrm{e}^{-t}\,dt, \qquad z! = \Gamma(z+1).$$

The identities $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(1) = 1$ yield indeed $\Gamma(n+1) = n(n-1)\cdots 1$. By pulling back like $\Gamma(z) = \Gamma(z+1)/z = \Gamma(z+2)/z(z+1) = \dots$, we extend the definition to all $z \in \mathbb{C}$. With Euler's Reflection Formula

$$\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$$

one may derive that Γ is analytic everywhere except for $z = -n$, $n = 0, 1, 2, \dots$ where it has simple poles with residue $(-1)^n/n!$. For the asymptotic behaviour of $\Gamma(z)$, see example 5.3.2.

The logarithmic derivative is known as the Digamma Function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \qquad \psi(z) \sim \log z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots \quad (z \to \infty \text{ in } |\arg(z)| \leqslant \pi - \delta)$$

For $z \in \mathbb{N}$ and Euler's Constant $\gamma = 0.5772156649\dots$ we have the result for the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \psi(n+1) + \gamma \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \cdots \quad (n \to \infty)$$

The Incomplete Gamma Functions are defined by the integrals

$$\Gamma(a,z) = \int_z^\infty t^{a-1}\,\mathrm{e}^{-t}\,dt, \qquad \gamma(a,z) = \int_0^z t^{a-1}\,\mathrm{e}^{-t}\,dt \quad (\mathrm{Re}(a) > 0).$$

Unless indicated otherwise, principal values are assumed with a branch cut along the negative real axis and the integration contours not crossing the negative real axis. Note that $\Gamma(0,z) = \mathrm{E}_1(z)$, $\Gamma(1,z) = \mathrm{e}^{-z}$, $\Gamma(\frac{1}{2}, z^2) = \sqrt{\pi}\,\mathrm{erfc}(z)$. Asymptotically for $z \to \infty$ and a fixed we have

$$\Gamma(a,z) = z^{a-1}\,\mathrm{e}^{-z}\left(\sum_{k=0}^{n-1} \frac{u_k}{z^k} + O(z^{-n})\right), \quad \text{in} \quad |\arg(z)| \leqslant \tfrac{3}{2}\pi - \delta.$$

where $u_0 = 1$, $u_k = (a-1)(a-2)\cdots(a-k)$.

A related function is $B(x,y)$, known as the Beta Function, and defined for $\mathrm{Re}\,x > 0$, $\mathrm{Re}\,y > 0$ by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt.$$

EXAMPLE 6.2.1. *Let $\alpha \in \mathbb{R}$ and $\beta, z \in \mathbb{C}$ with $\alpha > 0$ and $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,z > 0$, and a principal value power function. Then*

$$\int_0^\infty t^{\beta-1}\,\mathrm{e}^{-zt^\alpha}\,dt = \alpha^{-1} z^{-\beta/\alpha}\,\Gamma(\beta/\alpha).$$

EXAMPLE 6.2.2. *Let $z, \alpha \in \mathbb{C}$. Then*

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = 1 + \alpha z + \tfrac{1}{2}\alpha(\alpha-1)z^2 + \cdots, \quad \binom{\alpha}{n} = \frac{\alpha!}{n!(\alpha-n)!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

where the series is finite if $\alpha \in \mathbb{N} \cup \{0\}$. Otherwise, it converges absolutely for $|z| < 1$.

6.3 Dilogarithm and Exponential Integral

The Dilogarithm and the Exponential Integral are complex functions with much in common with the complex logarithm.

The Dilogarithm, defined by

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\ln(1-t)}{t}\,dt = \int_0^\infty \frac{zt}{\mathrm{e}^t - z}\,dt$$

has a branch point[2] in $z = 0$ and a branch cut along the negative real axis. Special values:

$$\mathrm{Li}_2(1) = \tfrac{1}{6}\pi, \qquad \mathrm{Li}_2(-1) = -\tfrac{1}{12}\pi.$$

Note that another definition is known, given by $\mathrm{dilog}(z) = \mathrm{Li}_2(1-z)$.

We define the Exponential Integral

$$\mathrm{E}_1(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,dt = -\gamma - \log(z) - \sum_{k=1}^{\infty} \frac{(-z)^k}{kk!} = -\gamma - \log(z) + \int_0^z \frac{1-\mathrm{e}^{-t}}{t}\,dt,$$

where $\gamma = 0.5772156649\ldots$ is Euler's Constant, and with a branch point in $z = 0$ and a branch cut along the negative real axis. Note that another definition is known, given by $\mathrm{Ei}(z) = -\mathrm{E}_1(-z)$.

6.4 Error Function

The Error Function is defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

The related Complementary Error Function is given by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \mathrm{e}^{-t^2}\,dt = 1 - \mathrm{erf}(z).$$

Special values:

$$\mathrm{erf}(\infty) = \mathrm{erfc}(0) = 1.$$

For large real x and any fixed $N \in \mathbb{N}$ we have asymptotically

$$\mathrm{erfc}(x) = \frac{\mathrm{e}^{-x^2}}{x\sqrt{\pi}} \left[1 + \sum_{n=1}^{N} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^n} + O(x^{-2N-1}) \right] \quad (x \to \infty).$$

[2] For other branches than the principal branch, there is a second branch point in $z = 1$.

Exercises Chapter 1

1.1

1. Determine real part, imaginary part, modulus and argument of

$$\frac{1-\mathrm{i}}{1+\mathrm{i}}, \quad \left(\frac{2+\mathrm{i}}{3-\mathrm{i}}\right)^2, \quad \frac{3-\mathrm{i}}{2+\mathrm{i}}+\frac{3+\mathrm{i}}{2-\mathrm{i}}.$$

2. (a) Show that $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2 + 2k\pi$, where $k = -1, 0$ or 1.
   (b) Show that $\overline{z_1}\,\overline{z_2} = \overline{z_1z_2}$ and that $1/\overline{z} = \overline{1/z}$.
   (c) Sketch in the complex plane the set of all z satisfying $(z-\mathrm{i})(\overline{z}+\mathrm{i}) = 4$.

3. (a) Prove the triangle inequality $|z_1 \pm z_2| \leqslant |z_1| + |z_2|$.
   (b) Derive from this the inequality $|z_1 \pm z_2| \geqslant ||z_1| - |z_2||$.

4. As in §1.1 of the lecture notes we define for a complex number $c = a + \mathrm{i}b$:

$$L_c = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

   (a) Prove that $L_{cz} = L_cL_z$ and $L_{c+z} = L_c + L_z$.
   (b) Determine the eigenvalues of L_c.

5. For which $z \in \mathbb{C}$ do we have

$$\text{a)} \quad \left|\frac{z+1}{z}\right|^2 = 1, \qquad \text{b)} \quad \left(\frac{z+1}{z}\right)^2 = 1,$$

$$\text{c)} \quad z^2 = |z|^2, \qquad \text{d)} \quad z + \frac{1}{z} \text{ is real?}$$

6. Sketch in the complex plane the set of all z with

$$\text{a)} \quad \arg\left(\frac{z-\mathrm{i}}{z+\mathrm{i}}\right) = \frac{\pi}{2}, \qquad \text{b)} \quad \operatorname{Im}\left(\frac{z-3}{z+2\mathrm{i}}\right) = 0,$$

$$\text{c)} \quad |z-3\mathrm{i}| = |4+2\mathrm{i}-z|, \quad \text{d)} \quad |z+1-\mathrm{i}|^2 = 1 \text{ and } \tfrac{1}{2}\pi \leqslant \arg z \leqslant \tfrac{3}{4}\pi.$$

7. Solve the following equations in $\mathbb{C}$:

   a) $z^3 = -\mathrm{i}$, b) $z^4 + 2z^2 + 4 = 0$,

   c) $z^5 - \mathrm{i}z^3 + \mathrm{i}z^2 + 1 = 0$, d) $(z + 2 - \mathrm{i})^6 = 27\mathrm{i}$.

8. Prove that

   (a) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z}_2)$.
   (b) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

   Give a geometrical interpretation of the second identity, which is known as the **parallelogram equality**.

9. For non-zero $z, w \in \mathbb{C}$ show that if $z\overline{w} - \overline{z}w = 0$, then $z = cw$ for some real c.

10. Cassini's Oval is a closed curve, described by $|z^2 - c^2| = 1$ with $c \in [0, 1]$. The limiting cases $c = 0$ and $c = 1$ correspond to the unit circle and Bernoulli's Lemniscate, respectively.

    (a) Find a parametrisation $r = R(\vartheta)$ of the curve in polar coordinates.
    (b) Show that the enclosed area equals $4E(c^2)$ where $E(k) = \int_0^{\frac{1}{2}\pi} \sqrt{1 - k^2 \sin^2 t}\, dt$.

1.2

1. What are the interior points and what are the boundary points of the sets mentioned in Exercise 1.1.6?

2. Which of the sets mentioned in Exercise 1.1.6 are open, closed, bounded?

1.3

1.

   (a) Show that $|z^2 + 2z + 3\mathrm{i}| \geqslant R^2 - 2R - 3 > 0$, if $|z| = R > 3$.
   HINT. Use two times the triangle inequality.
   (b) Prove that

   $$\left|\frac{z - 4}{z^2 + 2z + 3\mathrm{i}}\right| \to 0 \quad (|z| \to \infty).$$

2. Determine the sum of the series $\sum_{n=0}^{\infty} r^n \cos(n\vartheta)$, where $0 \leqslant r < 1$ and $\vartheta \in \mathbb{R}$.

3. Consider the function
$$f(z) = \frac{z^2 - 1}{|z-1|}.$$
   (a) Let $\vartheta \in \mathbb{R}$ be given. Determine $\lim_{r\to 0} f(1 + r\,e^{i\vartheta})$.
   (b) Does $\lim_{z\to 1} f(z)$ exist?
4. Prove that $f(z) = u(x, y) + iv(x, y)$ is continuous in $\mathbb{C}$ if and only if $u(x, y)$ and $v(x, y)$ are continuous in $\mathbb{R}^2$.
5. For which $a, b \in \mathbb{C}$ is $\displaystyle\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a^n b^k\Big)$ convergent? (Consider $|b| < 1, > 1, = 1$.) Determine the sum.

1.4

1. Define the function
$$f(z) = \frac{z^2 - 1}{z-1}.$$
   (a) Prove that $\ell = \lim_{z\to 1} f(z)$ exists.
   (b) If the function $g(z)$ is defined by
$$g(z) = f(z) \quad (z \neq 1), \qquad g(1) = \ell,$$
   prove, by using the definition of limit, that $g(z)$ is differentiable in $z = 1$.
2. Let n be an integer. Determine, by using the definition, the derivative of $f(z) = z^n$.
3. Show that $f(z) = z\overline{z}$ is differentiable in $z = 0$ but not holomorphic.
4. Show that the function $\cos|z|$ in $z = 0$ is differentiable but not holomorphic.
5. Show that $f(z)$ satisfies the Cauchy-Riemann-equations if and only if $w(x, y) = f(x + iy)$ satisfies $w_y = iw_x$.
6. Let $p(s, t)$ be a polynomial in the variables s and t. Show that $f(z) = p(z, \overline{z})$ is a holomorphic function if and only if p does not depend on t.
7. Show that if $f(z)$ is holomorphic and $|f(z)|$ is constant, then $f(z)$ is constant.
8. Prove that if $f(z, \overline{z})$ is holomorphic in z and differentiable in x and y, it is independent of $\overline{z}$.
9. Indicate where function $f(z) = z\,\mathrm{Im}(z) - \mathrm{Re}(z)$ is differentiable, and where holomorphic.

1.5

In this section, x and y will always denote the real and imaginary parts of z, so $z = x + \mathrm{i}y$.

1. Let the function $f(z) = 1/z$ be given.
   (a) Determine the functions $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$.
   (b) Verify that the functions u and v are harmonic and that the level curves intersect each other perpendicularly.
   (c) Sketch the level curves of u and v.
2. A function $f(z)$, everywhere holomorphic in $\mathbb{C}$, satisfies
   (a) $\operatorname{Re} f(z) = xy + \mathrm{e}^x \cos y$,
   (b) $f(0) = 1 + \mathrm{i}$.

