

IUTAM Symposium on Computational Aero-Acoustics  
for Aircraft Noise PredictionMean Flow Boundary Layer Effects of  
Hydrodynamic Instability of Impedance Wall

S.W. Rienstra, M. Darau

*Department of Mathematics and Computer Science,  
Eindhoven University of Technology,  
P.O. Box 513, 5600 MB Eindhoven, NL*

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**Abstract**

The Ingard-Myers condition, modelling the effect of an impedance wall under a mean flow by assuming a vanishing boundary layer, is known to lead to an ill-posed problem in time-domain. By analysing the stability of a mean flow, uniform except for a linear boundary layer of thickness  $h$ , in the incompressible limit, we show that the flow is absolutely unstable for  $h$  smaller than a critical  $h_c$  and convectively unstable or stable otherwise. This critical  $h_c$  is by nature independent of wave length or frequency and is a property of liner and mean flow only. An analytical approximation of  $h_c$  is given for a mass-spring-damper liner. For an aeronautically relevant example,  $h_c$  is shown to be extremely small, which explains why this instability has never been observed in industrial practice. A systematically regularised boundary condition, to replace the Ingard-Myers condition, is proposed that retains the effects of a finite  $h$ , such that the stability of the approximate problem correctly follows the stability of the real problem.

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**1. Introduction**

The problem we address is primarily a modelling problem, as we aim to clarify why a seemingly very thin mean flow boundary layer cannot be neglected. At the same time, the physical insight we provide may help to interpret recent experimental results.

Consider a liner of impedance  $Z(\omega)$  at a wall along a main flow  $(U_0, \rho_0, c_0)$  with boundary layer of thickness  $h$  and acoustic waves of typical wavelength  $\lambda$ . At the wall, with vanishing mean flow, the impedance relates the local Fourier transformed pressure  $\hat{p}(\omega)$  and normal velocity component  $\hat{v}(\omega) \cdot \mathbf{n}$  in the following way [1]

$$\hat{p} = Z(\hat{v} \cdot \mathbf{n})$$

(where normal vector  $\mathbf{n}$  points into the wall). This, however, is not a convenient boundary condition when the mean flow boundary layer is thin and the effective mean flow model is one with slip along the wall. In such a case the Ingard-Myers model [2, 3, 4] utilizes the fact that if  $h \ll \lambda$ , the sound waves don't see any difference between a finite boundary layer and a vortex sheet, so that the limit  $h \rightarrow 0$  can be taken, resulting into the celebrated Ingard boundary

condition [2] for mean flow along a straight wall in (say)  $x$ -direction

$$i\omega(\hat{\mathbf{v}} \cdot \mathbf{n}) = \left[ i\omega + U_0 \frac{\partial}{\partial x} \right] \left( \frac{\hat{p}}{Z} \right)$$

or its generalisation by Myers [3] for mean flow along a curved wall

$$i\omega(\hat{\mathbf{v}} \cdot \mathbf{n}) = [i\omega + \mathbf{V}_0 \cdot \nabla - \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{V}_0)] \left( \frac{\hat{p}}{Z} \right).$$

It is clear that both conditions are extremely useful for numerical calculations in those cases where the boundary layer is indeed negligible.

For a long time, however, there has been doubts [5, 6, 7] about a particular wave mode that exists along a lined wall with flow and the Ingard-Myers condition. This mode has some similarities with the Kelvin-Helmholtz instability of a free vortex sheet [8] and may therefore represent an instability, although the analysis is mathematically subtle [9, 10, 11, 12].

Since there was little or no indication that this instability was real, the problem seemed to be of minor practical importance, at least for calculations in frequency domain. However, once we approach the problem in time domain such that numerical errors generate perturbations of every frequency, it appears to our modeller's horror that the instability is at least in the model very real. The flow appears to be absolutely unstable [13, 9] and in fact it is worse: it is ill-posed [12]. Still, this absolute instability has not [14] or at least practically not [15] been reported in industrial reality, and only very rarely experimentally [16, 17, 18, 19] under special conditions. Although there is little doubt that the limit  $h \rightarrow 0$  is correct, there must be something wrong in our modelling assumptions. In particular, there must be a very small length scale in the problem, other than  $\lambda$ , on which  $h$  scales at the onset of instability. This is what we will consider here.

The present paper consists of three parts.