   Determine $f(z)$, expressed in z.
3. A function $f(z)$, holomorphic for $z \neq 0$, satisfies
   $$\operatorname{Re} f(z) = x - \frac{x}{x^2 + y^2}, \qquad f(1) = 0.$$
   Determine $f(z)$ expressed in z.
4. Let a be a real number. A function $f(z)$ is holomorphic everywhere and satisfies
   $$\operatorname{Re} f(z) = \mathrm{e}^{ax} \cos(2\pi y), \qquad \operatorname{Re} f(1) < 1, \operatorname{Im} f(1) = 2\pi.$$
   Determine a and $f(z)$, expressed in z.
5. A function $f(z)$, defined everywhere in $\mathbb{C}$, satisfies $\operatorname{Re} f(z) = x^2y^2$ for all $z \in \mathbb{C}$. Is it possible that this function is holomorphic? If yes, give an example of such a function; if no, explain why not.
6. Determine the holomorphic function $f(z)$ satisfying
   (a) $\operatorname{Re} f(z) = x(x^2 - 3y^2 - 6y - 4)$.
   (b) $f(0) = \mathrm{i}$.
7. Let the function $f(z)$ be holomorphic in the open set V. Show that:
   $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(x + \mathrm{i}y)|^2 = 4|f'(x + \mathrm{i}y)|^2.$$
8. Find holomorphic $f(z) = u(x, y) + \mathrm{i}v(x, y)$ explicitly as a function of z, if $f(0) = 0$, and
   $$u(x, y) = \frac{x^3 + x^2 - y^2 + xy^2}{(x + 1)^2 + y^2}.$$
   Use Cauchy-Riemann in shifted polar coordinates $z + 1 = r\,\mathrm{e}^{\mathrm{i}\vartheta}$: $\; ru_r = v_\vartheta, u_\vartheta = -rv_r$.

9. Find the holomorphic function $f(z) = f(x + \mathrm{i}y)$ that satisfies
$$\mathrm{Re}(f(z)) = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x},$$
for harmonic functions $\varphi(x, y)$ and $\psi(x, y)$.

10. How many level curves of $\mathrm{Re}(z^n)$ and $\mathrm{Im}(z^n)$, with $n \in \mathbb{N}$, pass the origin?

1.6

REMARK: it is not necessary to investigate convergence on the boundary of a region of convergence.

1. Determine region of convergence and sum of the following series.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{2n}}{2^n}$, b) $\displaystyle\sum_{n=0}^{\infty} (3^n + \mathrm{i}^n) z^n$, c) $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n!}(z-1)^n$, d) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(2n+1)!} z^{2n}$.

2. Determine radius of convergence of the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n! z^n}{(1+\mathrm{i})(1+2\mathrm{i})\cdots(1+n\mathrm{i})}$,

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{n^2}}{(n+1)!}$.

3. Determine the region of convergence and the sum of the series $\displaystyle\sum_{n=1}^{\infty} \exp\left[n(z - 1/z)\right]$.

4. The function $f(z)$ is given by
$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z-1}{z+1}\right)^n.$$
   (a) Determine the region of convergence G of the series.
   (b) Show that $f(z)$ is holomorphic in G.
   (c) Determine the corresponding series of $f'(z)$ and its sum.

5. Let the series be given
$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z\sqrt{3}-1}{z+\sqrt{3}}\right)^{2n+1}.$$
   (a) Determine $R > 0$ such that the series converges in the set $G = \{z \in \mathbb{C} \mid |z - \sqrt{3}| < R\}$.
   (b) Show that $f(z)$ is holomorphic in G.
   (c) Determine the series of $f'(z)$ and its sum.

6. The function $f(z)$ is given by

$$f(z) = \sum_{n=1}^{\infty} (n+1) \left(\frac{z}{z-1} \right)^n .$$

   (a) Determine the region of convergence of this series.

   (b) Determine $f(z)$.

7. Prove the following formulas

   (a) $\cos^2 z + \sin^2 z = 1$.

   (b) $\sin(z+w) = \sin z \cos w + \cos z \sin w$,

   (c) $\cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w$,

   (d) $\cosh^2 z - \sinh^2 z = 1$,

   (e) $|\sin z|^2 = \sin^2 x + \sinh^2 y$, where $z = x + \mathrm{i}y$.

8. Determine all zeros of cos, sin, cosh, sinh.

9. Solve the following equations in $\mathbb{C}$:

   a) $\mathrm{e}^z = 1 + \mathrm{i}$, b) $|\,\mathrm{e}^{\mathrm{i}z}\,| = 1$, c) $\cos z = 10$, d) $\sin z = 10$, e) $|\tanh z\,| = 1$.

10. Show that the function

$$\tan z = \frac{\sin z}{\cos z}$$

   assumes all values except i and $-$i.

11. (a) Determine the zeros in $\mathbb{C}$ of $\dfrac{z^m - 1}{z - 1}$, where m is a natural number.

   (b) Prove that

$$\prod_{k=1}^{m-1} \left(1 - \mathrm{e}^{2\pi \mathrm{i}k/m}\right) = m, \quad (m \in \mathbb{N}).$$

   NOTE: The expression $\displaystyle\prod_{k=1}^{m} a_k$ denotes the product $a_1 a_2 \cdots a_m$.

12. Let the mapping $w = \mathrm{e}^{\mathrm{i}z}$ be given. Determine the image in the complex w-plane of the set $\{z \in \mathbb{C} \,\big|\, -\frac{2}{3}\pi \leqslant \operatorname{Re} z \leqslant 0, 1 \leqslant \operatorname{Im} z \leqslant 3\}$.

13. Let the mapping $w = \cos z$ be given. Determine the image in the complex w-plane of the sets $\{z \in \mathbb{C} \,\big|\, 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\pi, \operatorname{Im} z \geqslant 0\}$ and $\{z \in \mathbb{C} \,\big|\, 0 \leqslant \operatorname{Re} z \leqslant \pi, \operatorname{Im} z \geqslant 0\}$.

Exercises Chapter 2

2.1

1. Sketch in the z-plane the closed curve that consists of the following parts:

   a) $z = t \ \ (0 \leqslant t \leqslant R)$, b) $z = R\,\mathrm{e}^{\mathrm{i}\varphi} \ \ (0 \leqslant \varphi \leqslant \pi/4)$,

   c) $z = \frac{1+\mathrm{i}}{\sqrt{2}}(R-t) \ \ (0 \leqslant t \leqslant R)$.

2. Give a parameter representation $z = \varphi(t)$ with $a \leqslant t \leqslant b$, (*i.e.* determine the function φ and the numbers a and b) of the following arcs:

   (a) the straight line segment that starts at $-1+\mathrm{i}$ and ends at $1-3\mathrm{i}$,

   (b) the part of the circular arc $|z-1| = \sqrt{2}$ that starts at i and ends at $-\mathrm{i}$ in the half plane $\operatorname{Re} z \leqslant 0$.

3. Let the function $f(z)$ be holomorphic in a domain G and assume that $f'(z) = 0$ for all $z \in G$. Prove that $f(z)$ is a constant on G.

   HINT.

   (a) Use the corresponding theorem for real functions.

   (b) The condition that G is connected, is essential. The function $f(z)$ defined on the open set $V = B_1(0) \cup B_1(3)$ by $f(z) = 0$ for $|z| < 1$ and $f(z) = 1$ for $|z-3| < 1$, satisfies $f'(z) = 0$ for all $z \in V$, but is not a constant!

4. Let the function $f(z)$ be holomorphic in a domain G. Prove the following properties:

   (a) If $f(z)$ is real for all $z \in G$, then $f(z)$ is constant on G.

   (b) If $|f(z)|$ is constant on G, then $f(z)$ is constant on G.

   (c) Show in both cases that the function is not necessarily constant if G is not connected.

2.2

1. Calculate $\int_K \overline{z}\, dz$ in the following cases:
   (a) K is the straight line segment from 0 to $4+2\mathrm{i}$,
   (b) K consists of the straight line segments from 0 to $2\mathrm{i}$ and from $2\mathrm{i}$ to $4+2\mathrm{i}$.
2. Calculate $\displaystyle\int_K \frac{dz}{z-\sqrt{3}}$, where K is the straight line segment from $-\mathrm{i}$ to i.
3. Calculate $\int_K \frac{dz}{z}$ in the following cases:
   (a) K is a quarter circular arc, with centre 0, from $1-\mathrm{i}$ to $1+\mathrm{i}$,
   (b) K is the straight line segment from $1-\mathrm{i}$ to $1+\mathrm{i}$,
   (c) K is a three-quarter circular arc, with centre 0, from $1-\mathrm{i}$ to $1+\mathrm{i}$.
4. Calculate $\int_C dz/z$, where the Jordan curve $C = C_1 + C_2 + C_3 + C_4$ consists of
   - C_1, the straight line segment from $z = 1$ to $z = 2$,
   - C_2, the semi circular arc in the upper half plane with centre 0 from $z = 2$ to $z = -2$,
   - C_3, the straight line segment from $z = -2$ to $z = -1$,
   - C_4, the semi circular arc in the upper half plane with centre 0 from $z = -1$ to $z = 1$.
5. Let $C_R = \{z \in \mathbb{C} \,\big|\, |z| = R\}$ and $C_R^* = \{z \in \mathbb{C} \,\big|\, |z| = R, \operatorname{Re} z \geqslant 0\}$. Prove that
   (a) $\displaystyle\left|\int_{C_R} \frac{dz}{z+1}\right| \leqslant \frac{2\pi R}{R-1} \quad (R > 1).$
   (b) $\displaystyle\left|\int_{C_R} \frac{dz}{z^2+1}\right| \to 0 \quad (R \to \infty).$
   (c) $\displaystyle\left|\int_{C_R^*} \frac{\mathrm{e}^{-z}}{(z+1)^2}\, dz\right| \leqslant \frac{\pi R}{(R-1)^2} \quad (R > 1).$
6. Calculate the integral (with a positively oriented contour)
   $$\frac{1}{2\pi\mathrm{i}} \int_{|z|=R} \frac{|z|}{z}\, dz$$
7. Let f and g be holomorphic in domain $\mathcal{D}$, while f' and g' are continuous. A curve $\gamma \subset \mathcal{D}$ runs from a to b. Prove that
   $$\int_\gamma f(z)g'(z)\, dz = f(b)g(b) - f(a)g(a) - \int_\gamma f'(z)g(z)\, dz.$$

2.3

1. Consider the function

$$f(z) = \frac{1}{z^2+1} \quad (z \neq \pm \mathrm{i}).$$

   (a) Determine $\int_{K_1} f(z)\,dz$, where K_1 is the half ellipse $x^2 + 4y^2 = 1$, $y \geqslant 0$, starting at $z = x + \mathrm{i}y = -1$ and ending at $z = 1$.

   (b) Show that

$$\int_{\substack{|z| = R \\ \operatorname{Im} z \geqslant 0}} f(z)\,dz \to 0 \quad (R \to \infty)$$

   (c) Determine $\int_{K_2} f(z)\,dz$, where K_2 is the half ellipse $x^2 + \frac{1}{4}y^2 = 1$, $y \geqslant 0$, starting at $z = x + \mathrm{i}y = -1$ and ending at $z = 1$.
   HINT. Connect this integral to the one in question b).

2. For a given $a \in \mathbb{C}$ we define the positively oriented curve $C_a = \{z \in \mathbb{C} \big| |z - a| = 2\}$. Let

$$f(a) = \int_{C_a} \frac{1}{z^2 - 1}\,dz$$

   for those values of a where the integral exists. Indicate for which $a \in \mathbb{C}$ the integral exists and determine the value of the integral for such a.

3. Consider, for $n \in \mathbb{N}$, $R > 1$, and a positive orientation of the integration contour, the integral

$$A_n = \int_{|z|=R} \frac{z^n}{z^{10} - 1}\,dz.$$

   (a) Find A_n for $0 \leqslant n \leqslant 8$. HINT: let $R \to \infty$.

   (b) Find A_9.

   (c) Find A_n for $n \geqslant 10$. HINT: show that $A_{10m+k} = A_{10(m-1)+k}$ for $m \geqslant 1, 0 \leqslant k \leqslant 9$.

2.4

1. Determine the residue of the function $f(z)$ in the point $z = a$ in the following cases:

a) $\dfrac{e^{z^2}}{z^5}$,	$a = 0$;	b) $\dfrac{e^z}{(z-1)^6}$,	$a = 1$;
c) $\dfrac{\sinh(\pi z)}{z^4}$,	$a = 0$;	d) $\dfrac{\sin(\pi z)}{(z-1)^5}$,	$a = 1$.

2. Determine the integrals
$$\int_C \frac{e^{z^2}}{z^2}\,dz \quad \text{and} \quad \int_C \frac{e^{z^2}}{z^5}\,dz,$$
where C is the circle $|z| = 1$, traversed in positive direction.

3. Determine
$$\int_C \frac{1}{z^2 - 1}\,dz,$$
where C is the circle $|z - 1| = 1$, traversed in positive direction.

4. The function f is defined by
$$f(z) = \frac{1}{2\pi i}\int_K \frac{|\zeta|^2}{\zeta - z}\,d\zeta, \qquad z \notin K.$$
Here is K the circle $|\zeta - 1| = 1$, traversed in positive direction.
   (a) Show that for $\zeta \in K$ we have
   $$\overline{\zeta} = \frac{\zeta}{\zeta - 1}.$$
   (b) Calculate $f(z)$.

5. Determine
$$\int_{|z|=2} \frac{\overline{z} + 1}{z^2 + 1}\,dz.$$
HINT. Replace $\overline{z}$ along the integration contour by a holomorphic function.

6. Determine
$$\int_{|z|=1} \frac{2\,\mathrm{Re}(z)}{2z + 1}\,dz.$$

2.5

1. Let the function f be holomorphic inside and on the Jordan curve K. Prove by using Cauchy's integral formula that f is constant inside K if f is constant on K.

2. Prove the mean value theorem for holomorphic functions:
If $f(z)$ is holomorphic inside and on the circle with centre $c \in \mathbb{C}$ and radius $R > 0$, then $f(c)$ is equal to the mean of $f(z)$ along that circle. In other words
$$f(c) = \frac{1}{2\pi}\int_0^{2\pi} f(c + R\,e^{i\vartheta})\,d\vartheta$$
Can you give a version of this theorem for harmonic real functions?

3. Let the following series be given

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$$

with sum $F(z)$ (as far as the series converges).

(a) Prove that the series is uniformly convergent for $|z| < \frac{1}{2}$.

(b) Show that $F(z)$ is holomorphic for $|z| < \frac{1}{2}$.

Note: $F(z)$ is related to $\cot(\pi z)$; see example 4.4.2 or exercise 4.4.3.

4. Let the series be given

$$\sum_{n=1}^{\infty} \frac{(1 - \mathrm{e}^{-z})^n}{n}$$

with sum $F(z)$ (as far as the series converges).

(a) Prove that the series converges uniformly on the set

$$U = \{z \in \mathbb{C} \,\big|\, |1 - \mathrm{e}^{-z}| < \tfrac{1}{2}\}.$$

(b) Show that $F(z) = z$ in U.

5. Let the series be given

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{2z}{1 - z^2} \right)^{2n+1}$$

with sum $F(z)$ (as far as the series converges).

(a) Prove that the series converges uniformly on the set $G : |z| < \frac{1}{3}$.

(b) Determine the series of $F'(z)$ and its sum.

6. The function $f(z)$ is given by

$$f(z) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-nz}}{n}.$$

(a) Determine the open set where the series converges..

(b) Determine $f'(z)$.

7. Show that the series

$$\sum_{n=0}^{\infty} \frac{\sin nz}{n!}$$

for all $z \in \mathbb{C}$ converges, and determine the sum.

8. Determine the Taylor series of $f(z)$ around $z = a$ in the following cases:

a) $f(z) = \dfrac{1}{z-1}$, $a = 3$; b) $f(z) = \cosh z$, $a = \frac{1}{2}\pi\mathrm{i}$;

c) $f(z) = \mathrm{e}^z + \dfrac{1}{(1-z)^2}$, $a = 0$; d) $f(z) = \dfrac{1}{(1-z)(1+z^2)}$, $a = 0$.

9. We define

$$K_1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1, \operatorname{Im}\zeta \geqslant 0\}, \qquad K_2 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1, \operatorname{Im}\zeta \leqslant 0\},$$

both traversed from $\zeta = -1$ to $\zeta = 1$. Further, we define for $m = 1, 2$ the function $f_m(z)$ for $z \notin K_m$ by

$$f_m(z) = \int_{K_m} \frac{\zeta}{\zeta - z}\, d\zeta.$$

(a) Show that $f_m(z)$ is holomorphic on $\mathbb{C}\backslash K_m$ for $m = 1, 2$.
(b) Give a formula for $f_1^{(n)}(z)$ for $n = 1, 2, 3 \ldots$, and calculate $f_1'(0)$.
(c) Determine the difference $f_1(z) - f_2(z)$.
(d) Determine the Taylor series around $z = 0$ of the functions $f_1(z)$ and $f_2(z)$. What is their radius of convergence?

10. Consider the Taylor series $\sum_{n=0}^{\infty} c_n z^n$ of the functions $\dfrac{z}{\sin z}$, $\tan z$ and $\sin(\sin z)$. Determine the coefficients c_0, c_1, c_2 and c_3 for each case.

11. Determine the radius of convergence of the Taylor series around $z = 0$ of the following functions

a) $f(z) = \dfrac{\cosh \pi - \cos(\pi z)}{\mathrm{e}^{\pi z} + 1}$; b) $f(z) = \dfrac{4z^2 - \pi^2}{(z-1)^6}$.

12. Let $f(z) = u(x, y) + \mathrm{i}v(x, y)$ be analytic in $z = a$, with $f'(a) = 0$ and $f''(a) \neq 0$. Show that a is a saddle point of u (and similarly of v).

2.6

1. Which of the following functions are entire?

a) $\dfrac{1}{\mathrm{e}^z}$; b) $\mathrm{e}^{1/z}$;

c) $\dfrac{\sin z}{z}$; d) $\dfrac{\sqrt{3}\cos z}{z^4} - \dfrac{\sin\left(z\sqrt{3}\right)}{z^5}$.

2. Assume that the function $f(z)$ is entire and that there are constants $M > 0$ and $n \in \mathbb{N}$ such that $|f(z)| \leqslant M|z|^n$ for all $z \in \mathbb{C}$. Prove that $f(z) = az^n$, for some $a \in \mathbb{C}$ with $|a| \leqslant M$.

3. Which functions $f(z)$ satisfy the following conditions?

   (a) $f(z)$ is an entire function.
   (b) $|f(z)| \leqslant |\mathrm{e}^z|$ for all $z \in \mathbb{C}$.

4. Determine the function $f(z)$ that satisfies:

   (a) $f(z)$ is an entire function;
   (b) $f(z)/z^2 \to 2$ for $|z| \to \infty$;
   (c) $f(0) = 0$, $f'(0) = 1$.

5. The functions $f(z)$ is analytic and non-zero inside and on the Jordan curve K, except for a zero of order k in a point a inside K. Determine the integral
$$\frac{1}{2\pi\mathrm{i}}\int_K \frac{f'(z)}{f(z)}\,dz,$$
where K is traversed in positive direction. (Compare with exercises 2.7.11 and 2.9.17.)

6. Is it possible for an entire function h to satisfy the following limits?
$$a)\ \lim_{z\to\infty} h(z) = 1, \qquad b)\ \lim_{z\to\infty} zh(z) = 1, \qquad c)\ \lim_{z\to\infty} z^2h(z) = 0.$$
Give h, for the cases possible, in its most general form.

2.7

Unless indicated otherwise, Jordan curves have a positive orientation.

1. Determine the Laurent series of the function $f(z)$ around centre $z = a$, which converges in the annular domain $r < |z - a| < R$, for the following cases:
$$\text{a)}\quad f(z) = \frac{13z + 9}{(z-3)(z+1)^2}, \quad a = 0, \quad r = 1, \quad R = 3;$$
$$\text{b)}\quad f(z) = \frac{1}{z^2(z-1)}, \quad a = -1, r = 1, \quad R = 2.$$

2. Determine the Laurent series, and their regions of convergence, of the function $f(z)$, around centre $z = a$, which converges in point $z = b$, in the following cases:
$$\text{a)}\quad f(z) = \frac{3z^2 + 15z - 6}{(z-2)(z+4)^2}, \quad a = 0, \quad b = 3\mathrm{i};$$
$$\text{b)}\quad f(z) = \frac{1}{z^2+1}, \quad a = \mathrm{i}, \quad b = 0;$$
$$\text{c)}\quad f(z) = \frac{2z+3}{z^2+3z-4}, \quad a = -2, \quad b = \mathrm{i};$$
$$\text{d)}\quad f(z) = \frac{1}{(z^2-1)^3}, \quad a = -1, \quad b = 1 + \mathrm{i}.$$

3. The following function is given

$$f(z) = e^z + \frac{1}{(z-1)^2}.$$

   (a) Determine the Laurent series of $f(z)$ around centre $z = 0$, which converges in $z = 2$.
   (b) Determine the Laurent series of $f(z)$ around centre $z = 1$, which converges in $z = 0$.

4. The function $f(z)$ is analytic in $\mathbb{C}$ except for a finite number of singularities. In the domain $\{z \in \mathbb{C} \mid |z-1| > 1\}$ $f(z)$ can be expanded in a Laurent series

$$f(z) = \sum_{n=1}^{\infty} (z-1)^{-2n} + \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}.$$

   Determine the Laurent series of $f(z)$ around centre $z = 0$, which converges in a reduced neighbourhood of $z = 0$. Determine also the region of convergence of this Laurent series.

5. (a) Let $\sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent series of the function $\tan z$, which converges in $z = \pi$. Determine its region of convergence and the coefficients c_1, c_2 and c_n for $n \leqslant 0$.
   (b) Let $\sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent series of the function $1/(e^z - 1)$, which converges in $z = 3\pi$. Determine its region of convergence and the coefficients c_{-1}, c_0 and c_1.

6. (a) Determine the constants a and b such that the function

$$\frac{z}{\sin z} - \frac{a}{z-\pi} - \frac{b}{z+\pi}$$

   has removable singularities in $z = \pi$ and $z = -\pi$.
   (b) Let $\sum_{n=-\infty}^{\infty} c_n z^n$ be the Laurent series of the function $z/\sin z$, which converges in $z = \frac{3}{2}\pi$. Determine the principal part $\sum_{n=1}^{\infty} c_{-n} z^{-n}$ of this Laurent series. Determine also the sum of the principal part.

7. Let the function be given

$$f(z) = \frac{e^z}{z(1-\cos z)}.$$

   (a) Determine the location and character of the singularities of $f(z)$.
   (b) Determine the principal part of the Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ of $f(z)$, which converges in $z = \pi$.
   (c) Determine the region of convergence of this Laurent series.

8. The functions $f(z)$ and $g(z)$ are defined by

$$f(z) = \int_L \frac{dt}{t-z}, \quad g(z) = \int_L \frac{dt}{t^2 - z^2},$$

   where $L = [-1, 1] \subseteq \mathbb{R}$.

   (a) Prove that $f(z)$ is analytic outside L.

(b) Determine the Laurent series of $f(z)$ around $z = 0$, which converges for $|z| > 1$.

(c) Express $g(z)$ in $f(z)$. Where is $g(z)$ analytic?

(d) Determine the Laurent series of $g(z)$ around $z = 0$, which converges for $|z| > 1$.

9. (a) Determine the residues of the functions $\frac{1}{\sin z}$ and $\sin \frac{1}{z}$ in $z = 0$.

   (b) Determine location and character of the singularities of the function $\frac{1}{\sin z} - \sin \frac{1}{z}$.

   (c) Determine the residue of the function $\frac{\sin(1/z)}{1-z^2}$ in $z = 0$.

10. Determine location and character of the singularities of the following functions:

$$\text{a) } f(z) = \frac{z\,\mathrm{e}^{1/z}\sin z}{(z-2)^2(z+\pi)^2} \qquad \text{b) } f(z) = z^2\,\mathrm{e}^{1/z} - (z^2 - 2z)\sin\left(\frac{1}{z-1}\right).$$

11. The function $f(z)$ is analytic and non-zero inside and on the Jordan curve K, except for a pole of order k in a point a inside K. Determine the integral

$$\frac{1}{2\pi\mathrm{i}}\int_K \frac{f'(z)}{f(z)}\,dz,$$

where K is traversed in positive direction. (Compare with exercises 2.6.5 and 2.9.17.)

12. Determine the following integrals:

$$\text{a) } \int_{|z|=1} z(z^2+1)\,\mathrm{e}^{1/z}\,dz; \qquad \text{b) } \int_{|z-\frac{1}{2}|=3} \left\{ z^2\,\mathrm{e}^{1/z} - (z^2-2z)\sin\left(\frac{1}{z-1}\right)\right\} dz;$$

$$\text{c) } \int_{|z|=R} \frac{\mathrm{e}^{1/z}}{(1-z)^2}\,dz; \qquad \text{d) } \int_{|z-\frac{1}{2}|=R} \frac{z}{z^2-1}(\mathrm{e}^{1/z^2} - 1)\,dz \quad (R > 0,\ R \neq \tfrac{1}{2}).$$

13.

$$\text{a) } \int_{|z|=1} \frac{\sin(\pi z)}{(z-\frac{1}{2})^2(z+2)}\,dz; \qquad \text{b) } \int_{|z-4\mathrm{i}|=5} \frac{\mathrm{e}^z}{z^2 \sin z}\,dz;$$

$$\text{c) } \int_{|z-\frac{1}{2}|=1} \frac{\cot(\pi z)}{z^2-1}\,dz; \qquad \text{d) } \int_{|z-\mathrm{i}\sqrt{2}|=1} \frac{1}{z \sin z \sinh z}\,dz.$$

14. Determine the function $f(z)$ that satisfies:

   (a) $f(z)$ is analytic in $\mathbb{C}$, except for a pole of order 2 in $z = -1$, and a pole of order 1 in $z = \mathrm{i}$ with residue $3\mathrm{i}$;

   (b) $f(z) \to 3$ for $|z| \to \infty$;

   (c) $f(0) = 0$, and $\int_{|z|=2} f(z) = -16\pi$.