Firstly, we will show that the above modelling anomaly may be explained, in an inviscid model with a mean shear flow vanishing at the wall, by the existence of a (non-zero) critical boundary layer thickness  $h_c$ , such that the boundary layer is absolutely unstable for  $0 < h < h_c$  and not absolutely unstable (possibly convectively unstable) for  $h > h_c$ . It appears that for any industrially common configuration,  $h_c$  is very small. (We were originally inspired [21] for the concept of a critical thickness by the results of Michalke [22, 23] for the spatially unstable free shear layer, but it should be noted that an absolute instability is a more complex phenomenon.)

Secondly, we will make an estimate in analytic form of  $h_c$  as a function of the problem parameters. This will be valid for a certain parameter range that includes the industrially interesting cases.

Thirdly, we will propose a corrected or regularised ‘‘Ingard-Myers’’ boundary condition, that replaces the boundary layer (like the Ingard-Myers limit) but includes otherwise neglected terms that account for the finite boundary layer thickness effects. This new boundary condition is physically closer to the full problem and predicts (more) correctly stable and unstable behaviour.

## 2. The problem

An inviscid 2D parallel mean flow  $U_0(y)$  (figure 1), with uniform mean pressure  $p_0$  and density  $\rho_0$ , and small isentropic perturbations

$$u = U_0 + \tilde{u}, \quad v = \tilde{v}, \quad p = p_0 + \tilde{p}, \quad \rho = \rho_0 + \tilde{\rho}, \quad (1)$$

satisfies the usual linearised Euler equations given by

$$\begin{aligned} \frac{1}{\rho_0 c_0^2} \left( \frac{\partial \tilde{p}}{\partial t} + U_0 \frac{\partial \tilde{p}}{\partial x} \right) + \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0, \\ \frac{\partial \tilde{u}}{\partial t} + U_0 \frac{\partial \tilde{u}}{\partial x} + \frac{dU_0}{dy} \tilde{v} + \frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial x} &= 0, \\ \frac{\partial \tilde{v}}{\partial t} + U_0 \frac{\partial \tilde{v}}{\partial x} + \frac{1}{\rho_0} \frac{\partial \tilde{p}}{\partial y} &= 0, \end{aligned} \quad (2)$$

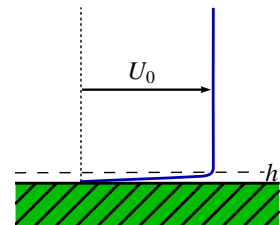


Figure 1: Mean flow.

where  $c_0$  is the sound speed and  $(\partial_t + U_0 \partial_x)(\tilde{p} - c_0^2 \tilde{\rho}) = 0$ . When we consider waves of the type

$$\tilde{p}(x, y, t) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \hat{p}(y; \alpha, \omega) e^{i\omega t - i\alpha x} d\alpha d\omega, \quad (3)$$

(similarly for  $\tilde{u}$ ,  $\tilde{v}$ ), the equations become

$$\begin{aligned} \frac{i(\omega - \alpha U_0) \hat{p}}{\rho_0 c_0^2} - i\alpha \hat{u} + \frac{d\hat{v}}{dy} &= 0, \\ i(\omega - \alpha U_0) \hat{u} + \frac{dU_0}{dy} \hat{v} - \frac{i\alpha}{\rho_0} \hat{p} &= 0, \\ i(\omega - \alpha U_0) \hat{v} + \frac{1}{\rho_0} \frac{d\hat{p}}{dy} &= 0. \end{aligned} \quad (4)$$

They may be further reduced to a form of the Pridmore-Brown equation [24] by eliminating  $\hat{v}$  and  $\hat{u}$

$$\frac{d^2 \hat{p}}{dy^2} + \frac{2\alpha \frac{d}{dy} U_0}{\omega - \alpha U_0} \frac{d\hat{p}}{dy} + \left( \frac{(\omega - \alpha U_0)^2}{c_0^2} - \alpha^2 \right) \hat{p} = 0. \quad (5)$$

At  $y = 0$  we have a uniform impedance boundary condition

$$-\frac{\hat{p}(0)}{\hat{v}(0)} = Z(\omega). \quad (6)$$

We select solutions of surface wave type, by assuming exponential decay for  $y \rightarrow \infty$ .

The mean flow is typically uniform everywhere, equal to  $U_\infty$ , except for a thin boundary layer of thickness  $h$ . We look for frequency ( $\omega$ ) and wavenumber ( $\alpha$ ) combinations that allow a solution. The stability of this solution will be investigated as a function of the problem parameters. In particular we will be interested in the critical thickness  $h = h_c$  below which the flow becomes absolutely unstable.