15. Determine the function $f(z)$ that satisfies:

    (a) $f(z)$ is holomorphic in $\mathbb{C}$, except for a pole of order 3 in $z = 0$ with residue 0;
    (b) $f(z) \to 0$ for $|z| \to \infty$;
    (c) $f(1) = 0$, and $\int_{-\mathrm{i}}^{i} f(z)\,dz = 2\mathrm{i}$ along an integration contour not through 0.

16. Determine the function $f(z)$ that satisfies:

    (a) $f(z)$ is analytic in $\mathbb{C}$, except for a pole of order 1 in $z = 0$ and a pole of order 2 in $z = 1$ with residue 2;
    (b) $f(z) \to 1$ for $|z| \to \infty$;
    (c) For the Laurent series $\sum_{n=-\infty}^{\infty} c_n z^n$ of $f(z)$, which converges for $|z| > 1$, we have $c_{-2} = -2, c_{-1} = 1$.

17. Determine the function $f(z)$ that satisfies:

    (a) $f(z)$ is analytic in $\mathbb{C}$, except for poles in $z = 0$ and $z = 1$;
    (b) $f(z)$ has a zero of order 2 in $z = -1$, a simple zero in $z = 2$, and no other zeros;
    (c) $f(z) \to 1$ for $|z| \to \infty$;
    (d) $\mathrm{Res}_{z=0}\, f(z)$ is positive.

18. Determine the function $f(z)$ that satisfies:

    (a) $f(z)$ is analytic in $\mathbb{C}$, except for a pole of order 2 in $z = 1$ and a simple pole in $z = 0$;
    (b) $zf(z) \to 1$ for $|z| \to \infty$;
    (c) $f(z)$ has two opposite zeros;
    (d) $\dfrac{1}{2\pi\mathrm{i}} \displaystyle\int_{|z|=R} \frac{f(z)}{z}\,dz = 4$ for $0 < R < 1$.

19. Let the function be given

$$f(z) = \frac{\mathrm{e}^{\pi \mathrm{i} z} - 1 + 2z^2}{z(z^2 - 1)}.$$

    Determine the singular points of $f(z)$ and show that these are all removable.

20. Show that the function

$$f(z) = \frac{\cos z}{\sin z} - \frac{1}{z} - \frac{1}{z - \pi} - \frac{1}{z + \pi}$$

    has removable singularities in $z = 0$, $z = \pi$ and $z = -\pi$, and can be expanded in a Taylor series around $z = 0$. What is the radius of convergence of this Taylor series? Which fractions have to be subtracted from $f(z)$ in order to have a Taylor series with radius of convergence equal to $n\pi$ (where $n \geqslant 2$ is).

21. Let the function be given

$$f(z) = \frac{1}{\cos z} + \frac{a}{z - \frac{1}{2}\pi} + \frac{b}{z + \frac{1}{2}\pi}.$$

(a) Determine the constants a and b and the function values $f(\pm\frac{1}{2}\pi)$ such that the singularities of $f(z)$ in $z = \pm\frac{1}{2}\pi$ are removable.

(b) Determine the radius of convergence of the Taylor series $\sum_{n=0}^{\infty} c_n z^n$ of $f(z)$ around $z = 0$, and determine the coefficients c_0, c_1 and c_2.

22. Determine the zeros and poles, and their multiplicities, of the following functions

a) $f(z) = \cos z - \cos(2z)$; b) $f(z) = \tan z - \sin z$;

c) $f(z) = \dfrac{\cos^2(\pi z)}{4z^2 - 1}$; d) $f(z) = \dfrac{(\mathrm{e}^{\pi \mathrm{i} z} + 1)\sin^2(\pi z)}{z^2(z^2-1)^2}$.

23. Determine the zeros and poles, and their multiplicities, of the following functions

a) $f(z) = \dfrac{(\mathrm{e}^{2z} - 1)^3}{z(1-\cos z)^2}$; b) $f(z) = \dfrac{(z^2 - \pi^2/4)\tan z}{z(\mathrm{e}^z - 1)^2)}$;

c) $f(z) = \dfrac{\cos z - \cos(2z)}{z(\mathrm{e}^z - 1)^3}$; d) $f(z) = \dfrac{\tan z - \sin z}{z(\mathrm{e}^{z^2} - 1)^2}$.

24. Determine the function $f(z)$ that satisfies:

(a) $f(z)$ is holomorphic in $\mathbb{C}$, except for a pole of order 2 in $z = 1$ and a pole of order 1 in $z = -1$ with residue 1;

(b) $f(z)/z \to 1$ for $|z| \to \infty$;

(c) $f(z)$ has a zero in $z = 0$ with multiplicity 3.

25. Determine the function $f(z)$ that satisfies:

(a) $f(z)$ is holomorphic in $\mathbb{C}$, except for a pole of order 1 in $z = 0$ and a pole of order 2 in $z = 2$ with residue 5;

(b) $f(z) \to 0$ for $|z| \to \infty$;

(c) $f(z)$ has zeros in $z = 1$ and in $z = -1$.

26. Determine the function $f(z)$ that satisfies:

(a) $f(z)$ is holomorphic in $\mathbb{C}$, except for simple poles in $z = 2$ and $z = -2$;

(b) $f(z)/z^3 \to 1$ for $|z| \to \infty$;

(c) $f(z)$ is odd, so $f(-z) = -f(z)$;

(d) $f(1) = 0$ and $\int_{|z|=4} f(z)\, dz = 6\pi \mathrm{i}$.

27. A rational function is the quotient of two polynomials. Prove the following theorem:

A function $f(z)$ is rational if and only if the following conditions are satisfied:

(a) *$f(z)$ is holomorphic in $\mathbb{C}$ except for a finite number of poles.*

(b) *There exist numbers $M > 0$, $R > 0$ and $k \in \mathbb{N}$ such that $|f(z)| \leqslant M|z|^k$ for $|z| \geqslant R$.*

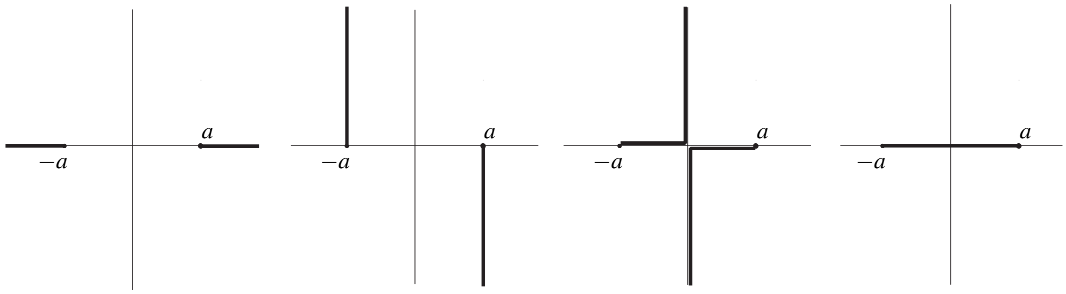
28. Prove Theorem 2.7.13.

2.9

Unless indicated otherwise, the principal value is understood of square root, logarithm and power functions.

1. Determine $\log \mathrm{i}$, i^{i} and $(\sqrt{\mathrm{i}})^{\log \mathrm{i}}$.
2. Under what condition is $\sqrt{z}\,\sqrt{w} = \sqrt{zw}$? Give an example of numbers z and w, where this relation is not valid.
3. Determine the branch cut of the function $f(z) = \sqrt{-z}$. What are $f(x)$ for $x \in (-\infty, 0]$, and $f(x + \mathrm{i}0)$ and $f(x - \mathrm{i}0)$ for $x \in [0, \infty)$?
4. Determine the branch cuts of the function $f(z) = \log(z + 1/z)$ (in other words, the inverse image of the branch cut of $\log(\,\cdot\,)$).
5. Determine
$$\int_K \sqrt{z}\, dz,$$
where $K = \{\mathrm{e}^{-\mathrm{i}t} \big| 0 \leqslant t \leqslant 2\pi\}$.
6. Determine
$$\int_{|z-1|=\frac{1}{2}} \frac{1}{z \log z}\, dz.$$
7. Determine
$$\int_{|z|=\frac{3}{2}} \frac{\log(z+2)}{\mathrm{e}^{2\pi z} - 1}\, dz.$$
8. Prove that for $|\arg z| < \pi$ we have: $\displaystyle\int_0^\infty \left(\frac{1}{t+1} - \frac{1}{t+z}\right) dt = \log z.$
9. If we try to define $\arctan z$ by
$$\arctan z = \int_0^z \frac{1}{w^2+1}\, dw,$$
then we need a branch cut. Can you indicate such a cut?
10. Determine in a similar way a cut for
$$f(z) = \int_0^z \frac{1}{\cos \zeta}\, d\zeta.$$
11. Is it possible for an entire function h to satisfy the following limits?
$$a)\ \lim_{z\to\infty} \frac{h(z)^2}{z} = 1, \qquad b)\ \lim_{z\to\infty} \frac{h(z)^2}{z^3} = 0.$$
Give h, for the cases possible, in its most general form.

12. Consider the complex square root $f(z) = \sqrt{a^2 - z^2}$, $a \in \mathbb{R}$ positive, with branch defined by $f(+0 + \mathrm{i}0) = a$, branch points $z = a$ and $z = -a$, and branch cuts as follows:



See example 2.9.8 for explicit representations. Determine the signs of $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ when z is varied along both sides of the branch cuts, and the real and imaginary axes.

13. Consider the complex square root $f(z) = \sqrt{\alpha^2 - z^2}$, $\alpha = a - \mathrm{i}b \in \mathbb{C}$ and $a, b > 0$, with branch defined by $f(0) = \alpha$. Determine the branch cuts, such that $\mathrm{Im}(f) \leqslant 0$ for all $z \in \mathbb{C}$. HINT. Consider $f(x + \mathrm{i}y)^2$ along the level curve $\mathrm{Im}(f) = 0$.

14. We define the complex logarithm $\log(z)$ as principal value (so with $\log(1) = 0$ and the branch cut along the negative real axis $(-\infty, 0]$). We define for positive real K the contour $\mathcal{C}$ from $-K - \mathrm{i}.0$ under the branch cut, right around the origin, and then above the branch cut back to $-K + \mathrm{i}.0$. See the figure below.

Determine
$$\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \log(z)\, dz.$$

15. $\sqrt{z^2} = \pm z$ has no meaning before it is defined. A possible definition (*cf.* example 2.9.4) is as follows. For $\varepsilon > 0$, and the principle value square root on the right-hand side, we define
$$\sqrt{z^2} = \lim_{\varepsilon \to 0} \sqrt{z^2 + \varepsilon^2}.$$
Determine the location of the branch cuts, and show that $\sqrt{z^2} = \mathrm{sign}(\mathrm{Re}\, z) z$.

16. Let function $f(z)$ be equal to a constant A for $\mathrm{Re}(z) > 0$ and constant B for $\mathrm{Re}(z) < 0$. We are looking for a representation $f(z) = f_+(z)/f_-(z)$ such that $f_\pm$ is analytic in $\mathrm{Im}(z) \gtrless 0$. From example 2.9.6, we can use $f_\pm(z) = C_\pm\, (z)_\pm^\alpha$ for certain α and $C_\pm$. Are they unique?

17. **The Argument Principle.** Let $f(z)$ be analytic inside and on a Jordan curve C except for a finite number of poles of finite order inside C. Prove that
$$\frac{1}{2\pi\mathrm{i}} \int_C \frac{f'(z)}{f(z)}\, dz = N - P$$
where N and P are, respectively, the number of zeros and poles of $f(z)$ inside C, with each zero and pole counted as many times as its multiplicity, *i.e.* order, indicates. (Compare with exercises 2.6.5 and 2.7.11.)