### 2.1. Dimension analysis and scaling

As the frequency and wave number at which the instability first appears is part of the problem, it is clear that  $h_c$  does not depend on  $\omega$  or  $\alpha$ . Furthermore, since the associated surface wave [6] is of hydrodynamic nature and inherently incompressible,  $h_c$  is only weakly depending on sound speed  $c_0$  and we can take  $M_0 = U_0/c_0 \rightarrow 0$ . As there are no other length scales in the fluid,  $h_c$  must scale on an inherent length scale of the liner. Suppose we have a liner of mass-spring-damper type with resistance  $R$ , inductance  $m$  and stiffness  $K$ , then

$$Z(\omega) = R + i\omega m - iK/\omega. \quad (7a)$$

If the liner is built from Helmholtz resonators [25] of cell depth  $L$  and

$$Z(\omega) = R + i\omega m - i\rho_0 c_0 \cotg(\omega L/c_0), \quad (7b)$$

and designed to work near the first cell resonance frequency, then  $\omega L/c_0$  is small for the relevant frequency range and  $K \approx \rho_0 c_0^2/L$ . Thus, we have 6 parameters ( $h_c, \rho_0, U_\infty, R, m, K$ ) and 3 dimensions (m, kg, s), so it follows from Buckingham's theorem that our problem has three dimensionless numbers, for example

$$\frac{R}{\rho_0 U_\infty}, \quad \frac{m}{\rho_0 h_c}, \quad \frac{K h_c}{\rho_0 U_\infty^2}. \quad (8)$$

Hence,  $h_c$  can be written (for example) in the form

$$h_c = \frac{\rho_0 U_\infty^2}{K} F\left(\frac{R}{\rho_0 U_\infty}, \frac{mK}{\rho_0^2 U_\infty^2}\right). \quad (9)$$

Later we will see that a proper reference length scale for  $h_c$ , i.e. one that preserves its order of magnitude, is a more complicated combination of these parameters. Since nondimensionalisation on arbitrary scaling values is not particularly useful, at least not here, we therefore deliberately leave the problem in *dimensional* form.

## 2.2. The model: incompressible piecewise linear shear flow

As the stability problem is essentially incompressible, we consider the incompressible limit, where Mach number  $M_0 = U_0/c_0 \rightarrow 0$ . Then the Pridmore-Brown equation reduces to

$$\frac{d^2 \hat{p}}{dy^2} + \frac{2\alpha \frac{d}{dy} U_0}{\omega - \alpha U_0} \frac{d\hat{p}}{dy} - \alpha^2 \hat{p} = 0. \quad (10)$$

If we assume a piecewise linear velocity profile of thickness  $h$

$$U_0(y) = \begin{cases} \frac{y}{h} U_\infty & \text{for } 0 \leq y \leq h \\ U_\infty & \text{for } h \leq y < \infty \end{cases} \quad (11)$$

we have an exact solution for our problem. For  $y \geq h$  we have

$$\hat{p} = A e^{-|\alpha|y}, \quad \text{where } |\alpha| = \sqrt{-i\alpha} \sqrt{i\alpha} = \pm \alpha \text{ if } \operatorname{Re}(\alpha) \geq 0. \quad (12)$$

where  $|\alpha|$  has branch cuts along  $(-\infty, 0)$  and  $(0, i\infty)$ . In the shear layer region  $(0, h)$  we have

$$\hat{p}(y) = C_1 e^{\alpha y} (h\omega - \alpha y U_\infty + U_\infty) + C_2 e^{-\alpha y} (h\omega - \alpha y U_\infty - U_\infty) \quad (13a)$$

$$\hat{u}(y) = \frac{\alpha h}{\rho_0} (C_1 e^{\alpha y} + C_2 e^{-\alpha y}) \quad (13b)$$

$$\hat{v}(y) = \frac{i\alpha h}{\rho_0} (C_1 e^{\alpha y} - C_2 e^{-\alpha y}). \quad (13c)$$

This last solution is originally due to Rayleigh [26], but has been used in a similar context of stability of flow along a flexible wall by Lingwood & Peake [27].

## 2.3. The dispersion relation

When we apply continuity of pressure and particle displacement at the interface  $y = h$ , and the impedance boundary condition at  $y = 0$ , we obtain the necessary relation between  $\omega$  and  $\alpha$  for a solution to exist. This is the dispersion relation of the waves of interest, given by

$$D(\alpha, \omega) = Z(\omega) - \frac{i\rho_0}{\alpha h} \frac{(U_\infty - h\omega)(|\alpha|(U_\infty + h\Omega) + \alpha h\Omega) e^{\alpha h} - (U_\infty + h\omega)(|\alpha|(U_\infty - h\Omega) + \alpha h\Omega) e^{-\alpha h}}{(|\alpha|(U_\infty + h\Omega) + \alpha h\Omega) e^{\alpha h} - (|\alpha|(U_\infty - h\Omega) + \alpha h\Omega) e^{-\alpha h}} = 0 \quad (14)$$

where

$$\Omega = \omega - \alpha U_\infty. \quad (15)$$

## 3. Stability analysis

We are essentially interested in any possible *spurious* absolutely unstable behaviour of our model, as this has by far the most dramatic consequences for numerical calculation in time-domain [13]. Of course, it is also of interest if the instability is physically genuine, like may be the case in [16, 17, 18, 19], but for aeronautical applications this is apparently very rare [15, 14].

To identify absolutely unstable behaviour we have to search for causal modes with vanishing group velocity (loosely speaking). For this we follow the method, originally developed by Briggs and Bers [28, 29] for plasma physical applications, but subsequently widely applied for fluid mechanical and aeroacoustical applications [30, 31, 27, 32, 9, 10].

If the impulse response of the system may be represented generically by a double Fourier integral

$$\Psi(x, y, t) = \frac{1}{(2\pi)^2} \int_{L_\omega} \int_{F_\alpha} \frac{\varphi(y)}{D(\alpha, \omega)} e^{i\omega t - i\alpha x} d\alpha d\omega, \quad (16)$$

the integration contours  $L_\omega$  and  $F_\alpha$  (figure 2) have to be located in domains of *absolute convergence* in the complex  $\omega$ - and  $\alpha$ -planes:

- For the  $\omega$ -integral,  $L_\omega$  should be below any poles  $\omega_j(\alpha)$  given by  $D(\alpha, \omega) = 0$ , where  $\alpha \in F_\alpha$ . This is due to causality that requires  $\Psi = 0$  for  $t < 0$  and the  $e^{i\omega t}$ -factor.
- For the  $\alpha$ -integral,  $F_\alpha$  should be in a strip along the real axis between the left and right running poles,  $\alpha^-(\omega)$  and  $\alpha^+(\omega)$  given by  $D(\alpha, \omega) = 0$ , for  $\omega \in L_\omega$ .

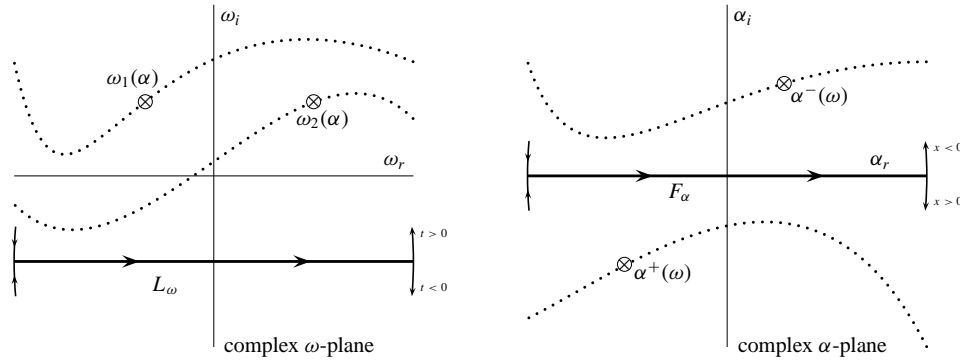


Figure 2: Paths of integration in  $\omega$ -plane and in  $\alpha$ -plane between *sketched* possible behaviour of poles.

The main idea is that we exploit the freedom we have in the location of  $L_\omega$  and  $F_\alpha$ . The first step is that we check that there exists a minimum imaginary part of the possible  $\omega_j$ :

$$\omega_{min} = \min_{\alpha \in \mathbb{R}} [\text{Im } \omega_j(\alpha)]. \tag{17}$$

This is relatively easy for a mass-spring-damper impedance, because the dispersion relation is equivalent to a third order polynomial in  $\omega$  with just 3 solutions, which can be traced without difficulty. See figure 3 for a typical case (note that we have to consider only  $\text{Re}(\alpha) > 0$  because of symmetry of  $D$ ). There is a minimum imaginary part, so