18. (a) Suppose $p(z)$ is a polynomial of even degree, with all its roots inside a disc of radius R. Explain why there is an analytic function $h(z)$ defined on the region $R < |z| < \infty$ which satisfies $h(z)^2 = p(z)$. We write $h(z) = \sqrt{p(z)}$. HINT. This requires an alternative definition (not necessarily principal value) of the square root.

    (b) Following the previous question, define the non-principal value square root $\sqrt{z^4 - z}$, that is analytic in $|z| > 1$. By expanding the integrand in a Laurent series, evaluate
$$\int_C \sqrt{z^4 - z}\, dz$$
where C is the circle of radius 2 with positive orientation. (The sign of your answer will depend on which branch of the square root you pick.)

19. Define the complex square root $w(z) = \sqrt{z^2 - 1}$ with $w(2) = \sqrt{3}$ and branch cuts along $(-\infty, -1]$ and $(-\infty, 1]$ such that there is effectively no discontinuity along $(-\infty, -1]$ and there is only a branch cut along $[-1, 1]$, outside of which w is analytic. Consider a positively oriented Jordan curve Γ that encircles the branch cut. Determine
$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (z^2 - \tfrac{9}{4}) w(z)\, dz$$
by constructing a Laurent series of the integrand for $|z| > 1$.
HINT. Show that $w(z) = z\sqrt{1 - 1/z^2}$, with $\sqrt{\ }$ denoting the principal value square root.

20. Define the complex logarithm $w(z) = \log(z - 1) - \log(z + 1)$ with $w(2) = \ln(\frac{1}{3})$ and branch cuts along $(-\infty, -1]$ and $(-\infty, 1]$. Note that across $(-\infty, -1]$ the jump of $2\pi \mathrm{i}$ of one logarithm is cancelled by a jump of $-2\pi \mathrm{i}$ of the other logarithm. Consequently, there is effectively no discontinuity along $(-\infty, -1]$ and there is only a single branch cut along $[-1, 1]$, outside of which w is everywhere analytic. Consider a Jordan curve Γ that encircles the branch cut. Determine the integral
$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma (z^2 - \tfrac{5}{6}) w(z)\, dz$$
by constructing a Laurent series of $w(z)$ around $z = 0$ convergent in $|z| > 1$.

21. Determine the branch cuts of (the standard versions of) $\operatorname{arsinh} z$, $\operatorname{arcosh} z$ and $\operatorname{artanh} z$.

22. Show that $\cos(\sqrt{z})$ (with the domain of $\sqrt{z}$ extended to $\mathbb{C}$) is entire, while $\sin(\sqrt{z})$ is not.

2.10

1. Find the maximum of $|f(z)|$ in the unit disc $|z| \leqslant 1$ for the functions $f(z)$ given by: (a) e^z; (b) $z^4 + z^2 + 1$; (c) $(2z + 1)/(2z - 1)$.

2. (**Schwarz's Lemma**). Let $f(z)$ be analytic for $|z| \leqslant R$, $f(0) = 0$, and $|f(z)| \leqslant M$. Prove that $|f(z)| \leqslant M|z|/R$. HINT. Apply the maximum modulus principle to $f(z)/z$.

3. (**Minimum Modulus Principle**). Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.
HINT. Apply the maximum modulus principle to $1/f(z)$.

Exercises Chapter 3

3.1

Unless indicated otherwise, Jordan curves have a positive orientation.

1. Determine the singular points and corresponding residues of the following functions

$$\text{a)}\quad \frac{z^2+1}{z(2z-1)(z+1)}, \qquad \text{b)}\quad \frac{\mathrm{e}^{\mathrm{i}z}}{z^2+4}, \qquad \text{c)}\quad \frac{\cos(\pi z)}{z(2z-1)(6z-1)}.$$

2. Determine the integrals

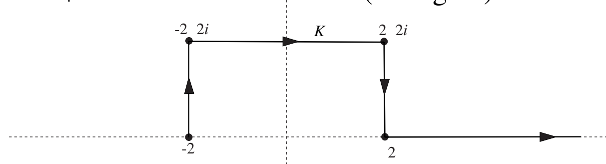
$$\text{a)}\quad \int_{|z|=3/2} \frac{\mathrm{e}^z}{(z+1)^2(z-2)}\,dz; \qquad \text{b)} \int_{|z|=2} \frac{\sin(\pi z)}{z^4-1}\,dz.$$

3. Determine the following integral for: a) $0 < R < 1$, b) $1 < R < 3$, c) $R > 3$.

$$\int_{|z|=R} \frac{dz}{(z^2+1)^8(z+3)}$$

4. In the complex plane, K is the integration contour consisting of the line segments from -2 to $-2+2\mathrm{i}$, from $-2+2\mathrm{i}$ to $2+2\mathrm{i}$, from $2+2\mathrm{i}$ to 2 and from 2 to ∞ (see figure). Determine the integral

$$\int_K \frac{dz}{z^2+2z+2}.$$



5. Classify the singularity in $z=0$ and determine the corresponding residue of

$$\text{a)}\quad (z+1)^2\cos\left(\tfrac{1}{z}\right), \qquad \text{b)}\quad \frac{1}{1+\frac{1}{z}}, \qquad \text{c)}\quad \frac{1}{(\log(1+z))^2},$$

where $\log(z)$ denotes the principal value logarithm with $\log(1)=0$.

6. Let $f(z)$ be analytic everywhere except in $z=0$, where $\pi\cot(\pi z)f(z)$ has residue r_0. Assume that there is a $K>0$ such that $|f(z)| \leqslant K/|z|^2$ for large enough $|z|$.
Consider for $N \in \mathbb{N}$ the rectangular contour Γ_N with vertices $(N+\frac{1}{2})(\pm 1 \pm \mathrm{i})$. Show that

$$\lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_N} \pi\cot(\pi z) f(z)\,dz = r_0 + \lim_{N\to\infty} \sum_{n=1}^{N} \big(f(n)+f(-n)\big) = 0.$$

Apply this to $f(z)=1/z^2$ to find the classic result $\sum_{n=1}^{\infty} n^{-2} = \frac{1}{6}\pi^2$.

3.2

1. Determine the following integrals:

a) $$\int_0^{2\pi} \frac{1 + 2\cos\varphi}{5 + 4\cos\varphi}\, d\varphi;$$ b) $$\int_0^{2\pi} \frac{2\cos\varphi}{(5 + 4\cos\varphi)^2}\, d\varphi;$$

c) $$\int_0^{2\pi} \frac{\sin\varphi}{4 + 3\mathrm{i}\sin\varphi}\, d\varphi;$$ d) $$\int_0^{2\pi} \frac{\cos(2\varphi)}{4 + 3\mathrm{i}\cos\varphi}\, d\varphi.$$

2.

a) $$\int_0^{2\pi} \frac{\cos(2\vartheta)}{12 + 5\mathrm{i}\cos\vartheta}\, d\vartheta;$$ b) $$\int_0^{2\pi} \frac{\cos(5\vartheta)}{3 + 2\sqrt{2}\cos\vartheta}\, d\vartheta;$$

c) $$\int_0^{2\pi} \frac{\sin\vartheta}{6 + 10\cos\vartheta + 8\mathrm{i}\sin\vartheta}\, d\vartheta;$$ d) $$\int_0^{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{(3 + 4\mathrm{i}\cos\vartheta)^2}\, d\vartheta.$$

3. Let $n = 0, 1, 2, \dots$. Determine the following integral by utilising Newton's binomium:

$$I_n = \int_0^{\pi/2} \cos^{2n}\varphi\, d\varphi,$$

4. Determine

a) $$\int_0^{2\pi} \cos\vartheta\, \cos(\sin\vartheta)\, d\vartheta,$$ b) $$\int_0^{2\pi} \sin\vartheta\, \cos(\sin\vartheta)\, d\vartheta.$$

Can you find the second integral also directly?

5. Determine

$$\int_0^{2\pi} \mathrm{e}^{2\cos\varphi}\, d\varphi.$$

6. Let $a > 0$ and $\varphi > 0$ be fixed real numbers. Determine the following integrals

a) $$\int_0^{2\pi} \frac{1}{1 + a^2 + 2a\cos\vartheta}\, d\vartheta;$$ b) $$\int_0^{2\pi} \frac{1}{\cosh\varphi + \cos\vartheta}\, d\vartheta.$$

7. Determine the following integrals

a) $$\int_0^{2\pi} \frac{1}{17 + 8\cos\vartheta}\, d\vartheta;$$ b) $$\int_0^{2\pi} \frac{\cos(2\vartheta)}{(5 - 4\cos\vartheta)^2}\, d\vartheta.$$

3.3

1. Determine the following integrals:

$$\text{a)}\quad \int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx; \qquad \text{b)}\quad \int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx;$$

$$\text{c)}\quad \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx; \qquad \text{d)}\quad \int_{-\infty}^{\infty} \frac{x}{(x-\mathrm{i}a)(x^2+1)}\,dx \quad (a \in \mathbb{R}).$$

HINT to c: show that $\frac{1}{x^6+1} = \frac{1}{2\mathrm{i}}\left(\frac{1}{x^3-\mathrm{i}} - \frac{1}{x^3+\mathrm{i}}\right)$, and then $\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \mathrm{Im}\left[\int_{-\infty}^{\infty} \frac{1}{x^3-\mathrm{i}}\,dx\right]$.

2. In the complex plane, $L(\alpha)$ is the half-line from $z = 0$, given by $\arg z = \alpha$. Determine the integral

$$I(\alpha) = \int_{L(\alpha)} \frac{1}{1-z^2}\,dz$$

for $\alpha = \frac{1}{2}\pi$ and for $0 < \alpha < \pi$.

3. Determine the integral

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,dx,$$

where $n = 1, 2, 3, \ldots$.

4. Determine for $a > 0$ the integral

$$I(a) = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos(2ax)\,dx.$$

HINT. Integrate the function e^{-z^2} along the rectangular integration contour with vertices $-R$, R, $R + \mathrm{i}a$, $-R + \mathrm{i}a$, where $R > 0$. Use the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,dx = \sqrt{\pi}.$$

5. (a) For which $a \in \mathbb{R}$ is the following integral convergent?

$$I(a) = \int_{-\infty}^{\infty} \frac{\cosh(ax)}{\cosh(x)}\,dx$$

   (b) Let K be the rectangular integration contour with vertices $-R$, R, $R + \pi\mathrm{i}$, $-R + \pi\mathrm{i}$, where $R > 0$. Determine the integral

$$\int_K \frac{\mathrm{e}^{az}}{\cosh z}\,dz.$$

   (c) Determine the integral $I(a)$ by using the previous result.

6. Determine the integrals (principal value square root and logarithm are assumed)

$$\text{a)}\quad \int_{-\infty}^{\infty} \frac{\sqrt{1+\mathrm{i}x}}{1+x^2}\,dx, \qquad \text{b)}\quad \int_{-\infty}^{\infty} \frac{\log(1-\mathrm{i}x)}{(1+x^2)^2}\,dx.$$

3.4

1. Let the following function be given

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}z}}{z^2 - z + 1}$$

   (a) Determine the points where $f(z)$ is not holomorphic.

   (b) Show that

$$|f(z)| \leqslant \frac{1}{R^2 - R - 1}$$

   for $|z| = R > 2$, $\operatorname{Im} z \geqslant 0$.

   (c) Determine

$$\lim_{R\to\infty} \int_{\substack{|z| = R \\ \operatorname{Im} z \geqslant 0}} f(z)\, dz$$

2. Determine the following integrals:

a) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^4 + 5x^2 + 4}\, dx;$ b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,\mathrm{e}^{-\mathrm{i}x}}{(x^2+1)^2}\, dx;$

c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\, \mathrm{e}^{\mathrm{i}ax}\, dx \quad (a \in \mathbb{R});$ d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx.$

3. Determine the following integrals:

a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 3\mathrm{i}x - 2}\, dx;$ b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x+\mathrm{i})(x-2\mathrm{i})}\, dx;$

c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x + \mathrm{i}a)^2}\, dx \quad (a > 0);$ d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{x^4 + x^2 + 1}\, dx.$

4. Determine for $a > 0$ the integral

$$I(a) = \int_0^{\infty} \frac{\sin^2 x}{x^2 + a^2}\, dx.$$

5. Determine for $z \notin \mathbb{R}$ the integral

$$\int_{-\infty}^{\infty} \frac{\cos t}{t - z}\, dt.$$

6. Determine the integral

$$\int_{\Omega} \frac{\cos(\pi z)}{z^2 - 1}\, dz,$$

where Ω is the path consisting of the interval $(-\infty, 0]$, the semi-circular arc $\{z \in \mathbb{C} \mid |z - 1| = 1, \operatorname{Im} z \geqslant 0\}$ traversed from $z = 0$ to $z = 2$, and the interval $[2, \infty)$.