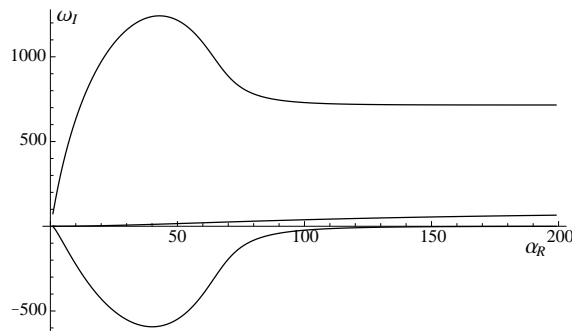


Figure 3: Plots of  $\text{Im}(\omega_j(\alpha))$  for  $\alpha \in \mathbb{R}$ . All have a minimum imaginary part so Briggs-Bers’ method is applicable. ( $\rho_0 = 1.22$ ,  $U_\infty = 82$ ,  $h = 0.01$ ,  $R = 100$ ,  $m = 0.1215$ ,  $K = 8166$ .)

Briggs-Bers’ method is applicable. Since  $\omega_{min} < 0$ , the flow is unstable.

Then we consider poles  $\alpha^+$  and  $\alpha^-$  in the  $\alpha$  plane, and plot  $\alpha^\pm(\omega)$ -images of the line  $\text{Im}(\omega) = c \geq \omega_{min}$ . Note that while  $c$  is increased, contour  $F_\alpha$  has to be deformed in order not to cross the poles, but always via the origin because of the branch cuts along the imaginary axis. As  $c$  is increased,  $\alpha^+$  and  $\alpha^-$  approach each other until they collide for  $\omega = \omega^*$  into  $\alpha = \alpha^*$ , where the  $F_\alpha$ -integration contour is pinched, unable to be further deformed; see figure 4 for a typical case. If  $\text{Im}(\omega^*) < 0$ , resp.  $> 0$ , then  $(\omega^*, \alpha^*)$  corresponds to an *absolute*, resp. *convective* instability. Since two solutions of  $D(\alpha, \omega) = 0$  coalesce, they satisfy the additional equation  $\frac{\partial}{\partial \alpha} D(\alpha, \omega) = 0$ .

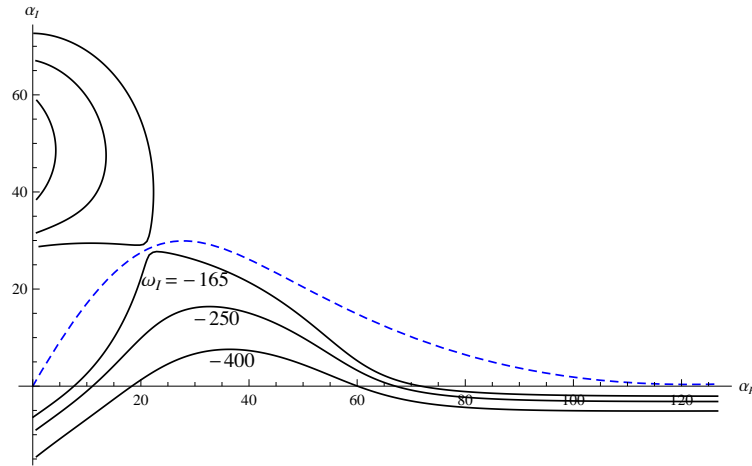


Figure 4: Plots of poles  $\alpha^+(\omega)$  and  $\alpha^-(\omega)$  for varying  $\text{Im}(\omega) = c$  until they collide for  $c = -165$ . So in this example (with  $\rho_0 = 1.22$ ,  $U_\infty = 82$ ,  $h = 0.01$ ,  $R = 100$ ,  $m = 0.1215$ ,  $K = 8166$ ) the flow is absolutely unstable.

### 3.1. A typical example from aeronautical applications

As a typical aeronautical example we consider a low Mach number mean flow  $U_\infty = 60$  m/s,  $\rho_0 = 1.225$  kg/m<sup>3</sup> and  $c_0 = 340$  m/s, with an impedance of Helmholtz resonator type [25]

$$Z(\omega) = R + i\omega m - i\rho_0 c_0 \cot\left(\frac{\omega L}{c_0}\right) \approx R + i\omega m - i\frac{\rho_0 c_0^2}{\omega L}, \quad (18)$$

which is chosen such that  $R = 2\rho_0 c_0 = 833$  kg/m<sup>2</sup>s, cell depth  $L = 3.5$  cm and  $m/\rho_0 = 25$  mm, with  $K = 4.0 \cdot 10^6$  kg/m<sup>2</sup>s<sup>2</sup> and  $m = 0.02$  kg/m<sup>2</sup>.

When we vary the boundary layer thickness  $h$ , and plot the imaginary part (= minus growth rate) of the found frequency  $\omega^*$ , we see that once  $h$  is small enough, the instability becomes absolute. See figure 5. We call the value of  $h$  where  $\text{Im}(\omega^*) = 0$  the critical thickness  $h_c$ , because for any  $h < h_c$  the instability is absolute. Note that  $\text{Im}(\omega^*) \rightarrow -\infty$  for  $h \downarrow 0$  so the growth rate becomes unbounded for  $h = 0$ , which confirms the ill-posedness of the Ingard-Myers limit, as observed by Brambley [12]. For the present example, the critical thickness  $h_c$  appears to be extremely small, namely

$$h_c = 8.2 \cdot 10^{-6} \text{ m} = 8.2 \text{ } \mu\text{m}, \quad \text{with } \omega^* = 14020.17 \text{ s}^{-1}, \quad \alpha^* = 466.268 + i5331.53 \text{ m}^{-1}. \quad (19)$$

It is clear that this is smaller than any practical boundary layer, so a real flow will not be unstable, in contrast to any model that adopts the Ingard-Myers limit, even though this is at first sight a very reasonable assumption if the boundary layers is only a fraction of any relevant acoustic wave length.

### 3.2. Approximation for large $R/\rho_0 U_\infty$ and large $\sqrt{mK}/R$

Insight is gained into the functional relationship between  $h_c$  and the other problem parameters by considering the asymptotic behaviour for large  $R/\rho_0 U_\infty$  and large  $\sqrt{mK}/R$ . If we define  $r = \frac{R}{\rho_0 U_\infty} \gg 1$  and assume  $\frac{\sqrt{mK}}{R} = O(r)$ , and scale  $\frac{m}{\rho_0 h_c} = O(r^4)$ ,  $\alpha h_c = O(r^{-1})$  and  $\frac{\omega h_c}{U_\infty} = O(r^{-2})$ , then we get to leading order from  $D(\alpha, \omega) = 0$  and  $D_\alpha(\alpha, \omega) = 0$

$$i\left(m\omega - \frac{K}{\omega}\right) + \left(R + i\rho_0 U_\infty \frac{\alpha h_c}{\frac{\omega h_c}{U_\infty} - \alpha^2 h_c^2}\right) + \dots = 0, \quad \left(\frac{i}{\frac{\omega h_c}{U_\infty} - \alpha^2 h_c^2} + \frac{2i\alpha^2 h_c^2}{\left(\frac{\omega h_c}{U_\infty} - \alpha^2 h_c^2\right)^2}\right) + \dots = 0 \quad (20)$$

With the condition that  $\omega$  is real, we have

$$\omega = \sqrt{\frac{K}{m}}, \quad \frac{\omega h_c}{U_\infty} + (\alpha h_c)^2 = 0, \quad \frac{R}{\rho_0 U_\infty} - \frac{i}{2\alpha h_c} = 0, \quad (21)$$