7. Determine the integrals

$$\int_0^\infty \cos(x^2)\,dx \quad \text{and} \quad \int_0^\infty \sin(x^2)\,dx.$$

HINT. Integrate e^{iz^2} along the curve K consisting of the line segment on the real axis from 0 to R (where $R > 0$), the circular arc with parameter representation $z = R\,e^{i\vartheta}$ with $0 \leqslant \vartheta \leqslant \frac{1}{4}\pi$, and the line segment from $R\,e^{\pi i/4}$ to 0. Use the integral

$$\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}.$$

8. Determine the integrals

a) $\displaystyle\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx;$ b) $\displaystyle\int_0^\infty \frac{\sin x}{x(1+x^2)}\,dx;$

c) $\displaystyle\int_0^\infty \frac{x-\sin x}{x^3}\,dx;$ d) $\displaystyle\int_0^\infty \frac{\cos(\frac{1}{2}\pi x)}{1-x^2}\,dx.$

9. Show that

a) $\displaystyle\int_0^\infty \frac{e^{-x}-\cos x}{x}\,dx = 0,$ b) $\displaystyle\int_0^\infty \frac{e^{-x}-\cos x+\sin x}{x^2}\,dx = \tfrac{1}{2}\pi.$

HINT a). Deform the contour for a part of the integrand to the positive imaginary axis.
HINT b). Apply partial integration, and use the results of a) and example 3.4.5.

10. Show that $\displaystyle\int_0^\infty e^{ix^{1/p}}\,dx = e^{\frac{1}{2}p\pi i}\,\Gamma(p+1)$ for $0 < p < 1$.

11. Let $n \in \mathbb{N}$, $n \geqslant 1$. Show that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2-n^2\pi^2}\,dx = 0$ and $\displaystyle\int_0^\infty \frac{\sin^2 x}{(x^2-n^2\pi^2)^2}\,dx = \frac{1}{4n^2\pi}.$

3.5

1. Let $n \in \mathbb{N}$. Determine, by using residue calculus, the inverse Laplace transforms of

a) $\dfrac{1}{s^2+1};$ b) $\dfrac{1}{s^4+1};$ c) $\dfrac{1}{s(s^2+1)};$

d) $\dfrac{1}{s^3+4s^2+4s};$ e) $\dfrac{1}{s^{n+1}};$ f) $\dfrac{1}{(s+\alpha)^{n+1}};$

3.6

1. Verify for which $\mu \in \mathbb{R}\backslash\mathbb{Z}$ the following integrals converge and determine the integrals

a) $\displaystyle\int_0^\infty \frac{x^{\mu-1}}{(x^2+1)^2}\,dx;$ b) $\displaystyle\int_0^\infty \frac{x^{\mu-1}}{x^3+1}\,dx;$

c) $\displaystyle\int_0^\infty \frac{x^{\mu-1}}{x^2+x+1}\,dx;$ d) $\displaystyle\int_0^\infty \frac{x^{\mu-1}}{x^2+2x+1}\,dx.$

2. Determine the following integrals by using residue calculus

a) $\displaystyle\int_0^\infty \frac{1}{(x+1)^3+1}\,dx$; b) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$;

c) $\displaystyle\int_0^\infty \frac{1}{(x^2+x+1)^2}\,dx$; d) $\displaystyle\int_0^\infty \frac{1}{x^4+1}\,dx$.

3. Determine the following integrals

a) $\displaystyle\int_0^\infty \frac{x\ln x}{(x+1)^3}\,dx$; b) $\displaystyle\int_0^\infty \frac{\ln x}{x^3+1}\,dx$;

c) $\displaystyle\int_0^\infty \frac{\ln x}{x^2+x+1}\,dx$; d) $\displaystyle\int_0^\infty \frac{\ln x}{x^4+1}\,dx$.

4. Determine the following integral

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx.$$

by considering the integral of $(\log z)^2/(z^2+1)$ along $(-\infty+\mathrm{i}0, \infty+\mathrm{i}0)$.

5. Determine for $0<\mu<2$ the following integral

$$\int_0^\infty \frac{x^{\mu-1}\ln x}{x^2+1}\,dx.$$

6. Determine for real ω the integral

$$\int_{-\infty}^\infty \frac{\mathrm{e}^{\mathrm{i}\omega x}}{\cosh x}\,dx$$

by using the result of example 3.6.1. Compare this with exercise 3.3.5.

7. Let $0<p<1$. Show that

$$\int_0^\infty \frac{1}{x^p(1+x)}\,dx = \frac{\pi}{\sin(p\pi)}$$

by considering the limit for $R\to\infty$ and $\varepsilon\downarrow 0$ of the complex integral

$$\int_{\mathcal{C}} \frac{z^{-p}}{z-1}\,dz$$

with a principal value power, along the positively oriented Jordan curve

$$\mathcal{C} = \{z\in\mathbb{C} \mid z = R\,\mathrm{e}^{\mathrm{i}\vartheta};\ -\pi+0<\vartheta<\pi-0\} \cup \{z\in\mathbb{C} \mid z = x+\mathrm{i}0\,;\ -R<x<-\varepsilon\}$$
$$\cup\,\{z\in\mathbb{C} \mid z = \varepsilon\,\mathrm{e}^{\mathrm{i}\vartheta}\,;\ -\pi+0<\vartheta<\pi-0\} \cup \{z\in\mathbb{C} \mid z = x-\mathrm{i}0\,;\ -R<x<-\varepsilon\}.$$

Exercises Chapter 4

4.1

1. Formulate Fourier series, its coefficients and Parseval's identity for function g with period L.
2. a) Show, if $f(x)$ is even on $[-\pi, \pi]$, that its Fourier series on $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(nx), \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx, \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

   b) Show, if $f(x)$ is odd on $[-\pi, \pi]$, that its Fourier series on $[-\pi, \pi]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

3. Determine the Fourier series on $[-1, 1]$ of $f(x) = x^2$. Comment on the convergence rate.
4. Determine the Fourier series on $[-1, 1]$ of $f(x) = |x|$.
5. Determine the Fourier series on $[0, \pi]$ of $f(x) = \sin x$ and $f(x) = \cos x$. Explain the different convergence rates.
6. Determine the Fourier series on $[0, 2\pi]$ of $f(x) = (\cos x)^n$, where $n \in \mathbb{N}$.
7. Determine the Fourier series on $[-L, L]$ of $f(x) = \mathrm{e}^{-|x|}$.
8. Determine the Fourier series on $[-\pi, \pi]$ of $f(t) = \mathrm{e}^{\mathrm{i}z \sin t}$. (Use section 6.1.)
9. Determine the Fourier series of the 2π-periodic $f(t) = (2 - \cos t)^{-1}$ by residue calculus.
10. A continuous function $f(x, y)$ is periodic, with the same period, in x and in y. Under what condition on a and b can the function $g(t) = f(at, bt)$ be expanded in a Fourier series?
11. Show, by applying Parseval's identity to the Fourier series on $[-\pi, \pi]$ of $f(x) = x^2$, that $\sum_{n=1}^{\infty} 1/n^4 = \frac{1}{90}\pi^4$.
12. Show, by applying Parseval's identity to the Fourier series on $[-\pi, \pi]$ of $f(x) = x^3 - \pi^2 x$, that $\sum_{n=1}^{\infty} 1/n^6 = \frac{1}{945}\pi^6$.

13. a) Show that $D_n(x)$ (known as the Dirichlet kernel) equals the following Fourier series

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2\pi\,\sin\frac{1}{2}x} = \frac{1}{2\pi}\sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}kx}\,.$$

b) Let f be a continuous function, defined on $[-\pi,\pi]$. Determine

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)D_n(x)\,dx.$$

14. We study the Gibb's phenomenon for the Fourier series of the sawtooth-type function

$$\left[\tfrac{1}{2}\pi - \tfrac{1}{2}x\right]_{0,2\pi} = \lim_{n\to\infty} S_n(x), \quad \text{where} \quad S_n(x) = \sum_{k=1}^{n-1}\frac{\sin(kx)}{k}.$$

a) Show that $S_n'(x) = \cos(\frac{1}{2}nx)\sin(\frac{1}{2}(n-1)x)/\sin(\frac{1}{2}x)$. Note that the first zero of $S_n'(x)$ on the right of 0, and hence the first maximum of $S_n(x)$, is at $y_n = \pi/n$, the zero of $\cos(\frac{1}{2}nx)$.

b) Show that $S_n(y_n)$ is just a Riemann sum of the function $\sin(z)/z$, approximating for $n\to\infty$ an integral. Determine this integral.

c) Find a numerical value of this integral. What is, for $n\to\infty$, the overshoot of $S_n(x)$ at $x = 0+$? (Answer: 18%).

15. Determine the (finite!) Fourier series on $[-\pi,\pi]$ of $f(x) = (\cos x)^4$. Use this result to find the integral $\int_{-\pi}^{\pi}(\cos x)^8\,dx$.

4.2

1. Determine the Fourier transform of $\mathrm{e}^{-\varepsilon|t|}\cos(\omega_0 t)$, with $\varepsilon > 0$.

2. Determine the Fourier transform of the block function $H(x-a) - H(x-b)$, where $a < b$.

3. While $\lambda \neq 0$, express the Fourier transform of $\mathrm{e}^{\mathrm{i}\alpha x} f(\lambda x + \mu)$ in terms of $\hat{f}(\omega)$, the Fourier transform of $f(x)$. Distinguish carefully between $\lambda > 0$ and $\lambda < 0$.

4. Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $= 0$ elsewhere. Determine from this

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

5. Find the Fourier transform of $f(x) = 1 - |x|$ if $|x| < 1$ and $= 0$ elsewhere. Determine then

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx.$$

6. Find, for $a > 0$ real, the Fourier transform of $f(x) = a^2 - x^2$ if $|x| < a$ and $= 0$ elsewhere. Conclude from this
$$\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx.$$

7. Determine, by applying Parseval's identity to the block function $H(x+1) - H(x-1)$,
$$\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx.$$

8. a) Show that a function can be written uniquely as the sum of an even and an odd function.
   b) Show that the Fourier transform of a real function f is real if f is even, imaginary if odd.

9. Find the Fourier transforms of the real functions $1/\sqrt{|x|}$ and $\operatorname{sign}(x)/\sqrt{|x|}$.
   HINT. You may use the identity $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{1}{4}\sqrt{2\pi}$.

10. Which causal (→ p. 87) function $f(t)$ has Fourier Transform $\hat{f}(\omega) = 1/\sqrt{\mathrm{i}\omega}$?
    Here, $\sqrt{\cdot}$ denotes the principal value square root. You may use the hint of exercise 4.2.9.

11. Find the spatial Fourier transform $\hat{u}(\alpha, t)$ of the Gaussian wave packet $u(x,t) = \mathrm{e}^{\mathrm{i}k_0(x-ct)-(x-ct)^2/\sigma^2}$.

4.3

1. Determine, by using the definition of the Gamma-function, the Laplace transform of the function $f(t) = t^{\lambda-1}\,\mathrm{e}^{ct}$, where $\lambda > 0$.

2. Determine the Laplace transform of the polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$.

3. Determine the Laplace transforms of $\sin(at)$ and $\sinh(at)$.

4. Show that the Laplace transform of $\operatorname{erf}(x)$ is $s^{-1}\,\mathrm{e}^{\frac{1}{4}s^2}\operatorname{erfc}(\frac{1}{2}s)$ for $\operatorname{Re} s > 0$.

5. Find the Laplace transform $F(s)$ of the Gaussian modulated oscillation $f(t) = \mathrm{e}^{\mathrm{i}\omega_0 t - t^2/\tau^2}$.

6. Show that the Laplace transform $F(s)$ of the T-periodic function $f(t)$ with $f(t) = f(t+T)$ is given by
$$F(s) = \frac{1}{1-\mathrm{e}^{-sT}}\int_0^T f(t)\,\mathrm{e}^{-st}\,dt.$$

7. Solve by Laplace transformation the following homogeneous initial value problem
$$y'' - 4y = 0, \quad y(0) = 0,\ y'(0) = 1.$$

8. Solve by Laplace transformation the following inhomogeneous initial value problem
$$y'' - 5y' + 6y = \mathrm{e}^{2t}, \quad y(0) = 0,\ y'(0) = 0.$$

4.4

1. Prove for $\lambda > 0, \notin \mathbb{N}$ and $\lfloor\lambda\rfloor =$ the greatest integer $\leqslant \lambda$, that

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi n\lambda)}{n} = 1 - 2(\lambda - \lfloor\lambda\rfloor)$$

by applying Poisson's Formula to the top-hat function $B(x) = H(x+1) - H(x-1)$. (Note: the result remains valid also for $\lambda < 0$.)