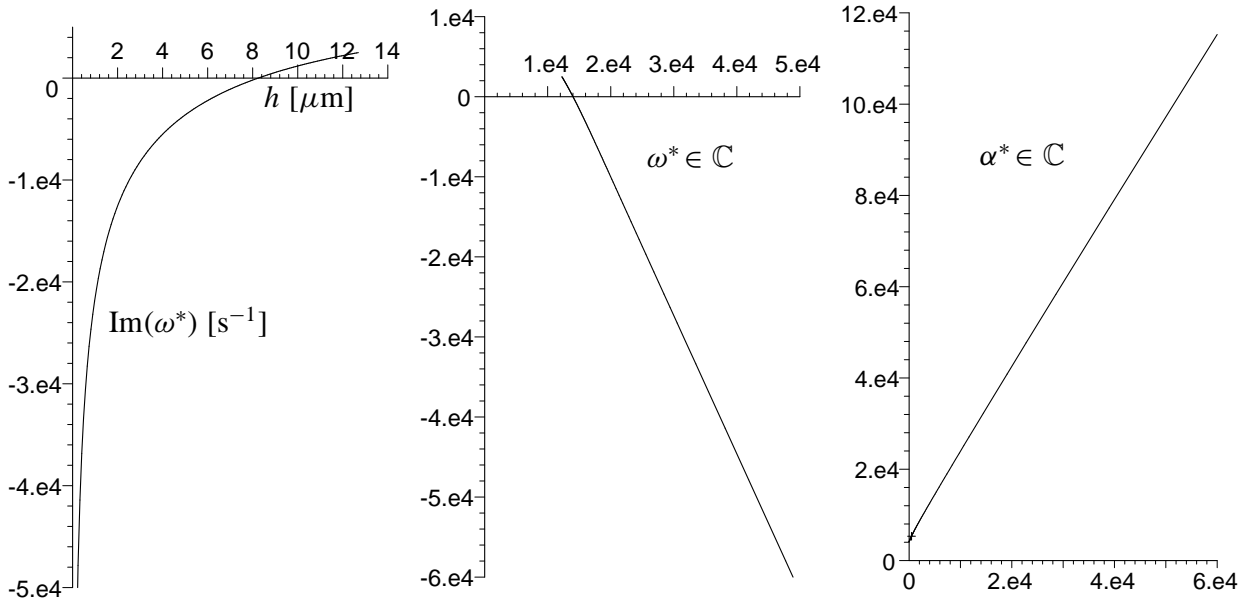


Figure 5: Growth rate  $\text{Im}(\omega^*)$  against  $h$  of potential absolute instability at vanishing group velocity (pinch point) is plotted together with the corresponding complex frequency  $\omega^*$  and wave number  $\alpha^*$ .

resulting into

$$h_c = \frac{1}{4} \left( \frac{\rho_0 U_\infty}{R} \right)^2 U_\infty \sqrt{\frac{m}{K}}. \quad (22)$$

This is confirmed by the numerical results given in figure 6. Here, quantity  $h_c R^2 \sqrt{K/m} / (\rho_0^2 U_\infty^3)$  is plotted against a varying  $R/\rho_0 U_\infty$  and a varying  $\sqrt{mK}/\rho_0 U_\infty$ , while otherwise the conditions are the same as in section 3.1. We see that for a rather large parameter range - including the above example (indicated by a dot) - this quantity remains between 0.2 and 0.25. So expression (22) appears to be a good estimate of  $h_c$  for  $R$ ,  $K$  and  $m$  not too close to zero.

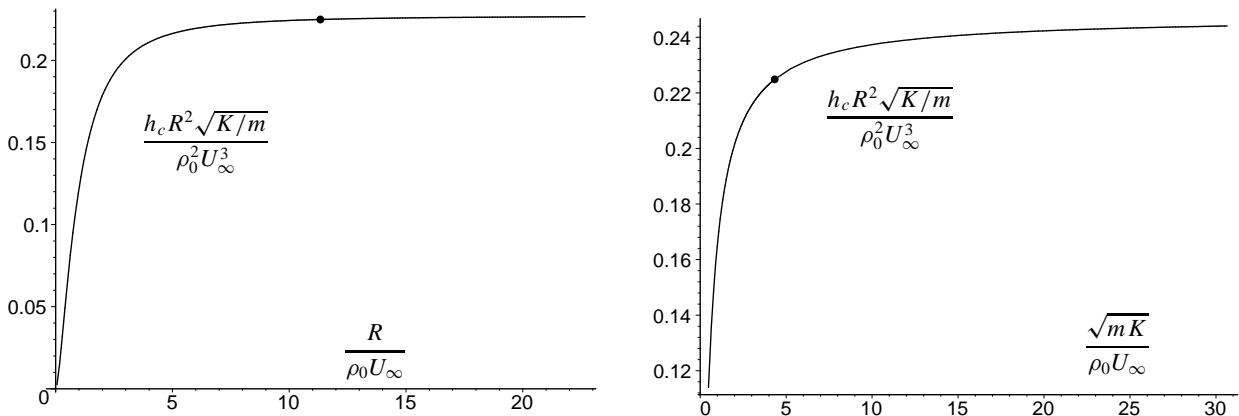


Figure 6: Variation in  $R$  and  $\sqrt{mK}$  with  $U_\infty = 60$ ,  $\rho_0 = 1.225$ ,  $K = 4 \cdot 10^6$ ,  $R = 2\rho_0 c_0$  and  $m = 0.02$ . The dot corresponds with the conditions of example 3.1.

#### 4. A regularised boundary condition

If we carefully consider the second order approximation for  $ah \rightarrow 0$  of both the nominator and denominator of dispersion relation  $D(\alpha, \omega) = 0$ , we find

$$Z(\omega) = \frac{\rho_0}{i} \cdot \frac{\Omega^2 + |\alpha|(\omega\Omega + \frac{1}{3}U_\infty^2\alpha^2)h + O(h^2)}{|\alpha|\omega + \alpha^2\Omega h + O(h^2)} \simeq \frac{i\Omega + \rho_0 \frac{|\alpha|}{i\Omega\rho_0} (i\omega i\Omega + \frac{1}{3}U_\infty^2(-i\alpha)^2)h}{-i\omega \frac{|\alpha|}{i\Omega\rho_0} + \frac{(-i\alpha)^2}{\rho_0}h}, \quad (23)$$

where  $\Omega = \omega - \alpha U_\infty$ . It should be noted that the solutions of *this* approximate dispersion relation have exactly the same behaviour with respect to the stability as the solutions of the original  $D(\alpha, \omega) = 0$ . Not only are all modes  $\omega_j(\alpha)$  bounded from below when  $\alpha \in \mathbb{R}$ , but also is the found  $h_c$  as a function of the problem parameters very similar to the “exact” one for the practical cases considered above. It therefore makes sense to consider an equivalent boundary condition that exactly produces this approximate dispersion relation and hence replaces the effect of the boundary layer (just like the Ingard-Myers limit) but now with a *finite*  $h$ . If we include a small but non-zero  $h$  the ill-posedness and associated absolute instability can be avoided. Most importantly, this is without sacrificing the physics but, on the contrary, by restoring a little bit of the inadvertently neglected physics!

If we identify the factor  $-i\alpha$  with an  $x$ -derivative, and at  $y = 0 \pm$  (that means: at  $y = \pm h$  for  $h \downarrow \uparrow 0$ )

$$\pm \hat{v} = \frac{|\alpha|}{i\Omega\rho_0} \hat{p} = -(\hat{v} \cdot \mathbf{n}), \quad (24)$$

for the normal vector  $\mathbf{n}$  pointing *into* the surface, then we have a “corrected” or “regularised” Ingard-Myers boundary condition

$$Z(\omega) = \frac{(i\omega + U_\infty \frac{\partial}{\partial x})\hat{p} - h\rho_0 \left( i\omega(i\omega + U_\infty \frac{\partial}{\partial x}) + \frac{1}{3}U_\infty^2 \frac{\partial^2}{\partial x^2} \right) (\hat{v} \cdot \mathbf{n})}{i\omega(\hat{v} \cdot \mathbf{n}) + \frac{h}{\rho_0} \frac{\partial^2}{\partial x^2} \hat{p}}, \quad (25)$$

which indeed reduces for  $h = 0$  to the Ingard approximation<sup>1</sup>, but now has the *physically correct* stability behaviour.

It should be noted that the above boundary condition is not unique. By identifying the factor  $|\alpha|$  with a  $\mp y$ -derivative, other forms that lead to the same dispersion relation are possible. Further research is underway to confirm the time-domain behaviour in CAA models.

#### 5. Conclusions

The stability of a mass-spring-damper liner with incompressible flow with piecewise linear velocity profile is analysed. The flow is found to be absolutely unstable for small but finite boundary layer  $h$ , say  $0 < h < h_c$ . In the limit of  $h \downarrow 0$  the growth rate tends to infinity and the flow may be called *hyper-unstable*, which confirms the ill-posedness of the Ingard-Myers limit.

The critical thickness  $h_c$  is a property of flow and liner, and has no relation with any acoustic wavelength. So neglecting the effect of a finite  $h$  (as is done when applying the Ingard-Myers limit) can not be justified by comparing  $h$  with a typical acoustic wavelength. An explicit approximate formula for  $h_c$  is formulated, which incidentally shows that the characteristic length scale for  $h_c$  is not easily guessed from the problem.

In industrial practice  $h_c$  is much smaller than any prevailing boundary layer thicknesses, which explains why the instability of the present kind has not yet been observed. At the same time this emphasises that  $h = 0$  is not an admissible modelling assumption, and a proper model (at least in time domain) will have to have a finite  $h > h_c$  in some way. Therefore, a corrected “Ingard-Myers” condition, including  $h$ , is proposed which is stable for  $h > h_c$ .

The linear profile has the great advantage of an exact solution, but of course the price to be paid is the absence of a critical<sup>2</sup> layer (since  $U_0'' \equiv 0$ ). This is subject of ongoing research.

<sup>1</sup>Note that the Myers generalisation for curved surfaces is far more complicated.

<sup>2</sup>A singularity of the solution at  $y = y_c$ , where  $\omega - \alpha U_0(y_c) = 0$ . No relation with  $h_c$ .



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