2. a) Prove the result of example 4.4.2, $p = 1$, directly by Fourier transformation of $\dfrac{1}{1+x^2}$, *i.e.* show that

$$\sum_{n=0}^{\infty}\frac{1}{1+n^2} = \frac{\pi}{2}\coth(\pi) + \frac{1}{2}.$$

b) Show, for $0 \leqslant \zeta \leqslant 2\pi$, the more general result

$$\sum_{n=0}^{\infty}\frac{\cos(n\zeta)}{1+n^2} = \frac{\pi\cosh(\pi - \zeta)}{2\sinh(\pi)} + \frac{1}{2}.$$

3. Show that the same expression as found in example 4.4.2 can be derived for $\mathrm{Re}(p) < 0$. (Consider $f(x) = H(-x)\,\mathrm{e}^{px}$.) Prove the famous representation for $z \in \mathbb{C}\backslash\mathbb{Z}$

$$\pi\cot(\pi z) = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2 - n^2} = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}.$$

HINT. For $z \in \mathbb{C}\backslash\mathbb{R}$ it is straightforward. Apply then exercise 2.5.3.

4. Prove, by Poisson's generalised summation formula applied to $p^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{x^2}{2p}}$, Jacobi's identity

$$\vartheta(z,\tau) = \sqrt{\mathrm{i}/\tau}\,\mathrm{e}^{-\mathrm{i}\pi z^2/\tau}\,\vartheta(z/\tau, -1/\tau)$$

for the theta function

$$\vartheta(z,\tau) = \sum_{n=-\infty}^{\infty}\mathrm{e}^{2\pi\mathrm{i}nz + \pi\mathrm{i}n^2\tau}.$$

HINT. Prove first $\vartheta(z, \mathrm{i}t) = \sqrt{1/t}\,\mathrm{e}^{-\pi z^2/t}\,\vartheta(\mathrm{i}z/t, \mathrm{i}/t)$, and use that $\vartheta(z,\tau) = \vartheta(-z,\tau)$.

Exercises Chapter 5

5.1

REMARK: in this section, ε will always denote a real and positive small parameter.

1. Prove, for functions in $\varepsilon \to 0$, that

   (a) If $f = O(\varphi)$ and $g = o(\psi)$, then $fg = o(\varphi\psi)$.
   (b) If $f = O(\varphi)$ and $g = o(\varphi)$, then $f + g = O(\varphi)$.
   (c) If $f = O(\varphi)$ and $\varphi = o(\psi)$, then $f = o(\psi)$.
   (d) If $f = o(\varphi)$ and $\varphi = O(\psi)$, then $f = o(\psi)$.
   (e) If $f = O(\varphi)$ and $\varphi = O(f)$, then $f = O_s(\varphi)$.

2. What values of α, if any, yield $f = O(\varepsilon^\alpha)$, $f = o(\varepsilon^\alpha)$, and $f = O_s(\varepsilon^\alpha)$ as $\varepsilon \to 0$?

   (a) $f = \sqrt{1+\varepsilon^2}$
   (b) $f = \varepsilon \sin(\varepsilon)$
   (c) $f = (1 - \mathrm{e}^\varepsilon)^{-1}$
   (d) $f = \ln(1+\varepsilon)$
   (e) $f = \varepsilon \ln(\varepsilon)$
   (f) $f = \sin(1/\varepsilon)$
   (g) $f = \sqrt{x+\varepsilon}$, where $0 \leqslant x \leqslant 1$

3. Determine asymptotic expansions for $\varepsilon \to 0$ with respect to $\{\varepsilon^n (\ln \varepsilon)^k\}$ of

   (a) $\varepsilon / \tan \varepsilon$,
   (b) $\varepsilon/(1-\varepsilon^\varepsilon)$,
   (c) $1/\log(\sin \varepsilon)$.
   (d) $(1 - \varepsilon + \varepsilon^2 \ln \varepsilon)/(1 - \varepsilon \ln \varepsilon - \varepsilon + \varepsilon^2 \ln \varepsilon)$.

4. Assuming $f \sim a\varepsilon^\alpha + b\varepsilon^\beta + \ldots$, find α, β (with $\alpha < \beta$) and nonzero a, b for

   (a) $f = 1/(1 - \mathrm{e}^\varepsilon)$
   (b) $f = \sinh(\sqrt{1+\varepsilon x})$ for $0 < x < \infty$.
   (c) $f = \int_0^\varepsilon \sin(x + \varepsilon x^2)\, dx$

5. Are the following sequences *asymptotic sequences* for $\varepsilon \to 0$? If not, arrange them such that they are, or explain why it is not possible to do so.

   (a) $\varphi_n = (\varepsilon^\varepsilon - 1)^n$ for $n = 0, 1, 2, 3, \ldots$
   (b) $\varphi_n = \sin(1/\varepsilon)^n$ for $n = 0, 1, 2, 3, \ldots$
   (c) $\varphi_n = 1/\varepsilon^{1/n}$ for $n = 1, 2, 3, \ldots$
   (d) $\varphi_1 = 1, \varphi_2 = \varepsilon, \varphi_3 = \varepsilon^2, \varphi_4 = \varepsilon \ln(\varepsilon), \varphi_5 = \varepsilon^2 \ln(\varepsilon), \varphi_6 = \varepsilon \ln^2(\varepsilon), \varphi_7 = \varepsilon^2 \ln^2(\varepsilon)$.
   (e) $\varphi_n = \varepsilon^{n\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$
   (f) $\varphi_n = \varepsilon^{n/\varepsilon}$ for $n = 0, 1, 2, 3, \ldots$

6. Find a one-term asymptotic approximation, for $\varepsilon \to 0$, of the form $f(x, \varepsilon) \sim \varphi(x)$ that holds for $-1 < x < 1$. Sketch $f(x, \varepsilon)$ and φ, and then explain why the approximation is not uniform for $-1 < x < 1$.

   (a) $f(x, \varepsilon) = x + \exp\big((x^2 - 1)/\varepsilon\big)$
   (b) $f(x, \varepsilon) = x + \tanh(x/\varepsilon)$
   (c) $f(x, \varepsilon) = x + 1/\cosh(x/\varepsilon)$

7. Determine, if possible, uniform asymptotic expansions for $\varepsilon \to 0$ and $x \in [0, 1]$ of

   (a) $\sin(\varepsilon x)$,
   (b) $1/(\varepsilon + x)$,
   (c) $x \log(\varepsilon x)$,
   (d) $\mathrm{e}^{-\sin(x)\varepsilon}$,
   (e) $\mathrm{e}^{-\sin(x)/\varepsilon}$.

8. Prove that a Taylor series in ε around $\varepsilon = 0$, with strictly positive radius of convergence and cut off to a finite number of terms, is at the same time an asymptotic expansion for $\varepsilon \to 0$.

9. Prove that $\log \varepsilon$ and $\sin(1/\varepsilon)$ possess no asymptotic expansions with respect to the asymptotic sequence $\{\varepsilon^n\}$.

10. Let $f(\varepsilon) \sim 1 + \sum_{n=1}^{\infty} a_n \varepsilon^n$ and $\log f(\varepsilon) \sim \sum_{n=0}^{\infty} b_n \varepsilon^n$. Show that $b_1 = a_1$ and $nb_n = na_n - (n-1)b_{n-1}a_1 - (n-2)b_{n-2}a_2 - \cdots - b_1 a_{n-1}$ for $n = 2, 3, \cdots$.

11. Consider the asymptotic sequence $\{\mu_n(\varepsilon)\}$ with $\mu_n(\varepsilon) = (\sin \varepsilon)^n$. Expand ε^k for $k \in \mathbb{N}$ in an asymptotic expansion with respect to $\{\mu_n\}$. The first 3 terms are sufficient.

5.2

1. Let φ be a smooth function (continuously differentiable) on $[0, a]$. Prove that for $\varepsilon \to 0$

   (a)

$$\int_0^a \frac{\varphi(t)}{t + \varepsilon}\, dt = -\varphi(0) \log \varepsilon + O(1)$$

(b)

$$\int_0^a \frac{\varphi(t)}{t^2+\varepsilon^2}\,dt = \frac{\pi}{2\varepsilon}\varphi(0) + O(\log\varepsilon)$$

HINT. Write $\varphi(t) = \varphi(t) - \varphi(0) + \varphi(0)$ and prove that $\exists(K > 0)$ with $|\varphi(t) - \varphi(0)| \leqslant Kt$.

2. Find the (leading order) asymptotic behaviour for $x \to \infty$ of

$$\int_0^\infty \frac{\log(1+1/t)}{t+x}\,dt.$$

HINT. Split the integration interval in $[0, \lambda] \cup [\lambda, \infty)$ with $\lambda = O_s(\sqrt{x})$.

3. Find, by applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of

$$\int_0^\infty e^{-xt} \sin t\,dt.$$

4. Find, by introducing the new variable $y = t^3$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of

$$\int_0^\infty \frac{e^{-xt^3}}{1+t}\,dt.$$

5. Find, by introducing the new variable $y = (\sin t)^4$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of

$$\int_0^{\frac{1}{2}\pi} \sqrt{\sin t}\, e^{-x(\sin t)^4}\,dt.$$

6. Find, by introducing $t = x(y+1)$ and applying Watson's Lemma, the asymptotic behaviour for $x \to \infty$ of the exponential integral (*cf.* section 6.3, and example 5.2.4)

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt.$$

7. Let $f(t)$ be continuous on $[0, \infty)$, analytic in $t = 0$ and bounded for $t \to \infty$. Let $\mu > 0$. Find the asymptotic behaviour for $z \to \infty$ of

$$F(z) = \int_0^\infty f(t)\, e^{-zt^{1/\mu}}\,dt.$$

HINT. Transform $t = y^\mu$ and apply Watson's Lemma.

8. A representation of the Airy function $Ai(x)$ is given by

$$Ai(x) = \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{\pi} \int_0^\infty e^{-x^{\frac{1}{2}}t^2} \cos\left(\tfrac{1}{3}t^3\right)dt.$$

Find its asymptotic behaviour for $x \to \infty$. HINT. Transform $t^2 = \tau$.

9. Find the asymptotic behaviour for $m \to \infty$ of

$$S(m) = \int_0^{\pi} \left(\frac{\sin t}{t}\right)^m dt$$

by introducing $\tau = \ln(t/\sin t)$ and showing that

$$S(m) = \int_0^{\infty} \mathrm{e}^{-m\tau} \frac{dt}{d\tau}\, d\tau, \qquad \frac{dt}{d\tau} = \frac{t \sin t}{\sin t - t\cos t} = \sqrt{\frac{3}{2\tau}} + \ldots \quad (\tau \to 0).$$

10. Generalise Watson's Lemma for integrals of the form

$$\int_0^{\infty} t^{\mu-1} \ln(t) f(t)\, \mathrm{e}^{-xt}\, dt$$

where f is analytic in $t = 0$. You may use the identity

$$\int_0^{\infty} t^{x-1} \ln(t)\, \mathrm{e}^{-t}\, dt = \Gamma'(x) = \psi(x)\Gamma(x).$$

11. Find an asymptotic expansion for $\varepsilon \to 0$ of the integral

$$\int_0^{\infty} \mathrm{e}^{-t}\, t^{\mu} f(\varepsilon t) dt$$

where $\mu > -1$, and $f(x)$ is analytic in $x = 0$ and exponentially bounded for $x \to \infty$.

12. Find an asymptotic expansion for $\varepsilon \to 0$ of the integral

$$\int_0^{\infty} \frac{\mathrm{e}^{-t}}{(1+\varepsilon t)^3} dt.$$

13. Consider for $x, a, s > 0$ the function $h(x; a, s) = \mathrm{e}^{-(a/x)^s}$. Show that for $s \to \infty$, a fixed and $x \in (0, \infty)$ it tends to the unit step function $H(x-a)$, by proving that for a smooth and integrable testfunction $\varphi(x)$ the following integral satisfies

$$\int_0^{\infty} h(x; a, s)\varphi(x)\, dx = \int_0^{\infty} H(x-a)\varphi(x)\, dx - s^{-1}\gamma\varphi(a) + o(s^{-1})$$

You may assume that there is some number $K > a$ such that $\varphi(x) = 0$ for $x > K$.

5.3

1. (a) Find, by introducing $t = x\sqrt{y+1}$ and applying Watson's Lemma,
   (b) Find, by introducing $t = x + y$ and applying Watson's Lemma,
   (c) Find, by introducing $t = xy$ and applying a version of Laplace's method,

   the asymptotic behaviour for $x \to \infty$ of the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \mathrm{e}^{-t^2}\, dt.$$

   Compare with the exact values of erfc(2) and erfc(4).

2. Find the asymptotic behaviour for $x \to \infty$ of the real function $K(x)$, defined for $x > 0$ by

$$K(x) = \int_{-\infty}^{\infty} \mathrm{e}^{t}\, \mathrm{e}^{-xh(t)}\, dt, \quad \text{where} \quad h(t) = \mathrm{e}^{t} - t.$$

3. Consider, for real a, b, x, and real and smooth g, h, the following integral

$$\int_a^b \mathrm{e}^{-xh(t)}\, g(t)\, dt$$

asymptotically for $x \to \infty$. Let $h(t)$ attain its minimum at the interior point $t_0 \in (a, b)$, while $h''(t_0) > 0$. Show that the first term of the asymptotic expansion is

$$g(t_0)\, \mathrm{e}^{-xh(t_0)} \sqrt{\frac{2\pi}{xh''(t_0)}}\,.$$

Write out the next few terms.

4. Find the asymptotic behaviour for $x \to \infty$ of

$$\int_0^1 t^x \sin(t)^2\, dt$$

5. Find, with $\alpha > 0$, the asymptotic behaviour for $x \to \infty$ of

$$\int_0^{\infty} \mathrm{e}^{-xt + x^{\alpha} \log(t)}\, dt$$

HINT. Transform $t = x^{\alpha-1} y$ and apply Laplace's method.

6. Verify that if in Laplace's Method $g(c) = 0$, the main contribution is from $g''(c)$, provided this is nonzero. Find the asymptotic behaviour for large x of the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-xt^2} \ln(1 + t + t^2) dt.$$

5.4

1. Find, by using the Method of Stationary Phase, the asymptotic behaviour of the n-th order Bessel function $J_n(x)$ for $x, n \to \infty$ at fixed ratio n/x. See example 5.4.3. Assume $n/x < 1$.

2. Find, by introducing $t = \sqrt{x}\, y$, the asymptotic behaviour for large x of the Airy function

$$Ai(-x) = \frac{1}{\pi} \int_0^{\infty} \cos(xt - \tfrac{1}{3}t^3) dt.$$

3. a) Show that, as $x \to \infty$,

$$\int_0^{\pi} f(t)\, e^{ix\psi(t)}\, dt \simeq f(0)\, e^{ixa+\frac{1}{6}\pi i}\left(\frac{1}{27bx}\right)^{\frac{1}{3}} \Gamma(\tfrac{1}{3})$$

where $f(t)$ and $\psi(t)$ are smooth, $\psi'(t) \neq 0$ for $t > 0$, $f(0) \neq 0$, $\psi(0) = a$, $\psi'(0) = \psi''(0) = 0$ and $\psi'''(0) = 6b > 0$.

b) Consider the Bessel function (section 6.1)

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(nt - x \sin t)\, dt.$$

Show that, as $n \to \infty$,

$$J_n(n) \simeq \frac{\Gamma(\frac{1}{3})}{\pi (48)^{\frac{1}{6}} n^{\frac{1}{3}}}$$

Mathematical Tripos Part II, 2013

4. a) Describe how the leading-order asymptotic behaviour as $x \to \infty$ of

$$I(x) = \int_a^b f(t)\, e^{ixg(t)}\, dt$$

may be found by the method of stationary phase, where f and g are real functions and the integral is taken along the real line. You should consider the cases for which:

(i) $g'(t)$ is non-zero in $[a, b)$ and has a simple zero at $t = b$.
(ii) $g'(t)$ is non-zero apart from having one simple zero at $t = t_0$, where $a < t_0 < b$.
(iii) $g'(t)$ has more than one simple zero in (a, b) with $g'(a) \neq 0$ and $g'(b) \neq 0$.

b) Use the method of stationary phase to find the leading-order asymptotic form as $x \to \infty$ of

$$J(x) = \int_0^1 \cos\big(x(t^4 - t^2)\big)\, dt.$$

Mathematical Tripos Part II, 2014

5.5

1. By Fourier transformation to x of the equation

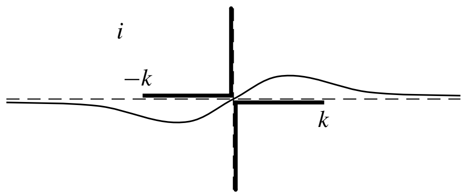
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + k^2 \varphi = 0$$

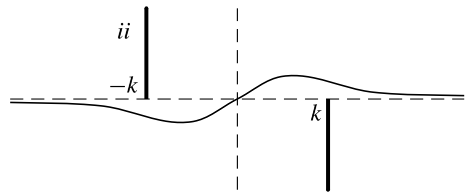
(where $k > 0$), solutions in $x \in (-\infty, \infty)$, $y \in [0, \infty)$, can be found of the form

$$\varphi(x, y) = \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x - i\gamma y}\, d\alpha, \quad \gamma(\alpha) = \sqrt{k^2 - \alpha^2}, \quad F(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_0(x)\, e^{i\alpha x}\, dx,$$

for given $\varphi_0(x) = \varphi(x, 0)$. In view of conditions of boundedness for $y \to \infty$ we choose a branch and branch cuts of γ such that $\mathrm{Im}(\gamma) \leqslant 0$ and $\gamma(0) = k$; see example 2.9.8-*iii*.

An integration contour is taken that never crosses the branch cuts (figure i).





Since we want to deform the contour into a steepest descent contour, it is more convenient to rotate the branch cuts away as in figure ii (example 2.9.8-*ii*). The integrand remains the same along the contour, so the integral does not change. We transform $\alpha = ku$, introduce $w(u) = \sqrt{1-u^2}$ analogous to γ, and rewrite

$$\varphi(x, y) = k \int_{-\infty}^{\infty} F(ku)\, \mathrm{e}^{-krh(u)}\, du,$$

where

$$h(u) = \mathrm{i}u \cos\vartheta + \mathrm{i}w(u)\sin\vartheta, \quad x = r\cos\vartheta, \quad y = r\sin\vartheta, \quad 0 \leqslant \vartheta \leqslant \pi.$$

Let F be given. Consider φ asymptotically for large kr.

(a) Determine the asymptotic behaviour of $\varphi(x, y)$ for $kr \to \infty$ by application of the Method of Stationary Phase, with the integration contour taken along the real axis.

(b) Find the saddle point u_0 of the phase function $\operatorname{Re} h(u)$. Note the sign of w.

(c) Determine the contour of steepest descent through u_0 for which u_0 is a minimum.
HINT. Consider $h(u) = h(u_0) + \lambda^2$ and solve for $u = u(\lambda)$. Make a plot.

(d) Determine the asymptotic behaviour of $\varphi(x, y)$ for $kr \to \infty$ by application of the Steepest Descent Method. Compare your result with the one found in a).

(e) The far field radiation pattern $D(\vartheta) = \lim_{kr\to\infty} \sqrt{kr}\,|\varphi(x, y)|$ is independent of r and signifies the angular dependence of the radiated field strength $|\varphi|$.
Determine $D(\vartheta)$ for a source of the form $\varphi_0(x) = \mathrm{e}^{-\mathrm{i}\alpha_0 x - \frac{1}{2}(x/L)^2}$, $\alpha_0 = \pm k\cos\vartheta_0$. (First, determine $F(\alpha)$ from $\varphi_0(x)$.) What happens if kL becomes large enough?

2. Consider the integral

$$I(x) = \int_0^1 \frac{1}{\sqrt{t-t^2}}\, \mathrm{e}^{\mathrm{i}x(t^2+t)}\, dt$$

for real $x > 0$. Find and sketch, in the complex t-plane, the paths of steepest descent through the endpoints $t = 0$ and $t = 1$ and through any saddle point(s). Obtain the leading order term in the asymptotic expansion of $I(x)$ for large positive x.

Mathematical Tripos Part II, 2016

The Names

Gottfried Wilhelm (von) Leibniz, 1646 - 1716
Brook Taylor, 1685 - 1731
James Stirling, 1692 - 1770
Daniel Bernoulli, 1700 - 1782
Leonhard Euler, 1707 - 1783
Jean-Baptiste le Rond d'Alembert, 1717 - 1783
Pierre-Simon Laplace, 1749 - 1827
Marc-Antoine Parseval des Chênes, 1755 - 1836
Jean-Baptiste Joseph Fourier, 1768 - 1830
Siméon Denis Poisson, 1781 - 1840
Friedrich Wilhelm Bessel, 1784 - 1846
Claude-Louis Navier, 1785 - 1836
Georg Simon Ohm, 1789 - 1854
Augustin-Louis Cauchy, 1789 - 1857
George Green, 1793 - 1841
Christian Andreas Doppler, 1803 - 1853
Johann Peter Gustav Lejeune Dirichlet, 1805 - 1859
Joseph Liouville 1809 - 1882
Ernst Eduard Kummer, 1810 - 1893
Pierre Alphonse Laurent, 1813 - 1854
Karl Theodor Wilhelm Weierstrass, 1815 - 1897
George Gabriel Stokes, 1819 - 1903
Hermann Ludwig Ferdinand von Helmholtz, 1821 - 1894
Georg Friedrich Bernhard Riemann, 1826 - 1866
Carl Gottfried Neumann, 1832 - 1925
Marie Ennemond Camille Jordan, 1838 - 1922
Hermann Hankel, 1839 - 1873
Josiah Willard Gibbs, 1839 - 1903
Osborne Reynolds, 1842 - 1912
Nikolay Yegorovich Zhukovsky (Joukowski), 1847 - 1921
Oliver Heaviside, 1850 - 1925
Jules Henri Poincaré, 1854 - 1912
Giacinto Morera, 1856 - 1909
Emil Johann Wiechert, 1861 - 1928
Jacques Salomon Hadamard, 1865 - 1963
Martin Wilhelm Kutta, 1867 - 1944
Alfred-Marie Liénard, 1869 - 1958
Thomas John l'Anson Bromwich, 1875 - 1929
Henri Léon Lebesgue, 1875 - 1941
Paul Richard Heinrich Blasius, 1883 - 1970
Petrus Josephus Wilhelmus Debije, 1884 - 1966
George Neville Watson, 1886 - 1965
Hendrik Anthony Kramers, 1894 - 1952
Ralph Kronig, 1904 - 1995
John Wilder Tukey, 1915 - 2000
James William Cooley, 1926 - 2016

Glossary

This glossary contains Dutch translations of English technical terms

complex analysis	=	complexe analyse, $\sim$ functietheorie
analytical function	=	analytische functie
analytic function	=	analytische functie
holomorphic function	=	holomorfe functie
entire function	=	gehele functie
conformal mapping	=	conforme afbeelding
convergence	=	convergentie
logarithm	=	logaritme
convergent series	=	convergente reeks
convergent sequence	=	convergente rij
absolute convergent	=	absoluut convergent
uniform convergent	=	uniform convergent
consequence	=	gevolg
series	=	reeks
sequence	=	rij
power series	=	machtreeks
theorem	=	stelling
lemma	=	hulpstelling, lemma
corollary	=	gevolg, corollarium
region	=	gebied
set	=	verzameling
domain	=	domein
neighbourhood	=	omgeving
curve	=	kromme
arc	=	boog
contour	=	contour
triangle inequality	=	driehoeksongelijkheid
differentiable	=	differentieerbaar
continuous	=	continu
argument	=	argument
radial coordinate	=	radiële coördinaat
radius of convergence	=	convergentiestraal
bounded	=	begrensd
constraint	=	beperking
constrained	=	beperkt
complex numbers $\mathbb{C}$	=	complexe getallen
real numbers $\mathbb{R}$	=	reële getallen
rational numbers $\mathbb{Q}$	=	rationale getallen
integer numbers $\mathbb{Z}$	=	gehele getallen
natural numbers $\mathbb{N}$	=	natuurlijke getallen
upper half plane	=	bovenhalfvlak
lower half plane	=	onderhalfvlak
right half plane	=	rechterhalfvlak
left half plane	=	linkerhalfvlak
simple	=	enkelvoudig
multiple	=	meervoud(ig)
branch	=	tak, vertakking
branch point	=	vertakkingspunt
branch cut	=	vertakkingssnede
principal value	=	hoofdwaarde
principal branch	=	hoofdtak
principle	=	principe
square root	=	(gewone) wortel, tweedemachts wortel
cubic root	=	derdemachts wortel
n-th root	=	n-de machtswortel
root, zero	=	nulpunt (van een functie)
block function	=	blokfunctie
top hat	=	hoge zijden (hoge hoed)